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BLOCK TOEPLITZ OPERATORS WITH FREQUENCY-MODULATED SEMI-ALMOST PERIODIC SYMBOLS

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This paper is concerned with the influence of frequency modulation on the semi-Fredholm properties of Toeplitz operators with oscillating matrix symbols. The main results give conditions on an orientation-preserving homeomorphism α of the real line that ensure the following: if b belongs to a certain class of oscillating matrix functions (periodic, almost periodic, or semi-almost periodic matrix functions) and the Toeplitz operator generated by the matrix function $b(x)$ is semi-Fredholm, then the Toeplitz operator with the matrix symbol $b(\alpha(x))$ is also semi-Fredholm.

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1. Introduction. Let $H^2(\mathbb{R})$ be the usual Hardy space of the real line, that is, the Hilbert space of all functions in $L^2(\mathbb{R})$ that can be represented in the form

$$f(x) = \int_0^\infty g(t)e^{itx}\,dt, \quad x \in \mathbb{R}, \tag{1.1}$$

with $g \in L^2(0,\infty)$, and let P be the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. For a function $a \in L^\infty(\mathbb{R})$, the Toeplitz operator $T(a)$ is the bounded linear operator on $H^2(\mathbb{R})$ that acts by the rule $f \mapsto P(af)$. The function a is in this context referred to as the symbol of the operator $T(a)$.

The algebra $\mathrm{AP}(\mathbb{R})$ of almost periodic functions is defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains the set $\{e_\lambda : \lambda \in \mathbb{R}\}$, where $e_\lambda(x) = e^{i\lambda x}$. We denote by $P_T(\mathbb{R})$ the set of all T-periodic functions in $\mathrm{AP}(\mathbb{R})$ and by $C(\overline{\mathbb{R}})$ the set of all continuous functions f on $\mathbb{R}$ that have finite one-side limits $f(-\infty)$ and $f(+\infty)$ at infinity. Finally, the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains $\mathrm{AP}(\mathbb{R}) \cup C(\overline{\mathbb{R}})$ is denoted by $\mathrm{SAP}(\mathbb{R})$ and is called the algebra of semi-almost periodic functions.

For each class of functions X introduced above we denote by $X_{n\times n}$ and X_n the sets of all $n\times n$ and $n\times 1$ matrix functions with entries in X, respectively. We also write $L^\infty_{n\times n}(\mathbb{R}) := [L^\infty(\mathbb{R})]_{n\times n}$, $H^2_n(\mathbb{R}) := [H^2(\mathbb{R})]_n$, and so on. For $a \in L^\infty_{n\times n}(\mathbb{R})$, block Toeplitz operator $T(a)$ is defined by $T(a) : H^2_n(\mathbb{R}) \to H^2_n(\mathbb{R})$, $f \mapsto P(af)$, where P is the orthogonal projection of $L^2_n(\mathbb{R})$ onto $H^2_n(\mathbb{R})$.

A bounded linear operator A on a Hilbert space H is said to be normally solvable if its range $\operatorname{im} A$ is closed. We put $\ker A = \{f \in H : Af = 0\}$ and $\operatorname{coker} A := H/\operatorname{im} A$. If A is normally solvable and $\dim\ker A < \infty$, then A is called a Φ_+-operator, and if A is normally solvable and $\dim\operatorname{coker} A < \infty$, then A is called a Φ_--operator. A Fredholm operator is an operator that is both Φ_- and Φ_+. The index of a Fredholm operator A is the integer $\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A$. The operator A is right (left) invertible if there is a bounded linear operator B on H such that $AB = I$ ($BA = I$), where I is identity operator on H, and the operator A is invertible if there is a bounded operator B on H such that $AB = BA = I$. Finally the operator A is said to be generalized invertible if there is a bounded operator B such that $ABA = A$. It is easy to see that if A is left (right) invertible, then A is a $\Phi_+(\Phi_-)$-operator. Moreover, it is well known that generalized invertibility is equivalent to normal solvability.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an orientation-preserving homeomorphism. In this paper, we study the following problem: if b is in one of the classes of oscillating matrix functions introduced above and $T(b)$ is $\Phi_\pm$, Fredholm, or normally solvable, is the operator $T(\gamma_\alpha b)$ with the symbol $(\gamma_\alpha b)(x) := b(\alpha(x))$ also $\Phi_\pm$, Fredholm, or normally solvable?

If b is in $C_{n\times n}(\overline{\mathbb{R}})$, then the answer to the above question is known to be positive. However, the symbols b currently emerging are often oscillating, that is, they belong to $\mathrm{AP}_{n\times n}(\mathbb{R})$ or even $\mathrm{SAP}_{n\times n}(\mathbb{R})$. For example, in [1, 13], it is shown that the inverse scattering method for the modified Korteweg-de Vries equation leads to a Riemann-Hilbert factorization problem and thus to a Toeplitz operator whose symbol is of the form $\gamma_\alpha b$ with $b \in \mathrm{SAP}_{2\times 2}(\mathbb{R})$ and $\alpha(x) = c_1x^3 + c_2x$. Although equations with such symbols can be tackled successfully (see, e.g., [5, 6, 7]), the semi-Fredholm properties of the operators involved are not yet understood sufficiently well. This paper is a first attempt of systematically exploring the change of the semi-Fredholm properties of Toeplitz operators with oscillating symbols caused by frequency modulation.

In connection with the problem studied here, we want to mention the profound investigations by Muhly and Xia [19, 20, 21]. Their context is the complex unit circle $\mathbb{T}$, and to translate their results to the real line, we have to employ the substitutions

$$\varphi : \mathbb{R} \longrightarrow \mathbb{T}, \quad \varphi(x) = \frac{x-i}{x+i}, \qquad \psi : \mathbb{T} \longrightarrow \mathbb{R}, \quad \psi(t) = i\frac{1+t}{1-t}. \tag{1.2}$$

If α is a homeomorphism of $\mathbb{R}$ onto itself, then the map σ given by $\sigma = \varphi\circ\alpha\circ\psi$ is a homeomorphism of $\mathbb{T}$ onto itself such that $\sigma(1) = 1$. The result of [20] implies that if σ is a bi-Lipschitz homeomorphism and σ' belongs to VMO, then $T(\gamma_\alpha b) - T(b)$ is compact for every $b \in L^\infty_{n\times n}(\mathbb{R})$. In particular, for such homeomorphisms α, passage from $T(b)$ to $T(\gamma_\alpha b)$ preserves Fredholmness for every $b \in L^\infty_{n\times n}(\mathbb{R})$. We here admit only semi-almost periodic matrix symbols b, which, however, allows us to consider homeomorphisms α that are far beyond those covered by Muhly and Xia's result. To see this, note that if σ is bi-Lipschitz and $\sigma(1) = 1$, then the function μ defined by $\mu(t) = (\sigma(t)-1)/(t-1)$ satisfies

$$0 < \inf_{t\in\mathbb{T}} |\mu(t)| \le \sup_{t\in\mathbb{T}} |\mu(t)| < \infty. \tag{1.3}$$

An elementary computation shows that

$$\alpha(x) = i\frac{1+\sigma(\varphi(x))}{1-\sigma(\varphi(x))} = \frac{x+i}{\mu(\varphi(x))} - i, \tag{1.4}$$

and hence

$$0 < \liminf_{|x|\to\infty} \left|\frac{\alpha(x)}{x}\right| \le \limsup_{|x|\to\infty} \left|\frac{\alpha(x)}{x}\right| < \infty. \tag{1.5}$$

The homeomorphisms α considered in this paper include those with logarithmic, polynomial, or exponential behavior at infinity (see Section 3), and it is clear that such homeomorphisms in general do not satisfy (1.5).

