IJMMS 2003:9, 575–586
PII. S0161171203006744
http://ijmms.hindawi.com

THE EIGENVALUE PROBLEM FOR THE p-LAPLACIAN-LIKE EQUATIONS

ZU-CHI CHEN and TAO LUO

Received 16 February 2001

We consider the eigenvalue problem for the following p-Laplacian-like equation: $-\mathrm{div}(a(|Du|^p)|Du|^{p-2}Du) = \lambda f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. When λ is small enough, a multiplicity result for eigenfunctions are obtained. Two examples from nonlinear quantized mechanics and capillary phenomena, respectively, are given for applications of the theorems.

2000 Mathematics Subject Classification: 35J60, 35P30.

1. Introduction. This paper is devoted to the study of the eigenvalue problem for the p-Laplacian-like equation

$$\begin{gathered} -\mathrm{div}\left(a\left(|Du|^p\right)|Du|^{p-2}Du\right) = \lambda f(x,u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $1 < p < n$, Ω is a bounded smooth domain in $\mathbb{R}^n$, and Du denotes the gradient of u, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}^+, \mathbb{R})$.

We call λ an eigenvalue of (1.1) provided (1.1), for this λ, has a nontrivial weak solution, say u_λ, which is then called an eigenfunction corresponding to λ. Denote

$$A(r) = \int_0^r a(s)\,ds, \qquad F(x,t) = \int_0^t f(x,s)\,ds. \tag{1.2}$$

We look for nontrivial solutions of (1.1), and this question is reduced to show, for some $\lambda \in \mathbb{R}$, the existence of critical points for the functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega A\left(|Du|^p\right)dx - \lambda\int_\Omega F(x,u)\,dx, \quad u \in E = W_0^{1,p}(\Omega). \tag{1.3}$$

In [5], Pielichowski discussed the existence and nonnegativity of the first eigenvalue and eigenfunction, in a weak sense, of the *p-Laplace* equations with some kind of nonlinear terms below

$$-\mathrm{div}\left(|Du|^{p-2}Du\right) + a(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u. \tag{1.4}$$

Under the assumption that $A(\sigma) \le \phi(\sigma(x))/|\psi(\sigma(x))| \le B(\sigma)$ where $A(\sigma)$, $B(\sigma)$ are constants, Garcia-Huidobro et al. [4] proved the existence of eigenvalues and eigenfunctions for the *p-Laplacian-like* equation in the radial form

$$\begin{gathered} [r^{n-1}\phi(u')]' + \lambda r^{n-1}\psi(u) = 0, \quad r \in (0,R), \\ u'(0) = 0, \qquad u(R) = 0. \end{gathered} \tag{1.5}$$

They used the fixed-point theorem and continuation to techniques. Recently, Boccardo [2] showed the existence of positive eigenfunctions to a kind of *p-Laplace-like* equations

$$\begin{gathered} -\operatorname{div}(M(x,u)Du) = \lambda u, \quad x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ \|u\|_{L^2(\Omega)} = r \quad r \in \mathbb{R}^+. \end{gathered} \tag{1.6}$$

We are especially interested in Ubilla's paper [7], which studied the solvability of the boundary value problem for *p-Laplacian-like* equation in the radial form

$$\begin{gathered} -\Big(a\Big(|u'(r)|^p\Big)|u'(r)|^{p-2}u'(r)\Big)' = f(u(r)) \quad r \in I = (0,1) \\ u(0) = u(1) = 0. \end{gathered} \tag{1.7}$$

Under the assumption that

$$a(|t|^p)|t|^{p-2}t \in C^1(R\backslash\{0\},R) \cap C(R,R), \quad (a(|t|^p)|t|^{p-2}t)' > 0, \quad \forall t \neq 0, \tag{1.8}$$

a multiplicity result was obtained by using energy relations and the shooting method. The key of our trick is to change this assumption into that the mapping $r \mapsto A(|r|^p)$ defined in (1.2) is strictly convex, and then consider the eigenvalue problem (1.1). Also, the method we used, the mountain pass theorem and the minimax principle, is different from [7] and some other related papers (see [7] and the references therein). We got the existence of two eigenfunctions u_λ, v_λ not necessarily radial ones. In addition, we found that the behaviors of these two eigenfunctions near $\lambda = 0$ are much different as $\lim_{\lambda\to 0+}\|u_\lambda\|_E = +\infty$, $\lim_{\lambda\to 0+}\|v_\lambda\|_E = 0$. Our idea comes partially from [1].

