IJMMS 2003:9, 539–547
PII. S0161171203202027
http://ijmms.hindawi.com

A BASIC INEQUALITY FOR SUBMANIFOLDS IN A COSYMPLECTIC SPACE FORM

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Received 7 February 2002

For submanifolds tangent to the structure vector field in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants of the submanifold, namely, its sectional curvature and scalar curvature on one side; and its main extrinsic invariant, namely, squared mean curvature on the other side. Some applications, including inequalities between the intrinsic invariant δ_M and the squared mean curvature, are given. The equality cases are also discussed.

2000 Mathematics Subject Classification: 53C40, 53D15.

1. Introduction. To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the natural interests of the submanifold theory. Let M be an n-dimensional Riemannian manifold. For each point $p \in M$, let $(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_pM\}$. Then, the well-defined intrinsic invariant δ_M of M introduced by Chen [4] is

$$\delta_M(p) = \tau(p) - (\inf K)(p), \tag{1.1}$$

where τ is the scalar curvature of M (see also [6]).

In [3], Chen established the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature for n-dimensional submanifolds M in a real space form $R(c)$ of constant sectional curvature c:

$$\delta_M \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{1.2}$$

The above inequality is also true for anti-invariant submanifolds in complex space forms $\widetilde{M}(4c)$ as remarked in [7]. In [5], he proved a general inequality for an arbitrary submanifold of a dimension greater than 2 in a complex space form. Applying this inequality, he showed that (1.2) is also valid for arbitrary submanifolds in the complex hyperbolic space $\mathrm{CH}^m(4c)$. He also established the basic inequality for a submanifold in a complex projective space CP^m.

A submanifold normal to the structure vector field ξ of a contact manifold is anti-invariant. Thus, the C-totally real submanifolds in a Sasakian manifold are anti-invariant as they are normal to ξ. An inequality similar to (1.2) for C-totally real submanifolds in a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature c is given in [8]. In [9], for submanifolds in a Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ, a basic inequality, along with some applications, is presented.

There is another interesting class of almost contact metric manifolds, namely, cosymplectic manifolds [10]. In this paper, submanifolds tangent to the structure vector field ξ in cosymplectic space forms are studied. Section 2 contains the necessary details of submanifolds and cosymplectic space forms for further use. In Section 3, for submanifolds tangent to the structure vector field ξ in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants, namely, its sectional curvature function K and its scalar curvature function τ of the submanifold on the one side, and its main extrinsic invariant, namely, its mean curvature function $\|H\|$ on the other side. In Section 4, we give some applications including inequalities between the intrinsic invariant δ_M and the extrinsic invariant $\|H\|$. We also discuss the equality cases.

2. Preliminaries. Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold [2] endowed with an almost contact structure (φ,ξ,η), that is, φ is a $(1,1)$ tensor field, ξ is a vector field, and η is 1-form such that

$$\varphi^2 = -I+\eta\otimes\xi, \qquad \eta(\xi)=1. \tag{2.1}$$

Then, $\varphi(\xi)=0$ and $\eta\circ\varphi=0$.

Let g be a compatible Riemannian metric with (φ,ξ,η), that is, $g(\varphi X,\varphi Y) = g(X,Y)-\eta(X)\eta(Y)$ or, equivalently, $g(X,\varphi Y) = -g(\varphi X,Y)$ and $g(X,\xi) = \eta(X)$ for all $X,Y\in T\tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (φ,ξ,η,g). An almost contact metric manifold is *cosymplectic* [2] if $\tilde{\nabla}_X\varphi=0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g. From the formula $\tilde{\nabla}_X\varphi=0$, it follows that $\tilde{\nabla}_X\xi=0$.

A plane section σ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a *φ-section* if $\sigma\perp\xi$ and $\varphi(\sigma)=\sigma$. The $(2m+1)$-dimensional almost contact manifold $\tilde{M}$ is of the *constant φ-sectional curvature* if the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ-section σ of $T_p\tilde{M}$ and the choice of a point $p\in\tilde{M}$. A cosymplectic manifold $\tilde{M}$ is of the constant φ-sectional curvature c if and only if its curvature tensor $\tilde{R}$ is of the form [10]

$$\begin{aligned}4\tilde{R}(X,Y,Z,W) = c\Big\{ & g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\ & +g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)\\ & -2g(X,\varphi Y)g(Z,\varphi W)\\ & -g(X,W)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)\eta(W)\\ & -g(Y,Z)\eta(X)\eta(W)+g(Y,W)\eta(X)\eta(Z)\Big\}.\end{aligned} \tag{2.2}$$

Let M be an $(n+1)$-dimensional submanifold of a manifold $\tilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.3}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇, and $\nabla^{\perp}$, respectively, are the Riemannian, induced Riemannian, and induced normal connections in $\tilde{M}$, M, and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$.

