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TWO-POINT DISTORTION THEOREMS FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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We give two-point distortion theorems for various subfamilies of analytic univalent functions. We also find the necessary and sufficient condition for these subclasses of analytic functions.

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1. Introduction. Let Ω be the family of functions $\omega(z)$ regular in the unit disc $D = \{z \mid |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

For arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $P(A,B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{1.1}$$

regular in D, such that $p(z) \in P(A,B)$ if and only if

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.2}$$

for some functions $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [6].

Moreover, let $C(A,B,b)$ denote the family of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.3}$$

regular in D, such that $f(z) \in C(A,B,b)$ if and only if

$$1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.4}$$

where $b \neq 0$, b is a complex number, for some functions $p(z) \in P(A,B)$ and all $z \in D$.

Next we consider the following class of functions defined in D.

Let $S^*(A,B,b)$ denote the family of functions

$$f(z) = z + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \tag{1.5}$$

regular in D, such that $f(z) \in S^*(A,B,b)$ if and only if

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=\frac{1+A\omega(z)}{1+B\omega(z)}, \tag{1.6}$$

where $b \neq 0$, b is a complex number, for some functions $\omega(z) \in \Omega$ and all $z \in D$.

We obtain the following subclasses of $C(A,B,b)$ by giving specific values to A, B, and b. For $A = 1$, $B = -1$, and $b = 1$, $C(1,-1,1)$ is the well-known class of convex functions [3, 4]. For $A = 1$, $B = -1$, and $b = 1-\alpha$ $(0 \le \alpha < 1)$, $C(1,-1,1-\alpha)$ is the class of convex functions of order α introduced by Robertson [9].

For $A = 1$, $B = -1$, $C(1,-1,b)$ is the class of convex functions of complex order; this class was introduced by Wiatrowski [12]. For $A = 1$, $B = -1$, and $b = e^{-i\lambda}\cos\lambda$, $|\lambda| < \pi/2$, $C(1,-1,e^{-i\lambda}\cos\lambda)$ is the class of functions for which $zf'(z)$ is λ-spirallike; this class was introduced by Robertson [10].

For $A=1$, $B=-1$, and $b=(1-\alpha)e^{-i\lambda}\cos\lambda$, $0 \le \alpha < 1$, $|\lambda| < \pi/2$, $C(1,-1,(1-\alpha)e^{-i\lambda}\cos\lambda)$ is the class of functions for which $zf'(z)$ is λ-spirallike of order α [1, 2, 7, 8, 11].

If we write $1+(1/b)z(f''(z)/f'(z)) = C(f'(z),f''(z),b)$, then we obtain the following classes:

(1) the class $C(1,0,b)$ defined by $|C(f'(z),f''(z),b)-1| < 1$,
(2) the class $C(\beta,0,b)$ defined by $|C(f'(z),f''(z),b)-1| < \beta$, $0 \le \beta < 1$,
(3) the class $C(\beta,-\beta,b)$ defined by

$$\left|\frac{C(f'(z),f''(z),b)-1}{C(f'(z),f''(z),b)+1}\right| < 1, \quad 0 < \beta, \tag{1.7}$$

(4) the class $C(1,(1-1/M),b)$ defined by $|C(f'(z),f''(z),b)-M| < M$, $M > 1$.

Similarly, the subclasses of $S^*(A,B,b)$ are obtained by giving specific values to A, B, and b. These subclasses are obtained in [1, 2, 7, 8, 11].

2. Preliminary lemmas. For the purpose of this paper, we give the following lemmas.

Lemma 2.1. *The necessary and sufficient condition for $f(z) \in C(A,B,b)$ is*

$$f(z)=\begin{cases}\displaystyle\int_0^z \left(1+B\omega(\zeta)\right)^{b(A-B)/B}d\zeta, & B \neq 0\\ \displaystyle\int_0^z e^{bA\omega(\zeta)}d\zeta, & B=0,\end{cases} \tag{2.1}$$

where $\omega(z) \in \Omega$.