2. Main results. Assume that

(A1) the mapping $r \mapsto B(r) = A(|r|^p)$ is strongly convex;

(A2) there exist constants $c_0 > 0$, $T > 0$ such that $A(t) \ge c_0 t$, for all $t \ge 0$ and $a(s) \le T$, for all $s \ge 0$;

(A3) there exist constants $b_0 > 0$, $b_1 > 0$ such that for all $x \in \Omega$,

$$|f(x,u)| \le b_0|u|^{r-1} + b_1|u|^{q-1}, \quad \text{for } 1 < q < p < r < p^*, \ p^* = \frac{np}{n-p}; \tag{2.1}$$

(A4) there exist constants t_0, θ such that $0 < \theta < c_0/pT$ where c_0, T are constants as in (A2), and

$$\theta f(x,t)t > F(x,t) > 0, \quad \forall x \in \overline{\Omega},\ 0 < t_0 < |t|; \tag{2.2}$$

(A5) for all $x \in \overline{\Omega}$, $t \geq 0$, $f(x,t) \geq 0$, it holds that

$$\lim_{t \to 0+} \frac{F(x,t)}{t^p} = +\infty. \tag{2.3}$$

Then we have the main results.

Theorem 2.1. *Under assumptions (A1) to (A5), there exists a number $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$, there exists an eigenfunction u_λ of (1.1) satisfying $\lim_{\lambda \to 0} \|u_\lambda\|_E = +\infty$.*

Theorem 2.2. *Assume (A1) to (A5) and $f(x,t) \geq 0$, then there is a number $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$, (1.1) has one eigenfunction u_λ behaving $\lim_{\lambda \to 0+} \|u_\lambda\|_E = 0$.*

3. Proof of the main results

Lemma 3.1. *Assume (A1) to (A4), then I_λ defined in (1.3) belongs to $C^1(E,R)$.*

Proof. Denote

$$I_A(u) = \frac{1}{p} \int_\Omega A(|Du|^p)\, dx, \quad I_F(u) = \lambda \int_\Omega F(x,u)\, dx, \quad u \in E \tag{3.1}$$

so $I_\lambda(u) = I_A(u) - I_F(u)$. We will then complete the proof by the following two claims.

Claim 1 ($I_A \in C^1(E,R)$). In fact, by (A1), for all $\lambda \in (0,1)$, $\varphi \in E$, we have

$$\begin{aligned} &\int_0^{|\lambda Du + (1-\lambda)(Du + D\varphi)|^p} a(s)\, ds \\ &\qquad \leq \lambda \int_0^{|Du|^p} a(s)\, ds + (1-\lambda) \int_0^{|Du + D\varphi|^p} a(s)\, ds, \end{aligned} \tag{3.2}$$

that is,

$$\begin{aligned} &\int_0^{|Du + (1-\lambda)D\varphi|^p} a(s)\, ds - \int_0^{|Du|^p} a(s)\, ds \\ &\qquad \leq (1-\lambda)\left(\int_0^{|Du + D\varphi|^p} a(s)\, ds - \int_0^{|Du|^p} a(s)\, ds \right). \end{aligned} \tag{3.3}$$

Set, in the above inequality, $1-\lambda = t$, we then have

$$\begin{aligned}\frac{I_A(u+t\varphi)-I_A(u)}{t} &= \frac{1}{tp}\int_\Omega \left(\int_0^{|Du+tD\varphi|^p} a(s)\,ds - \int_0^{|Du|^p} a(s)\,ds\right)dx \\ &\le \frac{1}{p}\int_\Omega \left(\int_0^{|Du+D\varphi|^p} a(s)\,ds - \int_0^{|Du|^p} a(s)\,ds\right)dx \\ &< +\infty,\end{aligned} \tag{3.4}$$

which is independent of t. Hence, we can apply the Lebesgue dominated convergence theorem to the equality