In the scalar case, that is, for $n = 1$, Toeplitz operators with strongly frequency-modulated symbols, that is, with frequency modulations caused by homeomorphisms that do not necessarily satisfy (1.5), were studied in [2]. The first main result of [2] states that frequency modulation can destroy Fredholmness: there exists $b \in \mathrm{AP}(\mathbb{R})$ such that $T(b)$ is invertible but $T(\gamma_\alpha b)$ is not even normally solvable. The second main result of [2] provides us with conditions on the homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ that ensure that $T(\gamma_\alpha b)$ is Fredholm whenever $b \in \mathrm{SAP}(\mathbb{R})$ and $T(b)$ is Fredholm. In the present paper, we will establish analogues of these conditions in the matrix case.

2. Factorization of u-periodic matrix functions.

Let $H^\infty(\mathbb{R})$ be the algebra of all functions $f \in L^\infty(\mathbb{R})$ that can be analytically extended to the upper half-plane such that $\sup_{\mathrm{Im}\, z>0} |f(z)| < \infty$. A function $u \in H^\infty(\mathbb{R})$ is called an inner function if $|u(x)| = 1$ for almost all $x \in \mathbb{R}$.

We denote by $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$, respectively, the usual Lebesgue and Hardy spaces of the unit circle $\mathbb{T}$. If u is an inner function, then the composition operator γ_u given by $(\gamma_u f) = f(u(x))$ sends functions defined on $\mathbb{T}$ to functions living on $\mathbb{R}$. A function on $\mathbb{R}$ of the form $f(u(x))$ is called a u-periodic function. Note that if $u(x) = e^{i\lambda x}$ $(\lambda > 0)$, then $f(u(x))$ is simply a periodic function with the period $2\pi/\lambda$. Important examples of inner functions are Blaschke products, that is, functions of the form

$$u(x) = \prod_{j\in I} \frac{|1-z_j^2|}{|1+z_j^2|} \frac{x-z_j}{x-\overline{z}_j}, \tag{2.1}$$

where $\mathrm{Im}\, z_j > 0$ and $\sum_{j\in I} \mathrm{Im}\, z_j/|z_j+i|^2 < \infty$. The set I is finite or countable. In the former case u is called a finite Blaschke product.

The following result can be easily derived from well-known results on composition operators on the unit circle (see [22, 25]) by conformally mapping the unit disk onto the upper half-plane.

Theorem 2.1. *Let $u(x)$ be an inner function. If $f(t)$ is a function in $L^2(\mathbb{T})$, then $(\gamma_u f)(x)/|x+i|$ is a function in $L^2(\mathbb{R})$. Moreover, if $f(t)$ is a function in $H^2(\mathbb{T})$, then $(\gamma_u f)(x)/(x+i)$ is a function belonging to $H^2(\mathbb{R})$ and if $f(t)$ is a function in $\overline{H^2(\mathbb{T})}$, then $(\gamma_u f)(x)/(x-i)$ is a function in $\overline{H^2(\mathbb{R})}$. Finally, if $f(t)$ is a function in $H^\infty(\mathbb{T})$ or $\overline{H^\infty(\mathbb{T})}$, then $(\gamma_u f)(x)$ is a function in $H^\infty(\mathbb{R})$ or $\overline{H^\infty(\mathbb{R})}$, respectively.*

We denote by P_0 the orthogonal projection of $L^2_n(\mathbb{T})$ onto $H^2_n(\mathbb{T})$, and for $g \in L^\infty_{n\times n}(\mathbb{T})$, we let $T(g)$ stand for the Toeplitz operator on $H^2_n(\mathbb{T}) : T(g)f = P_0(gf)$. A matrix function $g \in L^\infty_{n\times n}(\mathbb{T})$ is said to admit an L^2-factorization (see [4, 18]) if it can be represented in the form

$$g(t) = g_-(t)d(t)g_+(t), \tag{2.2}$$

where $d(t) = \operatorname{diag}(t^{\kappa_1},\dots,t^{\kappa_n})$ with integers $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_n$ (the so-called partial indices) and where the matrix functions $g_\pm$ satisfy the following conditions:

$$g_+^{\pm 1} \in H^2_{n\times n}(\mathbb{T}), \qquad g_-^{\pm 1} \in \overline{H^2_{n\times n}(\mathbb{T})}, \tag{2.3}$$

$$\text{the operator } K_g := g_- P_0 g_-^{-1} I \text{ is bounded on the space } L^2_n(\mathbb{T}). \tag{2.4}$$

An L^2-factorization of the matrix function g exists if and only if the Toeplitz operator $T(g)$ is Fredholm (see [4, 18, 26]).

Let now $u(x)$ be an inner function. We say that a matrix function $a \in L^\infty_{n\times n}(\mathbb{R})$ has an L^2-u-factorization if it is representable in the form

$$a(x) = a_-(x)d_u(x)a_+(x), \tag{2.5}$$

where $d_u = \operatorname{diag}(u^{\kappa_1},\dots,u^{\kappa_n})$ with integers $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_n$ and the matrix functions $a_\pm(x)$ are such that

$$\begin{aligned} &\frac{a_+^{\pm 1}(x)}{x+i} \text{ is a function in } H^2_{n\times n}(\mathbb{R}),\\ &\frac{a_-^{\pm 1}(x)}{x-i} \text{ is a function in } \overline{H^2_{n\times n}(\to R)}, \end{aligned} \tag{2.6}$$

and

$$\text{the operator } K_a := a_- P a_-^{-1} I \text{ is bounded on the space } L^2_n(\mathbb{R}). \tag{2.7}$$

The following result, which is in principle already in [10, 11], forms the foundation of our investigation.

Theorem 2.2. *Let $u(x)$ be an inner function. If the matrix function $g(t)$ admits an L^2-factorization, then the matrix function $a(x) := g(u(x))$ has an L^2-u-factorization and the operator $T(a)$ is generalized invertible on $H^2_n(\mathbb{R})$. Furthermore, if $\kappa_j \geq 0$ for $j = 1,\dots,n$, then $T(a)$ is left-invertible, if $\kappa_j \leq 0$ for $j = 1,\dots,n$, then $T(a)$ is right-invertible, and if $\kappa_j = 0$ for $j = 1,\dots,n$, then $T(a)$ is invertible. In all cases a generalized inverse is given by*

$$T^{(-1)}(a) = Pa_+^{-1}d_u^{-1}Pa_-^{-1}P. \tag{2.8}$$

Moreover, if $\kappa_n < 0$ and $u(x)$ is not a finite Blaschke product, then $\dim\ker T(a) = \infty$, if $\kappa_1 > 0$ and $u(x)$ is not a finite Blaschke product, then $\dim\operatorname{coker} T(a) = \infty$, and if $u(x)$ is a finite Blaschke product, then $T(a)$ is a Fredholm operator and

$$\begin{aligned} \dim\ker T(a) &= -\frac{1}{2\pi}\left(\sum_{j=l+1}^{n}\kappa_j\right)\arg u(x)\Big|_{-\infty}^{\infty}, \\ \dim\operatorname{coker} T(a) &= \frac{1}{2\pi}\left(\sum_{j=1}^{l}\kappa_j\right)\arg u(x)\Big|_{-\infty}^{\infty}, \\ \operatorname{ind} T(a) &= -\frac{1}{2\pi}\left(\sum_{j=1}^{n}\kappa_j\right)\arg u(x)\Big|_{-\infty}^{\infty}, \end{aligned} \tag{2.9}$$

where $\kappa_j \geq 0$ for $j = 1,2,\dots,l$ and $\kappa_j < 0$ for $j = l+1,\dots,n$, $l \leq n$.

