IJMMS 2003:17, 1047–1053
PII. S0161171203208164
http://ijmms.hindawi.com


# CLOSED ORBITS OF $(G,\tau)$-EXTENSION OF ERGODIC TORAL AUTOMORPHISMS

**MOHD. SALMI MD. NOORANI**

Received 21 August 2002

Let $A : T \to T$ be an ergodic automorphism of a finite-dimensional torus $T$. Also, let $G$ be the set of elements in $T$ with some fixed finite order. Then, $G$ acts on the right of $T$, and by denoting the restriction of $A$ to $G$ by $\tau$, we have $A(xg) = A(x)\tau(g)$ for all $x \in T$ and $g \in G$. Now, let $\tilde{A} : \tilde{T} \to \tilde{T}$ be the (ergodic) automorphism induced by the $G$-action on $T$. Let $\tilde{\tau}$ be an $\tilde{A}$-closed orbit (i.e., periodic orbit) and $\tau$ an $A$-closed orbit which is a lift of $\tilde{\tau}$. Then, the degree of $\tau$ over $\tilde{\tau}$ is defined by the integer $\deg(\tau/\tilde{\tau}) = \lambda(\tau)/\lambda(\tilde{\tau})$, where $\lambda(\ )$ denotes the (least) period of the respective closed orbits. Suppose that $\tau_1,\dots,\tau_t$ is the distinct $A$-closed orbits that covers $\tilde{\tau}$. Then, $\deg(\tau_1/\tilde{\tau}) + \cdots + \deg(\tau_t/\tilde{\tau}) = |G|$. Now, let $\underline{l} = (\deg(\tau_1/\tilde{\tau}),\dots,\deg(\tau_t/\tilde{\tau}))$. Then, the previous equation implies that the $t$-tuple $\underline{l}$ is a partition of the integer $|G|$ (after reordering if needed). In this case, we say that $\tilde{\tau}$ induces the partition $\underline{l}$ of the integer $|G|$. Our aim in this paper is to characterize this partition $\underline{l}$ for which $A_{\underline{l}} = \{\tilde{\tau} \subset \tilde{T} : \tilde{\tau}$ induces the partition $\underline{l}\}$ is nonempty and provides an asymptotic formula involving the closed orbits in such a set as their period goes to infinity.

2000 Mathematics Subject Classification: 37C25.

**1. Introduction.** In a joint paper [3], we studied how periodic orbits of a shift of finite type $X$ lift to a so-called homogeneous extension of $X$. The main result of that paper, which was motivated by a number-theoretical result of Heilbronn (see Cassels and Fröhlich [1]), is an asymptotic formula for the number of periodic orbits of $X$ according to how they lift to the associated homogeneous extension. As an application, we observe that the shift $X$ and its associated homogeneous extension is the symbolic model for a $(G,\tau)$-extension of a hyperbolic toral automorphism. In particular, we showed that the previously mentioned formula also holds for this latter dynamical system (see Noorani and Parry [3] for details).

Our aim in this paper is to extend this latter result to ergodic (quasihyperbolic) toral automorphisms. More precisely, we consider a $(G,\tau)$-extension of an ergodic toral automorphism $\tilde{A}$ and provide an analogous result to the one obtained in Noorani and Parry [3]. As in the hyperbolic case, the proof of this asymptotic formula is carried out in two steps. The first step is to bring in a so-called cyclic extension and obtain the associated Chebotarev theorem for this extension. The asymptotic formula achieved in this theorem is analogous to Merten's theorem of the analytic number theory (see Noorani [2]). The second step is then just a simple application of our result in [3].

**2.** $(G,\tau)$**-extensions and identifications.** Let $A$ be an ergodic automorphism of a $d$-dimensional torus $T$. Note that in the literature this is also known as a quasihyperbolic automorphism of $T$. We always regard $A$ to be an element of $\mathrm{GL}(d,Z)$ with $\det A = \pm 1$. As is well known, this in turn is equivalent to requiring $A$ not to have any eigenvalue being a root of unity.