$$\begin{aligned}\frac{I_A(u+t\varphi)-I_A(u)}{t} &= \frac{1}{p}\int_\Omega a\big(|Du|^p + \eta(|Du+tD\varphi|^p - |Du|^p)\big) \\ &\quad \cdot \frac{1}{t}\big(|Du+tD\varphi|^p - |Du|^p\big)\,dx, \quad \text{for some } \eta \in (0,1),\end{aligned} \tag{3.5}$$

and letting $t \to 0$, we then get

$$I'_A(u)\varphi = \int_\Omega a\big(|Du|^p\big)|Du|^{p-2}\cdot D\varphi\,dx. \tag{3.6}$$

Next, we show that I'_A is continuous in u. In the following, the constant C may vary line by line.

Suppose $\{u_m\} \subset E$ satisfying $\|u_m - u\|_E \to 0$ as $m \to \infty$. We then claim that $\|I'_A(u_m) - I'_A(u)\| \to 0$. In fact,

$$\begin{aligned}&\|I'_A(u_m) - I'_A(u)\| \\ &= \sup_{\varphi\in E}\frac{\left|\int_\Omega \Big(a\big(|Du_m|^p\big)|Du_m|^{p-2}Du_m\cdot D\varphi - a\big(|Du|^p\big)|Du|^{p-2}Du\cdot D\varphi\Big)dx\right|}{\|\varphi\|_E} \\ &\le \frac{1}{p}\|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega)},\end{aligned} \tag{3.7}$$

where

$$B'(r) \equiv DB(r) = pa\big(|r|^p\big)|r|^{p-2}r, \quad r \in \mathbb{R}^n,\ p' = \frac{p}{p-1}. \tag{3.8}$$

Because $u_m \to u$ in E, by Egorov theorem, for any $\eta > 0$ there exists $\Omega_\eta \subset \Omega$ such that $|\Omega\backslash\Omega_\eta| < \eta$ and u_m, Du_m converge uniformly to u, Du, respectively, in Ω_η. Also, Ω_η can be chosen large enough so that the following holds as well

$$\int_{\Omega\setminus\Omega_\eta} |Du|^p dx < \frac{\varepsilon}{2} \tag{3.9}$$

for any given $\varepsilon > 0$. By virtue of $Du_m \to Du$ in $L^p(\Omega)$, when m is large enough,

$$\begin{aligned}\int_{\Omega\setminus\Omega_\eta} |Du_m|^p dx &< C\left(\int_{\Omega\setminus\Omega_\eta} |Du_m - Du|^p dx + \int_{\Omega\setminus\Omega_\eta} |Du|^p dx\right)\\ &< C\left(\int_{\Omega} |Du_m - Du|^p dx + \varepsilon/2\right) < C\varepsilon.\end{aligned} \tag{3.10}$$

Then, by (A2), (3.8), and (3.10), when m is large enough, we obtain

$$\begin{aligned}&\left(\int_{\Omega\setminus\Omega_\eta} |B'(Du_m)|^{p/(p-1)} dx\right)^{(p-1)/p}\\ &\quad \le p\left(\int_{\Omega\setminus\Omega_\eta} \left(T|Du_m|^{p-1}\right)^{p/(p-1)} dx\right)^{(p-1)/p} \le T(C\varepsilon)^{(p-1)/p},\end{aligned} \tag{3.11}$$

that is, $\|B'(Du_m)\|^{p'}_{L^{p'}(\Omega\setminus\Omega_\eta)} \le C\varepsilon$. Similarly,

$$||B'(Du)||^{p'}_{L^{p'}(\Omega\setminus\Omega_\eta)} \le C\epsilon. \tag{3.12}$$

Noticing that

$$\begin{aligned}||B'(Du_m) - B'(Du)||^{p'}_{L^{p'}(\Omega)} &\le ||B'(Du_m) - B'(Du)||^{p'}_{L^{p'}(\Omega_\eta)}\\ &\quad + ||B'(Du_m)||^{p'}_{L^{p'}(\Omega\setminus\Omega_\eta)} + ||B'(Du)||^{p'}_{L^{p'}(\Omega\setminus\Omega_\eta)}.\end{aligned} \tag{3.13}$$

We then get $\|I'_A(u_m) - I'_A(u)\| \to 0$ as $m \to \infty$. Therefore, I'_A is continuous at the point u, that is, $I_A \in C^1(E,R)$.