Proof. Let $B \neq 0$ and let $f(z) \in C(A,B,b)$. From the definition of the class $C(A,B,b)$, we can write

$$1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}. \tag{2.2}$$

Equality (2.2) can be written in the form

$$\frac{f''(z)}{f'(z)} = b(A-B)\frac{\omega'(z)}{1+B\omega(z)} \tag{2.3}$$

by using Jack's lemma [5]. Integrating both sides of equality (2.3), we obtain

$$f(z) = \int_0^z \left(1+B\omega(\zeta)\right)^{b(A-B)/B} d\zeta. \tag{2.4}$$

Equality (2.4) shows that $f(z) \in C(A,B,b)$.
Conversely, if we take differentiation from equality (2.3), we obtain

$$f'(z) = \left(1+Bw(z)\right)^{b(A-B)/B}. \tag{2.5}$$

Differentiating both sides of equality (2.5), we obtain

$$z\frac{f''(z)}{f'(z)} = b(A-B)\frac{z\omega'(z)}{1+B\omega(z)}. \tag{2.6}$$

Using Jack's lemma [5] and after the simple calculations from (2.6), we obtain

$$1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}. \tag{2.7}$$

This equality shows that $f(z) \in C(A,B,b)$. Similarly, we obtain

$$f(z) = \int_0^z e^{bAw(\zeta)} d\zeta \Longleftrightarrow f(z) \in C(A,B,b), \quad B = 0. \tag{2.8}$$

□

Lemma 2.2. *Let $f(z) \in C(A,B,b) \Rightarrow zf'(z) \in S^*(A,B,b)$.*

Proof. Let

$$g(z) = zf'(z). \tag{2.9}$$

Taking a logarithmic derivative of (2.9), and after simple calculations, we get

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) = 1+\frac{1}{b}z\frac{f''(z)}{f'(z)}. \tag{2.10}$$

This shows that the lemma is true. □

Lemma 2.3. *The class $C(A,B,b)$ is invariant under the rotation so that $f(e^{i\alpha}z) \in C(A,B,b)$, $|\alpha| \leq 1$, whenever $f(z) \in C(A,B,b)$.*

Proof. Let $g(z) = f(e^{i\alpha}z)$. After the simple calculations from this equality we get

$$1+\frac{1}{b}z\frac{g''(z)}{g'(z)} = 1+\frac{1}{b}(e^{i\alpha}z)\frac{f''(e^{i\alpha}z)}{f'(e^{i\alpha}z)}, \qquad |\zeta| = |e^{i\alpha}z| < 1. \tag{2.11}$$

This shows that the lemma is true. □

We note that the class $S^*(A,B,b)$ is invariant under the rotation so that $f(e^{i\alpha}z) \in S^*(A,B,b)$, $|\alpha| \le 1$, whenever $f(z) \in S^*(A,B,b)$.

Lemma 2.4. *Let $f(z)$ be regular and analytic in D and normalized so that $f(0)=0$ and $f'(0)=1$. A necessary and sufficient condition for $f(z) \in C(A,B,b)$ is that for each member $g(z)$, $g(z) = z+a_2z^2+\cdots$, of $S^*(A,B,b)$, the equation*

$$g(z) = z\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right)^2, \quad z,\zeta \in D,\ z \ne \zeta,\ \zeta = \eta z,\ |\eta| \le 1, \tag{2.12}$$

must be satisfied.

Proof. Let $f(z) \in C(A,B,b)$, then this function is analytic, regular, and continuous in the unit disc D and by using Lemmas 2.2 and 2.3, equality (2.12) can be written in the form

$$g(z) = z\left(f'(z)\right)^2. \tag{2.13}$$

Taking the logarithmic derivative from equality (2.13) and after simple calculations, we get

$$1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = 1+\frac{1}{2b}\left(z\frac{g'(z)}{g(z)}-1\right) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.14}$$

If we consider equality (2.14), the definition of $C(A,B,b)$, and the definition of $S^*(A,B,b)$, we obtain that $g(z) \in S^*(A,B,b)$.