Proof. Let $g(t)$ admit an L^2-factorization (2.2), (2.3), and (2.4). Then the matrix function $a(x) = g(u(x))$ has the representation

$$a(x) = g_-(u(x))d(u(x))g_+(u(x)), \tag{2.10}$$

where $d(u(x)) = d_u(x) = \operatorname{diag}(u^{\kappa_1}(x),\dots,u^{\kappa_n}(x))$. According to Theorem 2.1,

$$\frac{a_+^{\pm 1}(x)}{x+i}, \qquad \frac{a_-^{\pm 1}(x)}{x-i}, \tag{2.11}$$

where $a_+(x) := g_+(u(x))$ and $a_-(x) := g_-(u(x))$, are matrix functions in $H^2_{n\times n}(\mathbb{R})$ and $\overline{H^2_{n\times n}}(\mathbb{R})$, respectively. We show that condition (2.7) is satisfied. By [18, Lemma 2.1], $d(t)g_+(t) = \tilde{g}_+(t)r(t)$, where $r(t)$ is a rational matrix function with $\det r$ invertible in $L^\infty(\mathbb{R})$ and $\tilde{g}_+^{\pm 1}(t)$ are functions in $H^2_{n\times n}(\mathbb{T})$. Consider the matrix function

$$g_0(t) := g_-(t)\tilde{g}_+(t) = g(t)r^{-1}(t). \tag{2.12}$$

Clearly, $g_0 \in L^\infty_{n\times n}(\mathbb{T})$. Since the factorization $g_0(t) = g_-(t)\tilde{g}_+(t)$ satisfies conditions (2.2), (2.3), and (2.4) with $d(t) = I_n$ (notice that condition (2.4) contains the matrix function $g_-(t)$ only), the operator $T(g_0)$ is invertible (see, e.g., [18, page 117]). This in turn implies a representation

$$g_0(t) = s(t)h(t), \tag{2.13}$$

where $s(t)$ is a sectorial matrix function and $h^{\pm 1} \in H^\infty_{n\times n}(\mathbb{T})$ (see [18, page 307]). Put $a_0(x) = g_0(u(x)) = s(u(x))h(u(x))$. Obviously, $s(u(x))$ is a sectorial function and $h^{\pm 1}(u(x))$ are functions in $H^\infty_{n\times n}(\mathbb{R})$. Thus (see [18, page 307]) the operator $T(a_0)$ is invertible. Consequently, the matrix function $a_0(x)$ has a factorization

$$a_0(x) = b_-(x)b_+(x), \tag{2.14}$$

where $b_\pm(x)$ satisfy conditions (2.6) and (2.7). On the other hand, it is well known (see, e.g., [18, page 60]) that a factorization of the form

$$a_0(x) = g_-(u(x))\tilde{g}_+(u(x)), \tag{2.15}$$

which satisfies only condition (2.6), is unique up to a constant matrix factor. It follows that the matrix function $a_-(x) := g_-(u(x))$ satisfies condition (2.7), and therefore representation (2.10) is really an L^2-u-factorization of the forms (2.5), (2.6), and (2.7).

Let now $\kappa_j \ge 0$ for $j = 1,\dots,n$. In this case $d_u \in H^\infty_{n\times n}(\mathbb{R})$ and the operator $T^{(-1)}(a)$ is bounded on the space $H^2_{n\times n}(\mathbb{R})$ due to (2.7). Hence

$$T^{(-1)}(a)T(a) = Pa_+^{-1}d_u^{-1}Pa_-^{-1}Pa_-d_ua_+P = Pa_+^{-1}d_u^{-1}Pd_ua_+P = I. \tag{2.16}$$

Here we used the identities

$$Pa_-^{-1}Pa_-P = P, \qquad Pa_+^{-1}d_u^{-1}Pd_ua_+P = P, \tag{2.17}$$

which can be easily verified taking into account that the functions $a_-(x)/(x-i)$ and $d_u(x)a_+(x)/(x+i)$ belong to $\overline{H^2_{n\times n}(\mathbb{R})}$ and $H^2_{n\times n}(\mathbb{R})$, respectively.

Using the property $d_u^{-1} \in H_\infty^{(n\times n)}(\mathbb{R})$, one can consider the case $\kappa_j \le 0$ $(j = 1,\dots,n)$ analogously.

We now pass to the general case. Thus, let $\kappa_j \ge 0$ for $j = 1,\dots,l$ and $\kappa_j < 0$ for $j = l+1,\dots,n$. We want to show that $T^{(-1)}(a)$ is a generalized inverse of $T(a)$. We put

$$\begin{aligned}
Z_1 = \{f \in H^2_n(\mathbb{R}) : {}& f(x) = a_+^{-1}(x)\varphi_1(x),\\
&\varphi_1(x) = (\psi_1(x),\dots,\psi_l(x),0,\dots,0),\\
&\psi_j(x)(x+i) \text{ is a function in } H^\infty(\mathbb{R})\},\\
Z_2 = \{f \in H^2_n(\mathbb{R}) : {}& f(x) = a_+^{-1}(x)\varphi_2(x),\\
&\varphi_2(x) = (0,\dots,0,\psi_{l+1}(x),\dots,\psi_n(x)),\\
&\psi_j(x)(x+i) \text{ is a function in } H^\infty(\mathbb{R})\}.
\end{aligned} \tag{2.18}$$

We claim that $T(a)T^{(-1)}(a)T(a)|Z_1 = T(a)|Z_1$. Indeed, we have

$$\begin{aligned} T(a)T^{(-1)}(a)T(a)(a_+^{-1}\varphi_1) &= T(a)Pa_+^{-1}d_u^{-1}Pa_-^{-1}Pa_-d_u\varphi_1 \\ &= T(a)Pa_+^{-1}d_u^{-1}Pd_u\varphi_1 \\ &= T(a)a_+^{-1}\varphi_1. \end{aligned} \tag{2.19}$$

Here we used equalities (2.17) and the inclusion $d_u\varphi_1 \in H_n^2(\mathbb{R})$.

Analogously, as $d_{u-} := \operatorname{diag}(0,\dots,u^{\kappa_{l+1}},\dots,u^{\kappa_n}) \in \overline{H_{n\times n}^\infty(\mathbb{R})}$ and $d_{u-}^{-1} \in H_{n\times n}^\infty(\mathbb{R})$, we get

$$\begin{aligned} &T(a)T^{(-1)}(a)T(a)(a_+^{-1}\varphi_2) \\ &\quad = T(a)Pa_+^{-1}d_u^{-1}Pa_-^{-1}Pa_-d_{u-}\varphi_2 \\ &\quad = Pa_-d_ua_+Pa_+^{-1}d_{u-}^{-1}Pd_{u-}\varphi_2 \\ &\quad = Pa_-Pd_{u-}\varphi_2 = Pa_-d_{u-}\varphi_2 \\ &\quad = T(a)(a_+^{-1}\varphi_2). \end{aligned} \tag{2.20}$$

Thus, $T(a)T^{(-1)}(a)T(a)|Z_2 = T(a)|Z_2$. Since, by (2.7), $T^{(-1)}(a)$ is bounded and the closure of $Z_1 \oplus Z_2$ is all of $H_n^2(\mathbb{R})$, it follows that $T(a)T^{(-1)}(a)T(a)|H_n^2(\mathbb{R}) = T(a)|H_n^2(\mathbb{R})$, which implies that $T^{(-1)}(a)$ is a generalized inverse of $T(a)$.