Claim 2 ($I_F \in C^1(E,R)$). The proof is similar to Claim 1 and we then omit it.

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Assume (A1) to (A4), then I_λ satisfies (PS) condition.*

Proof. From Lemma 3.1, we know that

$$I'_\lambda(u)\varphi = \int_\Omega [a(|Du|^p)|Du|^{p-2}Du\cdot D\varphi - \lambda f(x,u)\varphi]\,dx \quad \forall u,v \in E. \tag{3.14}$$

Suppose that $S = \{u_m\} \subset E$ satisfies that for some $M > 0$,

$$I_\lambda(u_m) \leq M, \quad \forall u_m \in S, \tag{3.15}$$

$$I'_\lambda(u_m) \longrightarrow 0. \tag{3.16}$$

We prove below that there exists a subsequence of $\{u_m\}$ converging strongly in E.

(a) At first, we show that S is bounded in E. From (3.16), for all $\varphi \in E$, it holds that

$$\int_\Omega \left[a\left(|Du_m|^p\right)|Du_m|^{p-2}Du_m \cdot D\varphi - \lambda f(x,u_m)\varphi\right]dx = o(1)\|\varphi\|_E. \tag{3.17}$$

Using (A4) and (A2), we have

$$\begin{aligned} I_\lambda(u_m) - \theta I'_\lambda(u_m)u_m &= \frac{1}{p}\int_\Omega A\left(|Du_m|^p\right)dx - \theta\int_\Omega a\left(|Du_m|^p\right)|Du_m|^p dx \\ &\quad + \lambda\int_\Omega [\theta f(x,u_m)u_m - F(x,u_m)]\,dx \\ &> \frac{1}{p}\int_\Omega A\left(|Du_m|^p\right)dx - \theta\int_\Omega a\left(|Du_m|^p\right)|Du_m|^p dx \\ &> \frac{c_0}{p}\int_\Omega |Du_m|^p dx - \theta\int_\Omega T|Du_m|^p dx. \end{aligned} \tag{3.18}$$

Combining this with (3.17) yields

$$\left(\frac{c_0}{p} - \theta T\right)\int_\Omega |Du_m|^p dx < M + o(1)\theta\|u_m\|_E, \tag{3.19}$$

which implies

$$\|u_m\|_E \leq C. \tag{3.20}$$

Hence, there exists a subsequence of S, still denoted by $\{u_m\}$, such that $u_m \rightharpoonup u$ in E and hence $Du_m \rightharpoonup Du$ in $L^p(\Omega)$, $u_m \to u$ in $L^s(\Omega)$, $1 < s < p^*$.

(b) Set

$$p_m(x) \equiv \left(a\left(|Du_m|^p\right)|Du_m|^{p-2}Du_m - a(|Du|^p)|Du|^{p-2}Du\right)(Du_m - Du), \tag{3.21}$$

then

$$\begin{aligned} I_m &\equiv \int_\Omega p_m(x)\,dx \\ &= \int_\Omega a\Big(|Du_m|^p\Big)|Du_m|^{p-2}Du_m(Du_m-Du)\,dx \\ &\quad - \int_\Omega a(|Du|^p)|Du|^{p-2}Du(Du_m-Du)\,dx \\ &\equiv I_m^{(1)}+I_m^{(2)}. \end{aligned} \tag{3.22}$$

We show below that $p_m(x)\to 0$ a.e. in Ω. As $Du_m \rightharpoonup Du$ in $L^p(\Omega)$, it is obvious that $I_m^{(2)}\to 0$. We choose in (3.17) $\varphi = u_m-u$, then

$$I_m^{(1)} = \lambda\int_\Omega f(x,u_m)(u_m-u)\,dx + o(1)\|u_m-u\|_E. \tag{3.23}$$