Conversely, let $g(z) \in S^*(A,B,b)$, then on simple calculations from equality (2.12), we get

$$1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}\right]+1-\frac{1}{b}. \tag{2.15}$$

If we write

$$F(z,\zeta) = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\zeta)}-\frac{z+\zeta}{z-\zeta}\right]+1-\frac{1}{b}, \tag{2.16}$$

equality (2.15) can be written in the form

$$F(z,\zeta) = 1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right). \tag{2.17}$$

On the other hand,

$$\lim_{\zeta\to z} F(z,\zeta) = 1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = 1+\frac{1}{b}\left(z\frac{g'(z)}{g(z)}-1\right) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.18}$$

Equality (2.18) shows that $f(z)\in C(A,B,b)$. □

Corollary 2.5. *If $f(z)\in C(A,B,b)$, then*

$$2\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]-1 = p(z) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.19}$$

Proof. If we take $\zeta = 0$ in $F(z,\zeta)$, we obtain

$$F(z,0) = \frac{1}{b}\left(2z\frac{f'(z)}{f(z)}-1\right)+1-\frac{1}{b} = p(z) = \frac{1+Aw(z)}{1+Bw(z)}. \tag{2.20}$$

This shows that the corollary is true. □

Corollary 2.6. *If $f(z)\in C(A,B,b)$, then the set of values of $(z(f'(z)/f(z)))$ is the closed disc with centre $C(r)$ and radius $g(r)$, where*

$$\begin{aligned} C(r) &= \frac{2-[2B^2+|b|(AB-B^2)]r^2}{2(1-B^2r^2)}, \\ g(r) &= \frac{|b|(A-B)r}{2(1-B^2r^2)}. \end{aligned} \tag{2.21}$$

The proof of this corollary is obtained by using (2.19) and the inequality

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{1-B^2r^2}, \quad p(z)\in P(A,B). \tag{2.22}$$

Inequality (2.22) was proved by Janowski [6].

Lemma 2.7. *If $f(z)\in C(A,B,b)$ and $h_\rho(z)$ is defined by*

$$h_\rho(z) = \frac{f(\rho((z+a)/(1+z\bar{a}))) - f(a)}{(1-|a|^2)f'(a)}, \quad a,z\in D, \quad \rho\in(0,1), \tag{2.23}$$

then $h_\rho(z)\in C(A,B,b)$.

Proof. Let $B \neq 0$. After simple calculations from (2.23), we obtain

$$
\begin{aligned}
1 &+ \frac{1}{b} z \frac{h_\rho''(z)}{h_\rho'(z)} \\
&= \frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\left[1+\frac{1}{b}\left(\rho\left(\frac{z+a}{1+z\overline{a}}\right)\right)\frac{f''(\rho((z+a)/(1+\overline{z}a)))}{f'(\rho((z+a)/(1+z\overline{a})))}\right] \\
&\quad + \left[1-\frac{1}{b}\frac{2z\overline{a}}{1+z\overline{a}}-\frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\right].
\end{aligned}
\tag{2.24}
$$

On the other hand, if we use Lemma 2.1, we can write

$$
\frac{f(\rho((z+a)/(1+z\overline{a})))-f(a)}{(1-|a|^2)f'(a)} = \int_0^z (1+B\omega(\zeta))^{b(A-B)/B} d\zeta. \tag{2.25}
$$