Let $m$ be fixed and $G$ be the set of elements in $T$ with order $m$, that is, $G = \{g \in T : g^m = e_G\}$. It is clear that the restriction of $A$ to $G$ is a group isomorphism. Denote this restriction by $\tau$. Then, by letting $G$ act on the right of $T$ and using multiplicative notation, we have

$$A(xg) = A(x)\tau(g), \quad \forall x \in T,\ g \in G. \tag{2.1}$$

In this case, we say that $G$ $\tau$-commute with $A$. Now, since $G$ acts freely on $T$ and $|G| < \infty$, we observe that $A$ induces an automorphism $\tilde{A}$ on the quotient manifold $\tilde{T}(= T/G)$, which is also an ergodic toral automorphism. Letting $\pi_G : T \to \tilde{T}$ be the covering map, we have $\pi_G A = \tilde{A}\pi_G$. We refer to $A$ as a $(G,\tau)$-extension of $\tilde{A}$. In this case, we also say that $A$ covers itself with a finite abelian group. Note that the eigenvalues of the associated matrix of $A$ and $\tilde{A}$ are the same, including multiplicites. This follows from the easily proven fact that both $A$ and $\tilde{A}$ share the same linear lift.

Recall that a partition of a positive integer $k$ is a collection of positive integers $l_1, l_2, \dots, l_t$ such that $k \ge l_1 \ge l_2 \ge \cdots \ge l_t \ge 1$ and $l_1 + \cdots + l_t = k$. In this case, we write $\underline{l}$ for the $t$-tuple $(l_1, \dots, l_t)$.

Let $\tilde{\tau}$ be an $\tilde{A}$-closed orbit with (least) period $\lambda(\tilde{\tau})$ and $\tau$ be an $A$-closed orbit with period $\lambda(\tau)$ such that $\pi_G(\tau) = \tilde{\tau}$. Then, the degree of $\tau$ over $\tilde{\tau}$ is defined by the integer

$$\deg\left(\frac{\tau}{\tilde{\tau}}\right) = \frac{\lambda(\tau)}{\lambda(\tilde{\tau})}. \tag{2.2}$$

Suppose that $\tau_1, \dots, \tau_t$ are the distinct $A$-closed orbits that cover $\tilde{\tau}$. Then, the following basic relation holds:

$$\deg\left(\frac{\tau_1}{\tilde{\tau}}\right) + \cdots + \deg\left(\frac{\tau_t}{\tilde{\tau}}\right) = |G|. \tag{2.3}$$

Note that the above equation gives us a partition of $|G|$. In this case, we say that $\tilde{\tau}$ induces the partition $\underline{l} = (l_1, l_2, \dots, l_t)$ of the integer $|G|$ if

$$\underline{l} = \left(\deg\left(\frac{\tau_1}{\tilde{\tau}}\right), \dots, \deg\left(\frac{\tau_t}{\tilde{\tau}}\right)\right) \quad \text{(after reordering if needed)}. \tag{2.4}$$

For each partition $\underline{l}$ of $G$, let $A_{\underline{l}} = \{\tilde{\tau} \subset \tilde{T} : \tilde{\tau} \text{ induces the partition } \underline{l}\}$. We are interested in an asymptotic formula involving the set $A_{\underline{l}}$. To achieve this, we need to identify $A$ and $\tilde{A}$ with a certain direct product dynamical system.