Suppose now that $\kappa_n < 0$ and that u is not a finite Blaschke product. Consider the following infinite set of functions:

$$g_j(x) = \frac{u^{-\kappa_n}(x) - u^{-\kappa_n}(z_j)}{x - z_j}, \quad \operatorname{Im} z_j > 0, \tag{2.21}$$

where $j = 1,2,\dots$ and $z_k \neq z_j$ for $k \neq j$. It is easy to see that an arbitrary finite subset of this set is linearly independent. Put $f_j(x) = a_+^{-1}(x)(0,\dots,0,g_j(x))$. Since $u^{\kappa_n}g_j$ is in $\overline{H^\infty(\mathbb{R})} \cap \overline{H^2(\mathbb{R})}$, we have

$$T(a)f_j = Pa_-(0,\dots,0,u^{\kappa_n}g_j) = 0. \tag{2.22}$$

Thus, $f_j \in \ker T(a)$, and it results that $\dim\ker T(a) = \infty$, as desired.

The case $\kappa_1 > 0$ can be disposed of by passage to adjoint operators.

Finally, let u be a finite Blaschke product. Then u is continuous on $\dot{\mathbb{R}}$, the one-point compactification of the real line. This implies that $a(x)$ admits an L^2-factorization (2.5), (2.6), and (2.7) with $d_u(x)$ replaced by

$$\operatorname{diag}\left(\left(\frac{x-i}{x+i}\right)^{\kappa_1},\dots,\left(\frac{x-i}{x+i}\right)^{\kappa_n}\right), \tag{2.23}$$

and hence the last assertion of our theorem is a consequence of well-known results (see [4, 18, 26]). □

For periodic matrix functions $a(x)$, Theorem 2.2 yields the following theorem.

Theorem 2.3. *Let the matrix function $a(x)$ be in $[P_T(\mathbb{R})]_{n\times n}$ and suppose that $\det a(x) \neq 0$ for $x \in \mathbb{R}$. Then $a(x)$ admits an L^2-u_T-factorization with $u_T(x) = e^{i(2\pi/T)x}$ and the operator $T(a)$ is generalized invertible. In addition, if $\kappa_j \geq 0$ for $j = 1,\dots,n$, then $T(a)$ is left-invertible, if $\kappa_j \leq 0$ for $j = 1,\dots,n$, then $T(a)$ is right-invertible, and if $\kappa_j = 0$ for $j = 1,\dots,n$, then $T(a)$ is invertible.*

Proof. Consider the matrix function

$$g(t) = a\left(-\frac{iT}{2\pi}\ln t\right), \quad t \in \mathbb{T}, \tag{2.24}$$

where $i^{-1}\ln t$ is taken in $[0, 2\pi)$. Since g is continuous on $\mathbb{T}$ and $\det g(t) \neq 0$ for $t \in \mathbb{T}$, it follows that $g(t)$ admits an L^2-factorization (2.2), (2.3), and (2.4) (again see, e.g., [4, 18]). As $a(x) = g(e^{i(2\pi/T)x})$, we therefore infer from Theorem 2.2 that $a(x)$ has an L^2-u_T-factorization, which gives all the assertions of the theorem. □

3. α-periodic matrix functions. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an orientation-preserving homeomorphism. Let further $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ be the Banach algebra of all functions of the form $h + f$ with $h \in H^\infty(\mathbb{R})$ and $f \in C(\dot{\mathbb{R}})$.

The main condition we impose on the homeomorphism α is that

$$e^{i\lambda\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}) \quad \forall \lambda > 0. \tag{3.1}$$

We remark that (3.1) holds if α satisfies the following four conditions:

$$\liminf_{x\to+\infty} \frac{x\alpha''(x)}{\alpha'(x)} > -2, \tag{3.2}$$

$$\begin{gathered} \lim_{x\to+\infty} \frac{\alpha''(x)}{(\alpha'(x))^2} = 0, \\ \lim_{x\to+\infty} x^{1/2} \frac{\alpha''(x)}{(\alpha'(x))^{3/2}} = 0, \end{gathered} \tag{3.3}$$

$$\lim_{x\to+\infty} (\alpha(x) + \alpha(-x)) = 0. \tag{3.4}$$

Clearly, condition (3.2) is equivalent to saying that the function $x^2\alpha'(x)$ is strictly monotonically increasing.

The sufficiency of (3.2), (3.3), and (3.4) for (3.1) follows from [12, Theorems 2.2 and 2.3] (see also [2]). Conditions (3.2), (3.3), and (3.4) are in fact true for large classes of functions. Here are some examples:

$$\begin{gathered} \alpha(x) = cx^\gamma, \quad \gamma > 0, \\ \alpha(x) = c\ln^\delta(x+1), \quad \delta > 1, \\ \alpha(x) = cx^\gamma \ln^\delta(x+1), \quad \gamma > 0,\ \delta \in (-\infty, \infty), \\ \alpha(x) = c_1 \exp(c_2 x^\gamma), \quad \gamma > 0. \end{gathered} \tag{3.5}$$

Here we cite the functions $\alpha(x)$ for $x > 0$ only, and the constants c, c_1, and c_2 are supposed to be positive.

It is known that every function in $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ that is invertible in $L^\infty(\mathbb{R})$ can be written as the product of an $H^\infty(\mathbb{R})$ function and an invertible function in $C(\dot{\mathbb{R}})$. Indeed, every such f is a product of an outer function f_+ with $|f_+| = |f|$ a.e. on $\mathbb{R}$ by a unimodular function in $H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$. According to [28], this unimodular function is a product of an inner function and yet another unimodular function w, belonging to $QC(= (H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})) \cap \overline{H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})})$. Due to [23], w is a product of $((x-i)/(x+i))^n$ and $\exp(i(u+\tilde{v}))$, for some integer n and real-valued functions $u, v \in C(\dot{\mathbb{R}})$; here $\tilde{}$ denotes the harmonic conjugate. It remains to observe that $\exp(i(u+\tilde{v})) = \exp(-v+iu) \cdot \exp(v+i\tilde{v})$ with the first multiple on the right-hand side being invertible in $C(\dot{\mathbb{R}})$ and the second in $H^\infty(\mathbb{R})$.

In [12], an explicit construction of the representation

$$e^{i\lambda\alpha(x)} = u_\lambda(x) c_\lambda(x), \tag{3.6}$$

where $\lambda > 0$, u_λ is an inner function, and $c_\lambda \in C(\dot{\mathbb{R}})$, was given under the assumption that α is subject to conditions (3.2), (3.3), and (3.4).

A bounded linear operator A acting on a Hilbert space H is said to admit a right (left) regularization if there exists a bounded linear operator R on H such that $AR = I + K$ ($RA = I + N$), where K and N are compact operators. The operator R is called a right (left) regularizer. The following facts are well known (see, e.g., [8, 9]).