After a brief computation from equality (2.25), we get

$$
\begin{aligned}
\frac{1+A\omega(z)}{1+B\omega(z)} &= \frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\left[1+\frac{1}{b}\left(\rho\left(\frac{z+a}{1+z\overline{a}}\right)\right)\frac{f''(\rho((z+a)/(1+z\overline{a})))}{f'(\rho((z+a)/(1+z\overline{a})))}\right] \\
&\quad + \left[1-\frac{1}{b}\frac{2z\overline{a}}{1+z\overline{a}}-\frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\right].
\end{aligned}
\tag{2.26}
$$

Let $B = 0$. Similarly,

$$
\begin{aligned}
&\frac{f(\rho((z+a)/(1+z\overline{a})))-f(a)}{(1-|a|^2)f'(a)} = \int_0^z e^{bA\omega(\zeta)} d\zeta \Longrightarrow \frac{1+A\omega(z)}{1+B\omega(z)} = 1+A\omega(z) \\
&\quad = \frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\left[1+\frac{1}{b}\left(\rho\left(\frac{z+a}{1+z\overline{a}}\right)\right)\frac{f''(\rho((z+a)/(1+z\overline{a})))}{f'(\rho((z+a)/(1+z\overline{a})))}\right] \\
&\quad\quad + \left[1-\frac{1}{b}\frac{2z\overline{a}}{1+z\overline{a}}-\frac{(1-|a|^2)z}{(z+a)(1+z\overline{a})}\right].
\end{aligned}
\tag{2.27}
$$

In (2.26) and (2.27), letting $z = e^{i\theta}$ and $\omega = \rho((e^{i\theta}+a)/(1+e^{i\theta}\overline{a}))$ gives

$$
\begin{aligned}
\frac{1+A\omega(z)}{1+B\omega(z)} &= \frac{(1-|a|^2)}{|1+ae^{-i\theta}|^2}\left[1+\frac{1}{b}\omega\frac{f''(\omega)}{f'(\omega)}\right] \\
&\quad + \left[1-\frac{1}{b}\frac{2e^{i\theta}\overline{a}}{1+e^{i\theta}\overline{a}}-\frac{(1-|a|^2)e^{i\theta}}{|1+e^{-i\theta}a|^2}\right] \\
&= 1+\frac{1}{b} z \frac{h_\rho''(z)}{h_\rho'(z)},
\end{aligned}
\tag{2.28}
$$

and we conclude that $h_\rho(z)$ is in (2.27) for every admissible ρ. From the compactness of $C(A,B,b)$ and (2.28), we infer that $h(z) = \lim_{\rho\to 1} h_\rho(z)$ is in $C(A,B,b)$. □

3. Two-point distortion for the class $C(A,B,b)$. In this section, we give two-point distortion theorems for the class $C(A,B,b)$.

Theorem 3.1. *Let $f(z) \in C(A,B,b)$. Then for $|z| = r$, $0 \le r < 1$,*

$$\begin{aligned} &\frac{(1+B|z|)^{(B-A)(|b|-\operatorname{Re} b)/2B}}{(1-B|z|)^{(B-A)(|b|+\operatorname{Re} b)/2B}} \\ &\qquad\le |f'(z)| \le \frac{(1-B|z|)^{(B-A)(|b|-\operatorname{Re} b)/2B}}{(1+B|z|)^{(B-A)(|b|+\operatorname{Re} b)/2B}}, \quad B \ne 0, \\ &\qquad e^{-A|b||z|} \le |f'(z)| \le e^{A|b||z|}, \quad B = 0. \end{aligned} \tag{3.1}$$

Proof. If we use the definition of the class $C(A,B,b)$, then we obtain

$$\operatorname{Re}\left(z\frac{f''(z)}{f'(z)}\right) \ge \frac{\operatorname{Re} b(B^2-AB)r^2 - |b|(A-B)r}{1-B^2r^2}, \quad B \ne 0, \tag{3.2}$$

since

$$\operatorname{Re} z\frac{f''(z)}{f'(z)} = r\frac{\partial}{\partial r}\log|f'(z)|, \quad |z| = r; \tag{3.3}$$

and using (3.2), we obtain

$$\frac{\partial}{\partial r}\log|f'(z)| \ge \frac{\operatorname{Re} b(B^2-AB)r - |b|(A-B)}{(1-B^2r^2)}. \tag{3.4}$$