Since $G$ is finite, it is clear that $\tau^n = \mathrm{id}$ for some positive integer $n$. We denote by $\hat{T}$ the direct product manifold $Z_n \times T$, where $Z_n$ is the cyclic group on $n$ elements. Now, define $\hat{A}:\hat{T}\to\hat{T}$ as $\hat{A}(s,x)=(s+1,A(x))$. The obvious projection map $\pi:\hat{T}\to T$ gives us $\pi\hat{A}=A\pi$. With this in mind, we refer to $\hat{A}$ as a $Z_n$-(cylic) extension of $A$. Note that $\hat{A}$ is also an extension of $\tilde{A}$ via the obvious projection map $\pi_{\hat{G}}$. It is helpful to have the following multicommutative diagram in mind:

$$\begin{tikzcd} \hat{T} \arrow[r, "\hat{A}"] \arrow[dd, "\pi_{\hat{G}}"] & \hat{T} \arrow[dr] & & \\ & & T \arrow[r, "A"] \arrow[dl] & T. \\ \tilde{T} \arrow[r, "\tilde{A}"] & \tilde{T} & & \end{tikzcd} \tag{2.5}$$

Now, define the semidirect product group $\hat{G}$ of $Z_n$ and $G$ as follows:

$$\hat{G}=Z_n\times_\tau G=\{(r,g):r\in Z_n,\ g\in G\} \tag{2.6}$$

with operation

$$(r,g)\cdot(s,h)=(r+s,g\cdot\tau^r(h)). \tag{2.7}$$

It is easy to see that the identity element of $\hat{G}$ is $(0,e_G)$ and the inverse of $(r,g)$ is $(-r,\tau^{-r}(g^{-1}))$.

We introduce a right $\hat{G}$-action on $\hat{T}$ by $(s,x)(r,g)=(r+s,x\tau^s(g))$ for each $(r,g)\in\hat{G}$ and $(s,x)\in\hat{T}$. Then, it is clear that $\hat{G}$ acts freely on $\hat{T}$ and more importantly, this action commutes with $\hat{A}$, that is, $\hat{A}\hat{g}=\hat{g}\hat{A}$ for all $\hat{g}\in\hat{G}$. Now, form the $\hat{G}$-orbit space $\hat{T}/\hat{G}$ with the appropriate induced automorphism. Then, it is not difficult to see that we can identify $\hat{T}/\hat{G}$ (together with the induced map) with $\tilde{A}:\tilde{T}\to\tilde{T}$. With this identification, we refer to $\hat{A}:\hat{T}\to\hat{T}$ as a free $\hat{G}$-extension of $\tilde{A}:\tilde{T}\to\tilde{T}$.

Let $\hat{H}$ be the subgroup $Z_n\times\{e_G\}$ of $\hat{G}$, and consider the action of $\hat{H}$ on the space $\hat{T}$. The $\hat{H}$-orbit space is

$$\hat{T}/\hat{H}=\{(s,x)\cdot Z_n\times\{e_G\}:x\in T,\ r\in Z_n\}. \tag{2.8}$$

Since $(s,x)(r,e_G)=(r+s,x\tau^r(e_G))=(r+s,x)$ for all $(s,x)\in\hat{T}$ and $(r,e_G)\in\hat{H}$, we deduce that $\hat{T}/\hat{H}$ can be identified with $T$. Moreover, it can be checked that the induced map on $\hat{T}/\hat{H}$ is essentially $A$. These identifications are used in the next section to obtain an auxiliary result, the so-called Chebotarev theorem for the $\hat{G}$-extension of $\tilde{A}$ (see Parry and Pollicott [4]).

**3. Lifting closed orbits of $\tilde{A}$ to $\hat{A}$.** In this section, we will, with the help of the above identifications, provide the answer to the first step of the problem mentioned in the introduction. Recall that $\tilde{A}$ is the ergodic toral automorphism induced by the $G$-action on $A$.