Theorem 3.1. (1) *An operator admits a right (left) regularization if and only if it is a Φ_+ (Φ_-)-operator. It admits both a right and a left regularization if and only if it is a Φ-operator.*

(2) *If A is a Φ_+, Φ_-, or Φ-operator and B is a Φ-operator, then AB and BA are Φ_+, Φ_-, or Φ-operators, respectively. In the last case,*

$$\operatorname{ind} AB = \operatorname{ind} BA = \operatorname{ind} A + \operatorname{ind} B. \tag{3.7}$$

(3) *If A is a Φ_+ (Φ_-)-operator, then every right (left) regularizer is a Φ_- (Φ_+)-operator. If A is a Φ-operator, then every right (left) regularizer R is also a left (right) regularizer and R is a Φ-operator.*

(4) *If A is a Φ_+, Φ_-, or Φ-operator and K is a compact operator, then $A + K$ is a Φ_+, Φ_-, or Φ-operator, respectively.*

We say that a matrix function $a(x)$ is α-periodical (and we write $a \in [P_{T,\alpha}(\mathbb{R})]_{n\times n}$), α-almost periodical ($a \in [\mathrm{AP}_\alpha(\mathbb{R})]_{n\times n}$), and α-semi-almost periodical ($a \in [\mathrm{SAP}_\alpha(\mathbb{R})]_{n\times n}$) if a has the form

$$a(x) = b(\alpha(x)) \tag{3.8}$$

with $b \in [P_T(\mathbb{R})]_{n\times n}$, $b \in \mathrm{AP}_{n\times n}(\mathbb{R})$, and $b \in \mathrm{SAP}_{n\times n}(\mathbb{R})$, respectively.

Here is the main result of this section.

Theorem 3.2. *Assume that $e^{i\lambda_T\alpha} \in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}})$ for $\lambda_T = 2\pi/T$. Let $a \in [P_{T,\alpha}(\mathbb{R})]_{n\times n}$ and suppose that $\inf_{x\in\mathbb{R}} |\det a(x)| > 0$. Then the matrix function b given by $a(x) = b(\alpha(x))$ admits an L^2-u_T-factorization with $u_T(x) = e^{i\lambda_T x}$ and*

(i) *if $\kappa_1 = \cdots = \kappa_n = 0$, then the operator $T(a)$ is a Φ-operator on the space $H_n^2(\mathbb{R})$;*

(ii) *if $\kappa_1 > 0$, $\kappa_2 \ge 0, \ldots, \kappa_n \ge 0$, then the operator $T(a)$ is a Φ_+-operator on the space $H_n^2(\mathbb{R})$ and $\dim \operatorname{coker} T(a) = \infty$;*

(iii) *if $\kappa_1 \le 0, \ldots, \kappa_{n-1} \le 0$, $\kappa_n < 0$, then $T(a)$ is a Φ_--operator on the space $H_n^2(\mathbb{R})$ and $\dim \ker T(a) = \infty$;*

(iv) *if $\kappa_1 > 0$, $\kappa_2 \ge 0, \ldots, \kappa_l \ge 0$, $\kappa_{l+1} \le 0, \ldots, \kappa_{n-1} \le 0$, $\kappa_n < 0$, then the operator $T(a)$ is neither a Φ_--operator nor a Φ_+-operator on the space $H_n^2(\mathbb{R})$.*

Proof. Since $b \in [P_T(\mathbb{R})]_{n\times n}$ and $\det b(x) \ne 0$ for $x \in \mathbb{R}$, Theorem 2.3 shows that the matrix function b admits an L^2-u_T-factorization. It is easy to see that there exists a continuous matrix function g on $\mathbb{T}$ such that $b(x) = g(e^{i\lambda_T x})$. As $\det g(t) \ne 0$ for $t \in \mathbb{T}$, the matrix function g possesses an L^2-factorization with the same numbers κ_j as in the L^2-u-factorization of the matrix function b. Using (3.6) for λ_T, we get the representation

$$a(x) = b(\alpha(x)) = g\left(e^{i\lambda_T\alpha(x)}\right), \tag{3.9}$$

whence

$$a(x) = g(u_{\lambda_T}(x)c_{\lambda_T}(x)), \tag{3.10}$$

where u_{λ_T} is an inner function and $c_{\lambda_T} \in C(\dot{\mathbb{R}})$. Define a_∞ by

$$a_\infty(x) = g(u_\infty(x)), \tag{3.11}$$

where u_∞ is the inner function $u_\infty(x) := u_{\lambda_T}(x)c_{\lambda_T}(\infty)$. The matrix function a_∞ is u_∞-periodic and Theorem 2.2 can be applied to the operator $T(a_\infty)$.

From (3.10) and (3.11) we get the representation

$$a(x) = a_\infty(x)c(x), \tag{3.12}$$

where $c(x) = I_n + a_\infty^{-1}(x)(a(x) - a_\infty(x))$ is a continuous matrix function because $\lim_{x\to\infty}(a(x) - a_\infty(x)) = O_n$. (Here I_n is the identity matrix and O_n is the zero matrix of order n.)

Thus, in view of the classical theorem on the compactness of the so-called Hankel operator $(I-P)cP$ for continuous matrix functions c (see, e.g., [18, page 172]),

$$T(a) = T(a_\infty)T(c) + K, \tag{3.13}$$

where K is a compact operator.

We now prove (i). In this case, by (3.11) and Theorem 2.2, the operator $T(a_\infty)$ is invertible. Since $\det c(x) \neq 0$ for $x \in \mathbb{R}$ (recall (3.12)), the operator $T(c)$ is a Φ-operator (see, e.g., [18, page 176]). By Theorem 3.1(3) and (4), we can therefore conclude from (3.13) that the operator $T(a)$ is also a Φ-operator.

Analogously, we can tackle (ii) and (iii): according to Theorem 2.2, $T(a_\infty)$ is a Φ_+ (Φ_-)-operator, and from the representation (3.13) and Theorem 3.1(2) and (4) we see that $T(a)$ is also a Φ_+ (Φ_-)-operator.

Now assume that $\dim\operatorname{coker} T(a) < \infty$ in the case (ii). Then $T(a)$ is a Φ-operator. Since $T(c)$ is a Φ-operator, it admits a right regularization $T(c)R = I + K_1$, where K_1 is compact (Theorem 3.1(1)). Thus, from (3.13) we get

$$T(a_\infty) = T(a)R - K_2, \tag{3.14}$$

where $K_2 = T(a_\infty)K_1 + K_1R$ is compact. Equality (3.14) shows that the operator $T(a_\infty)$ is a Φ-operator. But u_T cannot be a finite Blaschke product because

$$\lim_{x\to\pm\infty} \arg u_T(x) = \pm\infty \tag{3.15}$$

due to (3.6). Hence, $\dim\operatorname{coker} T(a) = \infty$ by Theorem 2.2. This contradiction proves that $\dim\operatorname{coker} T(a) = \infty$. Having recourse to left regularization, one can analogously show that $\dim\ker T(a) = \infty$ in case (iii).

We finally turn to the case (iv). Assume first that $T(a)$ is a Φ_+-operator. Multiplying equality (3.13) from the right by the operator R we get again (3.14), and by Theorem 3.1(2) and (3) the operator $T(a_\infty)$ must be a Φ_+-operator, which contradicts Theorem 2.2 (note that $\dim\ker T(a) = \infty$). Analogously, one can show that $T(a)$ is not a Φ_--operator. □

4. α-almost periodic matrix functions.

Let $\mathrm{APW}(\mathbb{R})$ be the collection of all $f \in \mathrm{AP}(\mathbb{R})$ that can be written in the form

$$f(x) = \sum_{j\in\mathbb{Z}} c_j e^{i\lambda_j x}, \tag{4.1}$$

where $\lambda_j \in \mathbb{R}$, $c_j \in \mathbb{C}$, and $\sum_{j\in\mathbb{Z}} |c_j| < \infty$.