By (A3) and the Sobolev imbedding theorem,

$$\begin{aligned} \left|\int_\Omega f(x,u_m)(u_m-u)\,dx\right| &\le \|f(x,u_m)\|_{r'}\|u_m-u\|_r, \quad r'=r/r-1 \\ &\le \Big(b_0\|u_m^{r-1}\|_{r'}+b_1\|u_m^{q-1}\|_{r'}\Big)\|u_m-u\|_r \\ &\le c\Big(\|u_m\|_E^{r-1}+\|u_m\|_E^{q-1}\Big)\|u_m-u\|_r \\ &\longrightarrow 0, \quad \text{as } m\longrightarrow\infty. \end{aligned} \tag{3.24}$$

Therefore, from (3.23), $I_m^{(1)}\to 0$ and so $I_m\to 0$ as $m\to\infty$. Because $B(r)$ is strictly convex, then for all $r_1,r_2\in\mathbb{R}^n$, it holds that

$$\big(B'(r_1)-B'(r_2)\big)\cdot(r_1-r_2)\ge 0, \tag{3.25}$$

where the equality sign holds if and only if $r_1=r_2$. From this and the definition of $p_m(x)$, we then get $p_m(x)\ge 0$, which with $I_m\to 0$ gives $p_m(x)\to 0$, a.e. $x\in\Omega$. So we can find $\Omega_0\subset\Omega$ such that $\operatorname{meas}(\Omega-\Omega_0)=0$, $u_m(x)\to u(x)$ and $p_m(x)\to 0$ on Ω_0.

(c) Based on (3.25) and the fact that $p_m(x)\ge 0$, very similar to the first part of the proof of [3, Lemma 1], we can get $Du_m(x)\to Du(x)$, for all $x\in\Omega_0$.

(d) At last, we prove $\|u_m-u\|_E\to 0$. From the step (c), $Du_m\to Du$, a.e. $x\in\Omega$. By Egorov theorem, for any $\delta>0$, there exists $\Omega_\delta\subset\Omega$ such that $\operatorname{meas}(\Omega-\Omega_\delta)<\delta$ and Du_m converges uniformly to Du on Ω_δ. Because $B(r)$ is convex, then for any $r_1,r_2\in\mathbb{R}^n$ we have

$$B'(r_1)\cdot(r_1-r_2)\ge B(r_1)-B(r_2). \tag{3.26}$$

Choosing $r_2 = 0$, with $B(0) = A(0) = 0$, then

$$B'(r_1)\cdot r_1 \ge B(r_1) = A\left(|r_1|^p\right) \ge c_0|r_1|^p. \tag{3.27}$$

Suppose $\Omega' \subset \Omega$, by (3.27) and (3.8). Using (A2) and Young's inequality, we get

$$\begin{aligned}
\frac{c_0}{p}\int_{\Omega'}|Du_m(x)|^p dx &\le \int_{\Omega'} a(|Du_m|^p)|Du_m|^p dx \\
&= \int_{\Omega'} p_m(x)\,dx + \int_{\Omega'} a\left(|Du_m|^p\right)|Du_m|^{p-2}Du_m\cdot Du\,dx \\
&\quad + \int_{\Omega'} a(|Du|^p)|Du|^{p-2}Du\cdot Du_m\,dx \\
&\quad - \int_{\Omega'} a(|Du|^p)|Du|^p dx \\
&\le \int_{\Omega'} p_m(x)\,dx + T\int_{\Omega'}|Du_m|^{p-1}|Du|\,dx \\
&\quad + T\int_{\Omega'}|Du|^{p-1}|Du_m|\,dx + T\int_{\Omega'}|Du|^p dx \\
&\le \int_{\Omega'} p_m(x)\,dx + \varepsilon_1\int_{\Omega'}|Du_m|^p dx + C(\varepsilon_1)\int_{\Omega'}|Du|^p dx \\
&\quad + \varepsilon_2\int_{\Omega'}|Du_m|^p dx + C(\varepsilon_2)\int_{\Omega'}|Du|^p dx \\
&\quad + T\int_{\Omega'}|Du|^p dx.
\end{aligned} \tag{3.28}$$

Setting $\varepsilon_1 = \varepsilon_2 = c_0/4p$ in the above inequality yields

$$\frac{c_0}{2p}\int_{\Omega'}|Du_m(x)|^p dx \le \int_{\Omega'} p_m(x)\,dx + C\int_{\Omega'}|Du|^p dx. \tag{3.29}$$