**3.1. Frobenius classes, $\zeta$ and $L$-functions.** Let $\tilde{\tau}$ be an $\tilde{A}$-closed orbit with least period $\lambda(\tilde{\tau})$ and $\hat{\tau}$ an $\hat{A}$-closed orbit covering $\tilde{\tau}$. Then, since the $\hat{G}$-action on $\hat{T}$ is free and commutes with $\hat{A}$, we deduce that there exists a unique element $\hat{\gamma} = \hat{\gamma}(\hat{\tau}) \in \hat{G}$ such that if $p \in \hat{\tau}$, then

$$p\hat{\gamma} = \hat{A}^{\lambda(\tilde{\tau})}(p). \tag{3.1}$$

This group element $\hat{\gamma}(\hat{\tau})$, which only depends on $\hat{\tau}$, is called the *Frobenius element* of $\hat{\tau}$. Moreover, if $\hat{\tau}'$ is another $\hat{A}$-closed orbit also covering $\tilde{\tau}$, it is easy to see that the Frobenius elements of $\hat{\tau}'$ and of $\hat{\tau}$ are conjugated with each other. We call the conjugacy class determined by $\tilde{\tau}$ the *Frobenius class* of $\tilde{\tau}$ and is denoted by $[\tilde{\tau}]$ (see [4, 5]).

The zeta function of $\tilde{A}$ is defined by

$$\tilde{\zeta}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{Fix}_m \tilde{A}, \tag{3.2}$$

where $\operatorname{Fix}_m \tilde{A} = \operatorname{Card}\{x \in \tilde{T} : \tilde{A}^m(x) = x\}$. Let $h$ be the topological entropy of $\tilde{A}$. Then, it is well known that $\tilde{\zeta}(z)$ is analytic and nonzero for $|z| < e^{-h}$. The following result, which is due to Waddington [8], gives the necessary information regarding the analytic properties of $\tilde{\zeta}(z)$ for $z$ beyond $e^{-h}$.

**Proposition 3.1.** *Let $\tilde{A}$ be an ergodic toral automorphism with topological entropy $h$. Then,*

$$\tilde{\zeta}(z) = \tilde{B}(z) \prod_{\rho \in U} \frac{1}{(1 - e^h \rho z)^{K(\rho)}}, \tag{3.3}$$

*where $\tilde{B}(z)$ is analytic and nonzero for $|z| < \tilde{R}e^{-h}$ (some $\tilde{R} > 1$) and the elements of $U$ are the terms in the expansion of $\prod_\lambda (1-\lambda)$ ($\lambda$ is an eigenvalue of $\tilde{A}$ of modulus one) such that if $\rho \in U$, then $K(\rho)$ is the coefficient of the term $\rho$ in the above expansion.*

Using the above result, we have (see Noorani [2, Proposition 5]) the following asymptotic formula which is motivated by Merten's theorem of the analytic number theory (see also Sharp [7]).

**Proposition 3.2.** *The $\tilde{A}$-closed orbits $\tilde{\tau}$ satisfy*

$$\sum_{\lambda(\tilde{\tau}) \le x} \log \left(1 - e^{-h\lambda(\tilde{\tau})}\right)^{-1} = m \log x + \log \nu + m\gamma + o(1), \quad \text{as } x \longrightarrow \infty, \tag{3.4}$$

*where $\gamma$ is Euler's constant and $m = 2^{d/2}$ ($d$ is the number of eigenvalues of $\tilde{A}$ of modulus one) and some constant $\nu$.*

Let $\chi$ be an irreducible representation of $\hat{G}$. The $L$-function of $\chi$ is defined as

$$\tilde{L}(z,\chi) = \exp \sum_{\tilde{\tau}} \sum_{n=1}^{\infty} \frac{\chi([\tilde{\tau}]) z^{\lambda(\tilde{\tau})n}}{n}, \tag{3.5}$$

where the first sum is taken over all $\tilde{A}$-closed orbits. By comparing the above expression with $\tilde{\zeta}(z)$, we deduce that $\tilde{L}(z,\chi)$ is nonzero and analytic for $|z| < e^{-h}$. In fact, $\tilde{L}(z,\chi)$ has a nonzero meromorphic extension to some disc of radius greater than $e^{-h}$. Let $\mathrm{Irr}(\hat{G})$ be the collection of all irreducible characters of $\hat{G}$. Then, as will be apparent later on, our main problem is to obtain the analytic properties of $\tilde{L}(z,\chi)$ for $z$ in a neighborhood of $e^{-h}$ for each $\chi \in \mathrm{Irr}(\hat{G})$. First, observe that when $\chi = \chi_0$, the principal character, $\tilde{L}(z,\chi_0) = \tilde{\zeta}(z)$. Moreover, from Serre [6], we have the following proposition.