The set $\mathrm{APW}(\mathbb{R})$ is a subalgebra of $\mathrm{AP}(\mathbb{R})$ and the closure of $\mathrm{APW}(\mathbb{R})$ in the norm of $L^\infty(\mathbb{R})$ is $\mathrm{AP}(\mathbb{R})$. If, in (4.1), $\lambda_j \geq 0$ $(\lambda_j \leq 0)$ for all j, we will write $f \in \mathrm{APW}^+(\mathbb{R})$ $(f \in \mathrm{APW}^-(\mathbb{R}))$. It is easy to see that

$$\mathrm{APW}^+(\mathbb{R}) = \mathrm{APW}(\mathbb{R}) \cap H^\infty(\mathbb{R}), \qquad \mathrm{APW}^-(\mathbb{R}) = \mathrm{APW}(\mathbb{R}) \cap \overline{H^\infty(\mathbb{R})}. \tag{4.2}$$

Given some class X of matrix functions, we write $f \in GX$ if $f \in X$ and $f^{-1} \in X$. The following two results will be needed later.

Theorem 4.1 (see [27]). *Let the matrix function $b \in \mathrm{GAPW}_{n\times n}(\mathbb{R})$ be positive definite for almost all $x \in \mathbb{R}$. Then*

(i) *$T(b)$ is invertible on $H^2_n(\mathbb{R})$,*

(ii) *there is an $h \in \mathrm{GAPW}^+_{n\times n}(\mathbb{R})$ such that $b = h^*h$.*

A full proof is given in [3, Corollary 9.15].

A matrix function $u \in L^\infty_{n\times n}(\mathbb{R})$ is called unitary valued if $u^*(x)u(x) = u(x)u^*(x) = I_n$ for almost all $x \in \mathbb{R}$.

Theorem 4.2 (see [14]). *Let $u \in \mathrm{APW}_{n\times n}(\mathbb{R})$ be unitary valued and let the operator $T(u)$ be left-invertible. Then there exists a matrix function $w \in \mathrm{APW}^+_{n\times n}(\mathbb{R})$ such that*

$$\|u - w\|_\infty \le 1 - \varepsilon \quad \textit{for some } \varepsilon > 0. \tag{4.3}$$

A full proof can be found in [3, Theorem 20.4]. A check of the proof shows that we can choose

$$\varepsilon = ||T_l^{-1}(u)||_2^{-1}, \tag{4.4}$$

where $T_l^{-1}(u)$ is a left inverse of $T(u)$.

A matrix function $s \in L^\infty_{n\times n}(\mathbb{R})$ is called strongly sectorial if $\|I_n - s\|_\infty < 1$.

Lemma 4.3. *Let $b \in \mathrm{APW}_{n\times n}(\mathbb{R})$ and suppose that the operator $T(b)$ is left-invertible. Then there exists a representation*

$$b = (w^*)^{-1} sh, \tag{4.5}$$

where $w \in \mathrm{APW}^+_{n\times n}(\mathbb{R})$, $h \in \mathrm{GAPW}^+_{n\times n}(\mathbb{R})$, and s is strongly sectorial, and where

$$\begin{gathered} \|w\|_\infty \le 2, \\ ||h^{-1}||_\infty = ||b^{-1}||_\infty, \end{gathered} \tag{4.6}$$

$$||I_n - s||_\infty \le 1 - \varepsilon_0 \quad \textit{with } \varepsilon_0 = \|b\|_\infty^{-1} ||T_l^{-1}(b)||_2^{-1}. \tag{4.7}$$

Here $\|\cdot\|_\infty$ denotes the norm of the operator of multiplication by a matrix function on the space $L^2_n(\mathbb{R})$ and $\|\cdot\|_2$ is the operator norm on $L^2_n(\mathbb{R})$.

Proof. Consider the positive definite matrix function b^*b. According to Theorem 4.1, there exists a matrix function $h \in \mathrm{GAPW}^+_{n\times n}(\mathbb{R})$ such that $b^*b = h^*h$. The matrix function

$$u = bh^{-1} \tag{4.8}$$

is unitary valued and the operator $T(u)$ is left-invertible, a left inverse being

$$T_l^{-1}(u) = T(h)T_l^{-1}(b). \tag{4.9}$$

Hence,

$$||T_l^{-1}(u)||_2 \le \|h\|_\infty ||T_l^{-1}(b)||_2 = \|b\|_\infty ||T_l^{-1}(b)||_2. \tag{4.10}$$

By virtue of Theorem 4.2, there exists a matrix function $w \in \mathrm{APW}^+_{n\times n}(\mathbb{R})$ such that

$$\|u - w\|_\infty \le 1 - \varepsilon, \tag{4.11}$$

where, by (4.4),

$$\varepsilon = ||T_l^{-1}(u)||_2^{-1} \ge \|b\|_\infty^{-1} ||T_l^{-1}(b)||^{-1} = \varepsilon_0. \tag{4.12}$$

The matrix function

$$s := w^* u \tag{4.13}$$

is strongly sectorial, and from (4.11), (4.12) we obtain inequality (4.7). But (4.8) and (4.13) give (4.5) since $b = uh = (w^*)^{-1} sh$. Now (4.6) are implications of (4.11) and (4.8). □

Lemma 4.4. *Let $b \in \mathrm{APW}_{n\times n}(\mathbb{R})$ and suppose that the operator $T(b)$ is left-invertible on $H^2_n(\mathbb{R})$. Let further condition (3.1) be satisfied. Put $a(x) = b(\alpha(x))$. Then the operator $T(a)$ is a Φ_+-operator on $H^2_n(\mathbb{R})$ and there exists a left regularizer $R_l(a)$ such that*

$$||R_l(a)||_2 \le 2\|b\|_\infty ||b^{-1}||_\infty ||T_l^{-1}(b)||_2. \tag{4.14}$$

Proof. From Lemma 4.3 we obtain (4.5), and it is easy to see that

$$T_l^{-1}(b) = T(h^{-1})T^{-1}(s)T(w^*), \tag{4.15}$$

where $h^{-1} \in \mathrm{APW}^+_{n\times n}(\mathbb{R})$ and $w^* \in \mathrm{APW}^-_{n\times n}(\mathbb{R})$. Therefore condition (3.1) implies that

$$\begin{aligned} \gamma_\alpha h^{-1} &\in H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}}), \\ \gamma_\alpha w^* &\in \overline{H^\infty_{n\times n}(\mathbb{R})} + C_{n\times n}(\dot{\mathbb{R}}). \end{aligned} \tag{4.16}$$

Further, condition (4.7) gives $\|I_n - \gamma_\alpha s\| \le 1 - \varepsilon_0$, and hence the operator $T(\gamma_\alpha s)$ is invertible and

$$||T^{-1}(\gamma_\alpha s)||_2 \le \varepsilon_0^{-1}. \tag{4.17}$$

We claim that the operator

$$R_l(a) = T(\gamma_\alpha h^{-1})T^{-1}(\gamma_\alpha s)T(\gamma_\alpha w^*) \tag{4.18}$$

is a left regularizer of the operator $T(a)$. Indeed, we have

$$\begin{aligned} R_l(a)T(a) &= R_l(a)T\Big(\big(\gamma_\alpha (w^*)^{-1}\big)(\gamma_\alpha s)(\gamma_\alpha h)\Big) \\ &= P(\gamma_\alpha h^{-1})T^{-1}(\gamma_\alpha s)P(\gamma_\alpha s)(\gamma_\alpha h)P - K_1, \end{aligned} \tag{4.19}$$

where $K_1 := P(\gamma_\alpha h^{-1})T^{-1}(\gamma_\alpha s)P(\gamma_\alpha w^*)(I-P)T(a)$ is compact (notice that the Hankel operator $Pc(I-P)$ is compact for every $c \in \overline{H^\infty_{n\times n}(\mathbb{R})} + C_{n\times n}(\dot{\mathbb{R}})$). Further,

$$\begin{aligned} R_l(a)T(a) &= P(\gamma_\alpha h^{-1})T^{-1}(\gamma_\alpha s)P(\gamma_\alpha s)P(\gamma_\alpha h)P + K_2 - K_1 \\ &= P(\gamma_\alpha h^{-1})P(\gamma_\alpha h)P + K_2 - K_1, \end{aligned} \tag{4.20}$$

where

$$K_2 := P(\gamma_\alpha h)T^{-1}(\gamma_\alpha s)P(\gamma_\alpha s)(I-P)(\gamma_\alpha h)P \tag{4.21}$$

is compact (because the Hankel operator $(I-P)dP$ is compact for arbitrary $d \in H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}})$). Finally,

$$R_l(a)T(a) = I - K_3K_2 - K_1, \tag{4.22}$$

where $K_3 := P(\gamma_\alpha h^{-1})(I-P)(\gamma_\alpha h)P$ is compact since $\gamma_\alpha h \in H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}})$. Thus, $R_l(a)$ is really a left regularizer of $T(a)$, and the estimate (4.14) follows from (4.6) and (4.17). □

We are now in a position to prove the main result of this section.