Let $|\Omega'|$ be small enough so that for a given $\varepsilon > 0$ there holds

$$\int_{\Omega'}|Du|^p dx < \varepsilon. \tag{3.30}$$

Since $I_m \to 0$ and $p_m(x) > 0$, then when m is large enough we have

$$\int_{\Omega'} p_m(x)\,dx \le \int_{\Omega} p_m(x)\,dx < \varepsilon. \tag{3.31}$$

Combining this with (3.29), we get $\int_{\Omega'} |Du_m(x)|^p dx < C\varepsilon$ when m become large enough. Noticing $Du_m \to Du$ uniformly on $\Omega\backslash\Omega'$, then

$$\begin{aligned}
||Du_m - Du||_{L^p(\Omega)} &= ||Du_m - Du||_{L^p(\Omega\backslash\Omega')} + ||Du_m - Du||_{L^p(\Omega')} \\
&\leq ||Du_m - Du||_{L^p(\Omega\backslash\Omega')} + ||Du_m||_{L^p(\Omega')} + ||Du||_{L^p(\Omega')} \\
&\leq C\varepsilon \quad \text{as } m \text{ is large enough.}
\end{aligned} \tag{3.32}$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 2.1. We complete the proof by three steps.

Step 1. In fact, from (A3) we find

$$\left|F(x,u)\right| \leq \frac{b_0}{r}|u|^r + \frac{b_1}{q}|u|^q, \quad x \in \Omega. \tag{3.33}$$

Condition (A2) and the Sobolev imbedding theorem yield

$$\begin{aligned}
I_\lambda(u) &\geq \frac{c_0}{p}\int_\Omega |Du|^p dx - \lambda\int_\Omega \left(\frac{b_0}{r}|u|^r + \frac{b_1}{q}|u|^q\right)dx \\
&\geq \frac{c_0}{p}\|u\|_E - k_0\lambda\|u\|_E^r - k_1\lambda\|u\|_E^q,
\end{aligned} \tag{3.34}$$

where $k_0 > 0$, $k_1 > 0$ are constants and independent of u.

Suppose $u \in E$ satisfying that $\|u\|_E = \lambda^{-\alpha}$, $0 < \alpha < 1/(r-p)$, then by (3.34) we have

$$I_\lambda(u) \geq \frac{c_0}{p}\lambda^{-\alpha p} - k_0\lambda^{1-\alpha r} - k_1\lambda^{1-\alpha q}. \tag{3.35}$$

Because $0 < \alpha < 1/(r-p)$, then $\alpha_\lambda \equiv (c_0/p)\lambda^{-\alpha p} - k_0\lambda^{1-\alpha r} - k_1\lambda^{1-\alpha q} \to +\infty$ as $\lambda \to 0^+$. Hence, there exists $\lambda^* > 0$ small enough such that $\alpha_\lambda > 0$ for all $\lambda \in (0,\lambda^*)$. Then, we get

$$I_\lambda(u) \geq \alpha_\lambda > 0 \quad \text{for } \|u\|_E = \rho_\lambda, \tag{3.36}$$

where $\rho_\lambda = \lambda^{-\alpha}$.

Step 2. Condition (A4) implies that

$$F(x,t) > d_0 t^{1/\theta} - d_1, \quad \forall (x,t) \in \overline{\Omega}\times R, \tag{3.37}$$

where d_0, d_1 are positive constants. Using (3.37) and condition (A2), we find

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{p}\int_\Omega A(t^p|Dv|^p)\,dx - \lambda\int_\Omega F(x,tv)\,dx \\ &\le \frac{T}{p}\int_\Omega t^p|Dv|^p dx - \lambda\int_\Omega (d_0 t^{1/\theta} v^{1/\theta} - d_1)\,dx \\ &= \frac{T}{p} t^p \|v\|_E^p - \lambda d_0 t^{1/\theta}\|v\|_{1/\theta}^{1/\theta} + \lambda\tilde{d}_1. \end{aligned} \tag{3.38}$$

Condition (A2) implies that $c_0 \le T$, and then by (A4) we get $p < 1/\theta$. Thus as $t \to +\infty$, $I_\lambda(tv) \to -\infty$.