Let $\{e_1,\dots,e_{n+1}\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector $H(p)$ at $p \in M$ is

$$H(p) = \frac{1}{n+1}\sum_{i=1}^{n+1} h(e_i,e_i). \tag{2.4}$$

The submanifold M is *totally geodesic* in $\tilde{M}$ if $h = 0$ and *minimal* if $H = 0$. We put

$$h_{ij}^r = g(h(e_i,e_j),e_r), \qquad \|h\|^2 = \sum_{i,j=1}^{n+1} g(h(e_i,e_j),h(e_i,e_j)), \tag{2.5}$$

where $\{e_{n+2},\dots,e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$ and $r = n+2,\dots,2m+1$.

3. A basic inequality. Let M be a submanifold of an almost contact metric manifold. For $X \in TM$, let

$$\varphi X = PX + FX, \quad PX \in TM,\ FX \in T^{\perp}M. \tag{3.1}$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies

$$g(X,PY) = -g(PX,Y), \quad X,Y \in TM. \tag{3.2}$$

For a plane section $\pi \subset T_pM$ at a point $p \in M$,

$$\alpha(\pi) = g(e_1,Pe_2)^2, \qquad \beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2 \tag{3.3}$$

are real numbers in the closed unit interval $[0,1]$, which are independent of the choice of the orthonormal basis $\{e_1,e_2\}$ of π.

We recall the following lemma from [3].

Lemma 3.1. *If $a_1,\dots,a_{n+1},a$ are $n+2$ $(n\ge 1)$ real numbers such that*

$$\left(\sum_{i=1}^{n+1} a_i\right)^2 = n\left(\sum_{i=1}^{n+1} a_i^2 + a\right), \tag{3.4}$$

then $2a_1a_2 \ge a$, with equality holding if and only if $a_1+a_2=a_3=\cdots=a_{n+1}$.

Now, we prove the following theorem.

Theorem 3.2. *Let M be an $(n+1)$-dimensional $(n\ge 2)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M. Then, for each point $p\in M$ and each plane section $\pi\subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \le{}& \frac{(n+1)^2(n-1)}{2n}\|H\|^2\\ &+\frac{c}{8}\left(3\|P\|^2 - 6\alpha(\pi)+2\beta(\pi)+(n+1)(n-2)\right).\end{aligned} \tag{3.5}$$

The equality in (3.5) holds at $p\in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2},\dots,e_{2m+1}\}$ of $T_p^{\perp}M$ such that

(a) $\pi = \operatorname{Span}\{e_1,e_2\}$,

(b) *the forms of the shape operators $A_r \equiv A_{e_r}$, $r=n+2,\dots,2m+1$, become*

$$\begin{aligned} A_{n+2} &= \begin{pmatrix} \lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & (\lambda+\mu)I_{n-1}\end{pmatrix},\\ A_r &= \begin{pmatrix} h_{11}^r & h_{12}^r & 0\\ h_{12}^r & -h_{11}^r & 0\\ 0 & 0 & 0_{n-1}\end{pmatrix}, \quad r=n+3,\dots,2m+1.\end{aligned} \tag{3.6}$$

Proof. In view of the Gauss equation and (2.2), the scalar curvature and the mean curvature of M are related by

$$2\tau = \frac{c}{4}\left(3\|P\|^2 + n(n-1)\right) + (n+1)^2\|H\|^2 - \|h\|^2, \tag{3.7}$$

where $\|P\|^2$ is given by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2 \tag{3.8}$$

for any local orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM. We introduce

$$\rho = 2\tau - \frac{(n+1)^2(n-1)}{n}\|H\|^2 - \frac{c}{4}\left(3\|P\|^2 + n(n-1)\right). \tag{3.9}$$

From (3.7) and (3.9), we get

$$(n+1)^2\|H\|^2 = n\left(\|h\|^2 + \rho\right). \tag{3.10}$$

Let p be a point of M and let $\pi \subset T_pM$ be a plane section at p. We choose an orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \dots, e_{2m+1}\}$ for the normal space $T_p^{\perp}M$ at p such that $\pi = \operatorname{Span}\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+2}; then from (3.10), we get

$$\left(\sum_{i=1}^{n+1} h_{ii}^{n+2}\right)^2 = n\left(\sum_{i=1}^{n+1}\left(h_{ii}^{n+2}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{n+2}\right)^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}\left(h_{ij}^{r}\right)^2 + \rho\right). \tag{3.11}$$