Integrating both sides of inequality (3.4) from 0 to r, we obtain

$$|f'(z)| \ge \frac{(1+B|z|)^{(B-A)(|b|-\operatorname{Re} b)/2B}}{(1-B|z|)^{(B-A)(|b|+\operatorname{Re} b)/2B}}. \tag{3.5}$$

Similarly, we obtain the bounds on the right-hand side of (3.1).
If $B = 0$, then we have

$$-|b|Ar \le \operatorname{Re} z\frac{f''(z)}{f'(z)} \le |b|Ar; \tag{3.6}$$

and using (3.3), we obtain

$$-|b|A \le \frac{\partial}{\partial r}\log|f'(z)| \le |b|A. \tag{3.7}$$

Integrating both sides of inequality (3.7) from 0 to r, we obtain the desired result. □

Theorem 3.2. *If $f(z) \in C(A,B,b)$, then, for $|z| = r$, $0 \le r < 1$,*

$$\begin{aligned}
&\frac{|z|(1+B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}}{(1-B|z|)^{(B-A)(|b|+\operatorname{Re} b)/4B}} \\
&\qquad \le |f(z)| \le \frac{|z|(1-B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}}{(1+B|z|)^{(B-A)(|b|+\operatorname{Re} b)/4B}}, \quad B \ne 0, \\
&\qquad\qquad |z|e^{-|b|A|z|/2} \le |f(z)| \le |z|e^{|b|A|z|/2}, \qquad B = 0.
\end{aligned} \tag{3.8}$$

Proof. If we use Corollaries 2.5 and 2.6 and the definition of the classes $C(A,B,b)$ and $P(A,B)$, we can write

$$\left|\left[2\left(1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)-1\right)\right]-\frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{1-B^2r^2}. \tag{3.9}$$

After the simple calculations from inequality (3.9), we get

$$\operatorname{Re} z\frac{f'(z)}{f(z)} \ge \frac{2-|b|(A-B)r-(2B^2-(B^2-AB)\operatorname{Re} b)r^2}{1-B^2r^2} \tag{3.10}$$

since

$$\operatorname{Re} z\frac{f'(z)}{f(z)} = r\frac{\partial}{\partial r}\log|f(z)|; \tag{3.11}$$

and using (3.10), we obtain

$$\frac{\partial}{\partial r}\log|f(z)| \ge \frac{2-|b|(A-B)r-(2B^2-(B^2-AB)\operatorname{Re} b)r^2}{2r(1-B^2r^2)}. \tag{3.12}$$

Integrating both sides of this inequality from 0 to r, we obtain

$$|f(z)| \ge \frac{|z|(1+B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}}{(1-B|z|)^{(B-A)(|b|+\operatorname{Re} b)/4B}}. \tag{3.13}$$

Similarly, we obtain the upper bounds in (3.8). Thus we end the proof. □

We note that the bounds in Theorems 3.1 and 3.2 are sharp because the extremal function is

$$f_*(z) = \begin{cases} e^{Abz}, & B \neq 0, \\ \dfrac{z(1-Bz)^{(B-A)(|b|-2\operatorname{Re} b)/4B}}{(1+Bz)^{(B-A)(|b|+\operatorname{Re} b)/4B}}, & B = 0, \end{cases} \qquad z = \left(\frac{r\left(r-\sqrt{\overline{b}/b}\right)}{1-r\sqrt{\overline{b}/b}}\right). \tag{3.14}$$