Step 3. By Lemma 3.2, I_λ satisfies the (PS) condition. Then, by the results of Steps 1 and 2, we can apply the mountain pass theorem to get that there exists a nontrivial critical point u_λ of I_λ such that

$$I_\lambda(u_\lambda) = c_\lambda \ge \alpha_\lambda > 0, \tag{3.39}$$

and then

$$\begin{aligned} I_\lambda(u_\lambda) &\le \frac{1}{p}\int_\Omega T|Du_\lambda|^p dx + \lambda\int_\Omega\Big(\frac{b_0}{r}|u_\lambda|^r + \frac{b_1}{q}|u_\lambda|^q\Big)dx \\ &= \frac{T}{p}||u_\lambda||_E^p + \frac{\lambda b_0}{r}||u_\lambda||_r^r + \frac{\lambda b_1}{q}||u_\lambda||_q^q \\ &\le \frac{T}{p}||u_\lambda||_E^p + \tilde{b}_0||u_\lambda||_E^r + \tilde{b}_1||u_\lambda||_E^q. \end{aligned} \tag{3.40}$$

Let $\lambda \to 0+$ in (3.40) as $\alpha_\lambda \to +\infty$, then we obtain $\|u_\lambda\|_E \to +\infty$. This completes the proof. □

Proof of Theorem 2.2. For $0 < \alpha < 1/p$, let $\|u\|_E = \lambda^\alpha$. By (3.34), we have

$$I_\lambda(u) \ge \frac{c_0}{p}\lambda^{\alpha p} - k_0\lambda^{1+\alpha r} - k_1\lambda^{1+\alpha q} = \lambda\Big(\frac{c_0}{p}\lambda^{\alpha p-1} - k_0\lambda^{\alpha r} - k_1\lambda^{\alpha q}\Big). \tag{3.41}$$

As $\alpha p - 1 < 0$, then there exists $\lambda^* > 0$ small enough so that $I_\lambda(u) > 0$ for $\lambda \in (0,\lambda^*)$, that is,

$$I_\lambda(u) > 0, \quad \forall\, 0 < \lambda < \lambda^*,\ \|u\|_E = \rho_\lambda, \tag{3.42}$$

where $\rho_\lambda = \lambda^\alpha$. Set $B_{\rho_\lambda} = \{u \in E : \|u\|_E < \rho_\lambda\}$, then for $u \in \overline{B}_{\rho_\lambda}$, by (3.34), we find

$$\begin{aligned} I_\lambda(u) &\ge \frac{c_0}{p}\|u\|_E^p - k_0\lambda\|u\|_E^r - k_1\lambda\|u\|_E^q \\ &\ge -k_0\lambda\rho_\lambda^r - k_1\lambda\rho_\lambda^q \ge -k_0(\lambda^*)^{1+r\alpha} - k_1(\lambda^*)^{1+q\alpha}, \end{aligned} \tag{3.43}$$

then I_λ is bounded blow on $\overline{B}_{\rho_\lambda}$. Choosing $v \in C_0^\infty(\Omega)$, $0 < v < 1$, $0 \le |Dv| \le 1$, $t \ge 0$, then

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{p}\int_\Omega A(t^p|Dv|^p)\,dx - \lambda\int_\Omega F(x,tv)\,dx \\ &\le \frac{T}{p}\int_\Omega t^p|Dv|^p dx - \lambda\int_\Omega F(x,tv)\,dx \\ &\le t^p\left[\frac{T}{p}\int_\Omega |Dv|^p dx - \lambda\frac{\inf_{x\in\overline{\Omega}}F(x,t)}{t^p}\int_\Omega \frac{F(x,tv)}{F(x,t)}dx\right]. \end{aligned} \tag{3.44}$$