Using Lemma 3.1, from (3.11) we obtain

$$h_{11}^{n+2}h_{22}^{n+2} \geq \frac{1}{2}\left\{\sum_{i\neq j}\left(h_{ij}^{n+2}\right)^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}\left(h_{ij}^{r}\right)^2 + \rho\right\}. \tag{3.12}$$

From the Gauss equation and (2.2), we also have

$$K(\pi) = \frac{c}{4}\left(1 + 3\alpha(\pi) - \beta(\pi)\right) + h_{11}^{n+2}h_{22}^{n+2} - \left(h_{12}^{n+2}\right)^2 + \sum_{r=n+3}^{2m+1}\left(h_{11}^{r}h_{22}^{r} - \left(h_{12}^{r}\right)^2\right). \tag{3.13}$$

Thus, we have

$$\begin{aligned} K(\pi) \geq{}& \frac{c}{4}\left(1 + 3\alpha(\pi) - \beta(\pi)\right) + \frac{1}{2}\rho + \sum_{r=n+2}^{2m+1}\sum_{j>2}\left\{\left(h_{1j}^{r}\right)^2 + \left(h_{2j}^{r}\right)^2\right\} \\ &+ \frac{1}{2}\sum_{i\neq j>2}\left(h_{ij}^{n+2}\right)^2 + \frac{1}{2}\sum_{r=n+3}^{2m+1}\sum_{i,j>2}\left(h_{ij}^{r}\right)^2 + \frac{1}{2}\sum_{r=n+3}^{2m+1}\left(h_{11}^{r} + h_{22}^{r}\right)^2, \end{aligned} \tag{3.14}$$

or

$$K(\pi) \geq \frac{c}{4}\left(1 + 3\alpha(\pi) - \beta(\pi)\right) + \frac{1}{2}\rho, \tag{3.15}$$

which, in view of (3.9), yields (3.5).

If the equality in (3.5) holds, then the inequalities given by (3.12) and (3.14) become equalities. In this case, we have

$$\begin{gathered} h_{1j}^{n+2}=0, \quad h_{2j}^{n+2}=0, \quad h_{ij}^{n+2}=0, \quad i\neq j>2; \\ h_{1j}^{r}=h_{2j}^{r}=h_{ij}^{r}=0, \quad r=n+3,\dots,2m+1;\ i,j=3,\dots,n+1; \\ h_{11}^{n+3}+h_{22}^{n+3}=\cdots=h_{11}^{2m+1}+h_{22}^{2m+1}=0. \end{gathered} \tag{3.16}$$

Furthermore, we may choose e_1 and e_2 so that $h_{12}^{n+2}=0$. Moreover, by applying Lemma 3.1, we also have

$$h_{11}^{n+2}+h_{22}^{n+2}=h_{33}^{n+2}=\cdots=h_{n+1\,n+1}^{n+2}. \tag{3.17}$$

Thus, choosing a suitable orthonormal basis $\{e_1,\dots,e_{2m+1}\}$, the shape operator of M becomes of the form given by (3.6). The converse is straightforward. □

4. Some applications. For the case $c=0$, from (3.5) we have the following pinching result.

Proposition 4.1. *Let M be an $(n+1)$-dimensional $(n>1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ with $c=0$ such that $\xi\in TM$. Then,*

$$\delta_M \leq \frac{(n+1)^2(n-1)}{2n}\|H\|^2. \tag{4.1}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ with $\xi\in TM$ is called a *semi-invariant submanifold* [1] of $\tilde{M}$ if $TM=\mathfrak{D}\oplus\mathfrak{D}^{\perp}\oplus\{\xi\}$, where $\mathfrak{D}=TM\cap\varphi(TM)$ and $\mathfrak{D}^{\perp}=TM\cap\varphi(T^{\perp}M)$. In fact, the condition $TM=\mathfrak{D}\oplus\mathfrak{D}^{\perp}\oplus\{\xi\}$ implies that the endomorphism P is an *f-structure* [12] on M with $\operatorname{rank}(P)=\dim(\mathfrak{D})$. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant* or *anti-invariant submanifold* according as the anti-invariant distribution $\mathfrak{D}^{\perp}$ is $\{0\}$ or the invariant distribution $\mathfrak{D}$ is $\{0\}$ [1, 12].