Corollary 3.3. *Let $f(z) \in C(A,B,b)$. Then*

$$\begin{aligned} \alpha F_1(u,v) &\leq \left|f(u)-f(v)\right| \leq \alpha F_2(u,v), \quad B \neq 0, \\ \alpha G_1(u,v) &\leq \left|f(u)-f(v)\right| \leq \alpha G_2(u,v), \quad B = 0, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \alpha &= (1-|v|^2)\frac{|u-v|}{|1-\overline{v}u|}, \\ F_1(u,v) &= \frac{(1+B|z|)^{3(B-A)(|b|-\operatorname{Re} b)/4B}}{(1-B|z|)^{3(B-A)(|b|+\operatorname{Re} b)/4B}}, \\ F_2(u,v) &= \frac{(1-B|z|)^{3(B-A)(|b|-\operatorname{Re} b)/4B}}{(1+B|z|)^{3(B-A)(|b|+\operatorname{Re} b)/4B}}, \\ G_1(u,v) &= e^{-(3/2)|b|A(|u-v|/|1-u\overline{v}|)}, \\ G_2(u,v) &= e^{(3/2)|b|A(|u-v|/|1-u\overline{v}|)}. \end{aligned} \tag{3.16}$$

Proof. If we consider Lemmas 2.1 and 2.7 and Theorem 3.2, then we can write

$$\begin{aligned} \frac{|z|(1+B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}}{(1-B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}} &\leq \left|\frac{f\left((z+a)/(1+z\overline{a})\right)-f(a)}{(1-|a|^2)f'(a)}\right| \\ &\leq \frac{|z|(1-B|z|)^{(B-A)(|b|-\operatorname{Re} b)/4B}}{(1+B|z|)^{(B-A)(|b|+\operatorname{Re} b)/4B}}, \quad B \neq 0, \\ |z|e^{-|b|A|z|/2} &\leq \left|\frac{f\left((z+a)/(1+z\overline{a})\right)-f(a)}{(1-|a|^2)f'(a)}\right| \\ &\leq |z|e^{-|b|A|z|/2}, \quad B = 0. \end{aligned} \tag{3.17}$$

Inequalities (3.17) can be written in the form

$$\begin{aligned}(1-|a|^2)\,|f'(a)\,|M_1(|z|) &\le \left|f\left(\frac{z+a}{1+z\bar{a}}\right)-f(a)\right| \\ &\le (1-|a|^2)|f'(a)|M_2(|z|), \quad B\neq 0, \\ (1-|a|^2)|f'(a)|N_1(|z|) &\le \left|f\left(\frac{z+a}{1+z\bar{a}}\right)-f(a)\right| \\ &\le (1-|a|^2)\,|f'(a)\,|N_2(|z|), \quad B=0,\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}M_1(|z|) &= \frac{|z|(1+B|z|)^{(B-A)(|b|-\mathrm{Re}\,b)/4B}}{(1-B|z|)^{(B-A)(|b|+\mathrm{Re}\,b)/4B}}, \\ M_2(|z|) &= \frac{|z|(1-B|z|)^{(B-A)(|b|-\mathrm{Re}\,b)/4B}}{(1+B|z|)^{(B-A)(|b|+\mathrm{Re}\,b)/4B}}, \\ N_1(|z|) &= |z|e^{-|b|A|z|/2}, \\ N_2(|z|) &= |z|e^{-|b|A|z|/2}.\end{aligned} \tag{3.19}$$

If we take $v=a$, $u=(z+v)/(1+z\bar{v})$, or $z=(u-v)/(1-u\cdot\bar{v})$, and if we use Theorem 3.1 in inequalities (3.18), we obtain the desired result. □

We note that these inequalities are sharp because the extremal function is

$$f_*(z)=\begin{cases} e^{Abz}, & B\neq 0 \\ \dfrac{z(1-Bz)^{(B-A)(|b|-2\mathrm{Re}\,b)/4B}}{(1+Bz)^{(B-A)(|b|+\mathrm{Re}\,b)/4B}}, & B=0.\end{cases} \tag{3.20}$$

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