Theorem 4.5. *Let $b \in \mathrm{AP}_{n\times n}(\mathbb{R})$ and suppose that condition (3.1) holds. Put $a(x) = b(\alpha(x))$. Then*

(i) *if $T(b)$ is invertible, then $T(a)$ is a Φ-operator;*
(ii) *if $T(b)$ is left-invertible, then $T(a)$ is a Φ_+-operator;*
(iii) *if $T(b)$ is right-invertible, then $T(a)$ is a Φ_--operator.*

Proof. We begin with the proof of (ii). Let $\{b_j\}_{j=1}^\infty \subset \mathrm{APW}_{n\times n}(\mathbb{R})$ be a sequence such that

$$\lim_{j\to\infty} \|b_j - b\|_\infty = 0. \tag{4.23}$$

Then, obviously,

$$\lim_{j\to\infty} \|\gamma_\alpha b_j - \gamma_\alpha b\|_\infty = 0. \tag{4.24}$$

By virtue of (4.23), the operators $T(b_j)$ are left-invertible for all sufficiently large j and

$$\|T_l^{-1}(b_j)\|_2 \le 2\|T_l^{-1}(b)\|_2. \tag{4.25}$$

For these j's we infer from Lemma 4.4 that

$$\begin{aligned}\|R_l(\gamma_\alpha b_j)\|_2 &\le 2\|b_j\|_\infty \|b_j^{-1}\|_\infty \|T_l^{-1}(b_j)\|_2 \\ &\le C\|b\|_\infty \|b^{-1}\|_\infty \|T_l^{-1}(b)\|_2,\end{aligned} \tag{4.26}$$

where C is some constant independent of j. We have

$$\begin{aligned}R_l(\gamma_\alpha b_j)T(a) &= R_l(\gamma_\alpha b_j)T(\gamma_\alpha b_j) + R_l(\gamma_\alpha b_j)T(\gamma_\alpha b - \gamma_\alpha b_j) \\ &= I + R_l(\gamma_\alpha b_j)T(\gamma_\alpha b - \gamma_\alpha b_j) + K_j,\end{aligned} \tag{4.27}$$

where K_j are compact operators. By virtue of (4.24) and (4.26), the norm of the operator

$$E_j := R_l(\gamma_\alpha b_j)T(\gamma_\alpha b - \gamma_\alpha b_j) \tag{4.28}$$

can be made as small as desired by choosing j large enough. Thus, for sufficiently large j, the operator $I + E_j$ is invertible. From equality (4.27) we deduce that

$$(I+E_j)^{-1}R_l(\gamma_\alpha b_j)T(a) = I + (I+E_j)^{-1}K_j. \tag{4.29}$$

Consequently, $T(a)$ has a left regularizer and hence $T(a)$ is a Φ_+-operator.

Assertion (iii) follows from (ii) by passage to adjoints, and (i) obviously results from (ii) and (iii). □

It should be noted that if $b \in AP_{n\times n}(\mathbb{R})$ and $T(b)$ is a Φ_-, Φ_+, or Φ-operator, then $T(b)$ is right-invertible, left-invertible, or invertible, respectively (see [16]). Thus, both Fredholmness and semi-Fredholmness of Toeplitz operators with symbols $AP_{n\times n}(\mathbb{R})$ are preserved after the change of variables $x \to \alpha(x)$ in the symbol *provided* (3.1) *holds.*

5. α-semi-almost periodic matrix functions. From [24] we know that every matrix function $b \in SAP_{n\times n}(\mathbb{R})$ can be represented in the form

$$b(x) = v(x)b_l(x) + (1-v(x))b_r(x) + c_0(x), \tag{5.1}$$

where $b_l, b_r \in AP_{n\times n}(\mathbb{R})$, $c_0 \in C_{n\times n}(\dot{\mathbb{R}})$ with $c_0(\infty) = 0$, and v is a scalar function from $C(\overline{\mathbb{R}})$ such that

$$v(-\infty) = 1, \qquad v(+\infty) = 0. \tag{5.2}$$

The matrix functions b_l and b_r are determined uniquely and are called, respectively, the left and right almost periodic representatives of b.

In this section we change condition (3.1) to the conditions

$$
\begin{aligned}
ve^{i\lambda\alpha} &\in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}) \quad \forall \lambda > 0, \\
(1-v)e^{i\lambda\alpha} &\in H^\infty(\mathbb{R}) + C(\dot{\mathbb{R}}) \quad \forall \lambda > 0,
\end{aligned}
\tag{5.3}
$$

where $v \in C(\mathbb{R})$ is a fixed function subject to (5.2). For a fixed number $\delta > 1$, we introduce the functions

$$
\begin{aligned}
\beta_+(x) &:= \alpha(x) - (\log x)^\delta, \quad x > 1, \\
\beta_-(x) &:= \alpha(-x) + (\log x)^\delta, \quad x > 1.
\end{aligned}
\tag{5.4}
$$

We call a function β regular if it satisfies conditions (3.2), (3.3) with α replaced by β.

Sufficient conditions for (5.3) are given by the following result of [2].

Theorem 5.1. *If the functions β_+ and β_- are regular for some $\delta > 1$, then the homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ satisfies conditions (5.3).*

Put

$$
AP^+(\mathbb{R}) := AP(\mathbb{R}) \cap H^\infty(\mathbb{R}), \qquad AP^-(\mathbb{R}) := AP(\mathbb{R}) \cap \overline{H^\infty(\mathbb{R})}. \tag{5.5}
$$

We say that a matrix function $b \in AP_{n\times n}(\mathbb{R})$ admits a canonical AP-factorization if it can be represented in the form

$$
b(x) = \tilde{b}_-(x)\tilde{b}_+(x) \tag{5.6}
$$

with $\tilde{b}_\pm(x) \in GAP^\pm_{n\times n}(\mathbb{R})$. The Bohr mean value of a matrix function f in $AP_{n\times n}(\mathbb{R})$ is defined by

$$
\mathbf{M}(f) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx. \tag{5.7}
$$

The matrix

$$
\mathbf{d}(b) := \mathbf{M}\left(\tilde{b}_-\right)\mathbf{M}\left(\tilde{b}_+\right) \in \mathbb{C}^{n\times n} \tag{5.8}
$$

is called the geometric mean of the matrix function (5.6). It is well known (see, e.g., [16] or [3, Section 8.2]) that $\mathbf{d}(B)$ is independent of the particular choice of the canonical AP-factorization. Furthermore, it is easy to see that if we have (5.6), then b can be written in the form

$$
b(x) = b_-(x)\mathbf{d}(b)b_+(x), \tag{5.9}
$$

where $b_\pm \in GAP^\pm_{n\times n}(\mathbb{R})$ and

$$
\mathbf{M}(b_+) = \mathbf{M}(b_-) = I. \tag{5.10}
$$

Here is the main result of this section.