Now, we establish two inequalities in the following theorems, which are analogous to that of (1.2).

Theorem 4.2. *Let M be an $(n+1)$-dimensional $(n>1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M. If $c<0$, then*

$$\delta_M \leq \frac{(n+1)^2(n-1)}{2n}\|H\|^2+\frac{1}{2}(n+1)(n-2)\frac{c}{4}. \tag{4.2}$$

The equality in (4.2) holds if and only if M is a semi-invariant submanifold with $\dim(\mathfrak{D}) = 2$.

Proof. Since $c < 0$, in order to estimate δ_M, we minimize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (3.5). For an orthonormal basis $\{e_1,\dots,e_{n+1}\}$ of T_pM with $\pi = \operatorname{span}\{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^{n+1} g(e_i,\varphi e_j)^2 + 2\sum_{j=3}^{n+1}\left\{g(e_1,\varphi e_j)^2 + g(e_2,\varphi e_j)^2\right\}. \tag{4.3}$$

Thus, we see that the minimum value of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is zero provided $\pi = \operatorname{span}\{e_1,e_2\}$ is orthogonal to ξ and $\operatorname{span}\{\varphi e_j \mid j = 3,\dots,n\}$ is orthogonal to the tangent space T_pM. Thus, we have (4.2) with equality case holding if and only if M is semi-invariant such that $\dim(\mathfrak{D}) = 2$ with $\beta = 0$. □

Theorem 4.3. *Let M be an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that $\xi \in TM$. If $c > 0$, then*

$$\delta_M \le \frac{(n+1)^2(n-1)}{2n}\|H\|^2 + \frac{1}{2}n(n+2)\frac{c}{4}. \tag{4.4}$$

The equality in (4.4) holds if and only if M is an invariant submanifold.

Proof. Since $c > 0$, in order to estimate δ_M, we maximize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (3.5). We observe that the maximum of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is attained for $\|P\|^2 = n$, $\alpha(\pi) = 0$, and $\beta(\pi) = 1$, that is, M is an invariant and $\xi \in \pi$. Thus, we obtain (4.4) with equality case if and only if M is invariant with $\beta = 1$. □

In last, we prove the following theorem.

Theorem 4.4. *If M is an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that $c > 0$, $\xi \in TM$ and*

$$\delta_M = \frac{(n+1)^2(n-1)}{2n}\|H\|^2 + \frac{1}{2}n(n+2)\frac{c}{4}, \tag{4.5}$$

then M is a totally geodesic cosymplectic space form $M(c)$.

Proof. In view of Theorem 4.3, M is an odd-dimensional invariant submanifold of the cosymplectic space form $\tilde{M}(c)$. For every point $p \in M$, we can choose an orthonormal basis $\{e_1 = \xi, e_2,\dots,e_{n+1}\}$ for T_pM and $\{e_{n+2},\dots,e_{2m+1}\}$ for $T_p^{\perp}M$ such that A_r $(r = n+2,\dots,2m+1)$ take the form (3.6). Since M is an invariant submanifold of a cosymplectic manifold, therefore, it is minimal and $A_r\varphi+\varphi A_r=0$, $r=n+2,\dots,2m+1$ [11]. Thus, all the shape operators take the form

$$A_r=\begin{pmatrix} c_r & d_r & 0\\ d_r & -c_r & 0\\ 0 & 0 & 0_{n-1}\end{pmatrix},\quad r=n+2,\dots,2m+1. \tag{4.6}$$

Since $A_r\varphi e_1=0$, $r=n+2,\dots,2m+1$, from $A_r\varphi+\varphi A_r=0$, we get $\varphi A_r e_1=0$. Applying φ to this equation, we obtain $A_re_1=\eta(A_re_1)\xi=\eta(A_re_1)e_1$; thus, $d_r=0$, $r=n+2,\dots,2m+1$. This implies that $A_re_2=-c_re_2$. Applying φ to both sides, in view of $A_r\varphi+\varphi A_r=0$ we get $A_r\varphi e_2=c_r\varphi e_2$. Since φe_2 is orthogonal to ξ and e_2 and φ has a maximal rank, the principal curvature c_r is zero. Hence, M becomes totally geodesic. As in [12, Proposition 1.3, page 313], it is easy to show that M is a cosymplectic manifold of the constant φ-sectional curvature c. □

Acknowledgment. This work was supported by the Korea Science & Engineering Foundation grant R01-2001-00003.

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