From (A5), we know that $f(x,t) \ge 0$, for all $x \in \overline{\Omega}$, $t \ge 0$ and hence $F(x,tv)/F(x,t) \le 1$. By (3.44), (A5), and applying the dominated convergence theorem to (3.44), we find that there exist $\delta > 0$, $0 < t < \delta$, $tv \in B_{\rho_\lambda}$ such that

$$I_\lambda(tv) < 0. \tag{3.45}$$

Because I_λ satisfies the (PS) condition, the minimax theorem on $\overline{B}_{\rho_\lambda}$ claims that I_λ has a nontrivial critical point $u_\lambda \in B_{\rho_\lambda}$, which is a local minimum and $I_\lambda(u_\lambda) < 0$. Then $\|u_\lambda\|_E < \rho_\lambda = \lambda^\alpha$, $\|u_\lambda\|_E \to 0$ as $\lambda \to 0+$. This ends the proof. □

4. Examples

Example 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. We consider the *p-Laplacian* problem from nonlinear quantized mechanics as

$$\begin{gathered} -\operatorname{div}(|Du|^{p-2}Du) = \lambda(|u|^{q-2}u + |u|^{r-2}u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.1}$$

where $\lambda > 0$, $1 < p < n$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $1 < q < p < r < p^*$, $p^* = np/(n-p)$. In this case, $a(s) = 1$, $B(r) = |r|^p$ is strictly convex, and conditions (A2) and (A4) are satisfied for $c_0 = T = 1$ while $0 < \theta < 1/p$. Obviously, (A3) also holds. These conditions have been posted directly on the given functions in some papers which dealt with the solvability of the boundary value or eigenvalue problem for the *p-Laplacian* equation (see [6] and the references therein). Then, by virtue of Theorems 2.1 and 2.2, when λ is small enough, problem (4.1) possesses at least two eigenfunctions u_λ and v_λ, and

$$\lim_{\lambda\to 0}||u_\lambda||_E = +\infty, \qquad \lim_{\lambda\to 0}||v_\lambda||_E = 0. \tag{4.2}$$

Example 4.2. Consider the eigenvalue problem for generalized capillarity equation originated from the capillary phenomena

$$-\operatorname{div}\left(\left(1+\frac{|Du|^p}{\sqrt{1+|Du|^{2p}}}\right)|Du|^{p-2}Du\right)=\lambda\left(|u|^{q-2}u+|u|^{r-2}u\right),\quad x\in\Omega,$$
$$u(x)=0,\quad x\in\partial\Omega, \tag{4.3}$$

where $\lambda > 0$, $1 < q < p$, $2p < r < p^*$, $p^* = np/(n-p)$. We also can check that (A1) to (A5) are satisfied. By Theorems 2.1 and 2.2, there exist two eigenfunctions u_λ and v_λ and $\lim_{\lambda\to 0}\|u_\lambda\|_E = +\infty$, $\lim_{\lambda\to 0}\|v_\lambda\|_E = 0$.

Acknowledgment. The paper was supported by National Natural Science Foundation (NNSF) of China (project no. 10071080 and 10101024).

References

[1] A. Ambrosetti, H. Brezis, and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[2] L. Boccardo, *Positive eigenfunctions for some unbounded differential operators*, Recent Trends in Nonlinear Analysis, Progress in Nonlinear Differential Equations and Their Applications, vol. 40, Birkhäuser, Basel, 2000, pp. 41–45.

[3] Z.-C. Chen and X.-D. Yan, *A class of pseudo-monotone operators and its applications in PDE*, Acta Math. Sinica (N.S.) **13** (1997), no. 4, 517–526.

[4] M. Garcia-Huidobro, R. Manásevich, and K. Schmitt, *On principal eigenvalues of p-Laplacian-like operators*, J. Differential Equations **130** (1996), no. 1, 235–246.

[5] W. Pielichowski, *On the first eigenvalue of a quasilinear elliptic operator*, Selected Problems of Mathematics, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, vol. 6, Cracow University of Technology, Kraków, 1995, pp. 235–241.

[6] Y.-T. Shen and S.-S. Yan, *Variational Method in Quasilinear Elliptic Equations*, Press of Southern China University of Science and Technology, Guang Zhou, 1995 (Chinese).

[7] P. Ubilla, *Multiplicity results for the 1-dimensional generalized p-Laplacian*, J. Math. Anal. Appl. **190** (1995), no. 2, 611–623.

Zu-Chi Chen: Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China

E-mail address: chenzc@ustc.edu.cn

Tao Luo: Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China