Theorem 5.2. *Let α be a homeomorphism satisfying conditions (5.3) and let $b \in \mathrm{GSAP}_{n\times n}(\mathbb{R})$ be a matrix function whose left and right almost periodic representatives b_{l} and b_{r} admit canonical* AP*-factorizations. Put $a(x) = b(\alpha(x))$. Then the operator $T(a)$ is a Φ-operator if and only if*

$$\mathrm{sp}\,(\mathbf{d}^{-1}(b_{\mathrm{r}})\mathbf{d}(b_{\mathrm{l}})) \cap (-\infty, 0] = \emptyset, \tag{5.11}$$

where $\mathrm{sp}(\cdot)$ denotes the spectrum (= set of eigenvalues) of matrices in $\mathbb{C}^{n\times n}$.

Proof. We write the matrix functions b_{l} and b_{r} in the form (5.9), (5.10):

$$b_{\mathrm{l}}(x) = b_{l-}(x)\mathbf{d}(b_{\mathrm{l}})b_{l+}(x), \qquad b_{\mathrm{r}}(x) = b_{r-}(x)\mathbf{d}(b_{\mathrm{r}})b_{r+}(x). \tag{5.12}$$

Suppose first that

$$b_{l\pm} \in \mathrm{GAPW}^{\pm}_{n\times n}(\mathbb{R}), \qquad b_{r\pm} \in \mathrm{GAPW}^{\pm}_{n\times n}(\mathbb{R}). \tag{5.13}$$

Then we have the series representations

$$b_{l(r)\pm}(x) = I + \sum c^{\pm}_{jl(r)} e^{i\lambda^{\pm}_{jl(r)}x}, \qquad b^{-1}_{l(r)\pm}(x) = I + \sum d^{\pm}_{jl(r)} e^{i\kappa^{\pm}_{jl(r)}x}, \tag{5.14}$$

where $\lambda^{+}_{jl(r)} > 0$, $\kappa^{+}_{jl(r)} > 0$, $\lambda^{-}_{jl(r)} < 0$, $\kappa^{-}_{jl(r)} < 0$,

$$\sum |c^{\pm}_{jl(r)}| < \infty, \qquad \sum |d^{\pm}_{jl(r)}| < \infty. \tag{5.15}$$

By virtue of conditions (5.3),

$$\begin{aligned}
f_+ &:= v \cdot \gamma_\alpha b_{l+} + (1-v)\cdot \gamma_\alpha b_{r+} \in H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}}),\\
g_+ &:= v \cdot \gamma_\alpha b^{-1}_{l+} + (1-v)\cdot \gamma_\alpha b^{-1}_{r+} \in H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}}),\\
f_- &:= v \cdot \gamma_\alpha b_{l-} + (1-v)\cdot \gamma_\alpha b_{r-} \in \overline{H^\infty_{n\times n}(\mathbb{R})} + C_{n\times n}(\dot{\mathbb{R}}),\\
g_- &:= v \cdot \gamma_\alpha b^{-1}_{l-} + (1-v)\cdot \gamma_\alpha b^{-1}_{r-} \in \overline{H^\infty_{n\times n}(\mathbb{R})} + C_{n\times n}(\dot{\mathbb{R}}).
\end{aligned} \tag{5.16}$$

Moreover, since $b \in \mathrm{GAP}_{n\times n}(\mathbb{R})$, there exist matrix functions c_+, c_- in $C_{n\times n}(\mathbb{R})$ such that $c_-(\pm\infty) = 0$, $c_+(\pm\infty) = 0$,

$$\begin{aligned}
\widetilde{f}_+ &:= f_+ + c_+ \in G(H^\infty_{n\times n}(\mathbb{R}) + C_{n\times n}(\dot{\mathbb{R}})),\\
\widetilde{f}_- &:= f_- + c_- \in G\Big(\overline{H^\infty_{n\times n}(\mathbb{R})} + C_{n\times n}(\dot{\mathbb{R}})\Big).
\end{aligned} \tag{5.17}$$

Clearly, the matrix function $a = \gamma_\alpha b$ can be represented in the form

$$a(x) = \widetilde{f}_-(x)d(x)\widetilde{f}_+(x), \tag{5.18}$$

where $d \in C_{n\times n}(\overline{\mathbb{R}})$ and $d(-\infty) = \mathbf{d}(b_{\mathrm{l}})$, $d(+\infty) = \mathbf{d}(b_{\mathrm{r}})$. Suppose that condition (5.11) is satisfied. Then $T(d)$ is a Φ-operator (see [18, page 196]) with a regularizer $R(d)$. Proceeding as in the proof of Lemma 4.4, one can show that

$$R(a) := T\left(\tilde{f}_+^{-1}\right)R(d)T\left(\tilde{f}_-^{-1}\right) \tag{5.19}$$

is a regularizer of $T(a)$, which implies that $T(a)$ is a Φ-operator.

We now remove the extra assumption (5.13), that is, we merely assume that

$$b_{l\pm} \in \mathrm{GAP}^{+}_{n\times n}(\mathbb{R}), \qquad b_{r\pm} \in \mathrm{GAP}^{\pm}_{n\times n}(\mathbb{R}). \tag{5.20}$$

Since the sets $\mathrm{APW}^{\pm}_{n\times n}(\mathbb{R})$ are dense in the spaces $\mathrm{AP}^{\pm}_{n\times n}(\mathbb{R})$, we obtain from what has already been proved that the inclusions (5.16) and (5.17) are also true in the general case, which implies that the operator (5.19) is a regularizer in the general case, too. At this point we have proved the sufficiency of (5.11).

Finally, suppose that $T(a)$ is a Φ-operator and that $R(a)$ is a regularizer. It is easy to see that the operator $R(d) := T(\tilde{f}_+)R(a)T(\tilde{f}_-)$ is a regularizer of $T(d)$. Thus, $T(d)$ is also a Φ-operator, and the well-known Fredholm criterion for Toeplitz operators with symbols in $C(\overline{\mathbb{R}})$ (see, e.g., [18, page 196]) gives (5.11). □

Let $\mathrm{SAPW}^{n\times n}(\mathbb{R})$ be the set of all functions $b \in \mathrm{SAP}_{n\times n}(\mathbb{R})$ whose almost periodic representatives b_{l} and b_{r} are in $\mathrm{APW}_{n\times n}(\mathbb{R})$.

Theorem 5.3. *Let the homeomorphism α satisfy conditions (5.3) and let b be a matrix function in* $\mathrm{SAPW}_{n\times n}(\mathbb{R})$. *Put $a(x) = b(\alpha(x))$. If $T(b)$ is a Φ-operator, then $T(a)$ is also a Φ-operator.*

Proof. If $T(b)$ with $b \in \mathrm{SAPW}_{n\times n}(\mathbb{R})$ is a Φ-operator, then $T(b_{\mathrm{l}})$ and $T(b_{\mathrm{r}})$ are invertible operators (see [14, 17], [3, Corollary 10.5]). Consequently, the matrix functions b_{l} and b_{r} admit canonical AP-factorizations with factors in $\mathrm{APW}^{\pm}_{n\times n}(\mathbb{R})$ (see [14, 15] and [3, Lemma 9.7]). By virtue of Theorem 5.2, $T(a)$ is therefore a Φ-operator. □

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