**Proposition 3.3.** *Let $J = \{\eta_0,\eta_1,\ldots,\eta_{n-1}\}$ be the irreducible characters of the cyclic group $Z_n$. Then, there exists 1-dimensional characters $\chi_i \in \mathrm{Irr}(\hat{G})$, $i = 0,\ldots,n-1$ such that for each $r \in Z_n$, $\chi_i(r,g) = \chi_i(r,e_G)$ for all $g \in G$. Moreover, $\chi_i(r,e_G) = \eta_i(r)$ for all $r \in Z_n$.*

It is straightforward to check that the $L$-functions for these special characters satisfy the following corollary.

**Corollary 3.4.** *Let $K = \{\chi_0,\chi_1,\ldots,\chi_{n-1}\}$ be as above. Then, $\tilde{L}(z,\chi_i) = \tilde{\zeta}(\omega^i z)$ for all $i = 0,1,\ldots,n-1$, where $\omega$ is a primitive $n$th root of unity.*

Note that the above result says that $\tilde{L}(z,\chi_i)$, $i = 1,\ldots,n$ are analytic and nonzero in a neighborhood of $z = e^{-h}$. In fact, this holds true for each $\chi_i \in \mathrm{Irr}(\hat{G})$ $\chi \neq \chi_0$ (see [2, 4]).

**3.2. The Chebotarev theorem.** Let $C$ be a conjugacy class of $\hat{G}$. To capture the $\tilde{A}$-closed orbits $\tilde{\tau}$ with Frobenius class $[\tilde{\tau}] = C$, we introduce the following zeta function:

$$\tilde{\zeta}_C(z) = \prod_{[\tilde{\tau}]=C} \left(1 - z^{\lambda(\tilde{\tau})}\right)^{-1}, \tag{3.6}$$

where the product is taken over all $\tilde{A}$-closed orbits with Frobenius class $C$. Note that $\tilde{\zeta}_C(z)$ is just the restriction of $\tilde{\zeta}(z)$ to the $\tilde{A}$-closed orbits whose Frobenius class equals $C$. Let $g \in \hat{G}$. By the orthogonality relation for irreducible characters, we have

$$\frac{|G|}{|C|} \log \tilde{\zeta}_C(z) = \sum_{\chi} \chi(g^{-1}) \sum_{n=1}^{\infty} \sum_{\tilde{\tau}} \chi([\tilde{\tau}]) \frac{z^{\lambda(\tilde{\tau})n}}{n}. \tag{3.7}$$

Equivalently, we have

$$\begin{aligned}\frac{|G|}{|C|}\sum_{\substack{\lambda(\tilde{\tau})\le x\\ [\tilde{\tau}]=C}}\log\left(1-z^{\lambda(\tilde{\tau})}\right)^{-1}&=\sum_{\chi}\chi(g^{-1})\sum_{n=1}^{\infty}\sum_{\lambda(\tilde{\tau})\le x}\chi([\tilde{\tau}])\frac{z^{\lambda(\tilde{\tau})n}}{n}\\&=\sum_{n=1}^{\infty}\sum_{\lambda(\tilde{\tau})\le x}\log\left(1-z^{\lambda(\tilde{\tau})}\right)^{-1}\\&\quad+\sum_{\chi\neq\chi_0}\chi(g^{-1})\sum_{n=1}^{\infty}\sum_{\lambda(\tilde{\tau})\le x}\chi([\tilde{\tau}])\frac{z^{\lambda(\tilde{\tau})n}}{n}.\end{aligned}\tag{3.8}$$

In a neighborhood of $z=e^{-h}$, we have by Proposition 3.2,

$$\begin{aligned}\frac{|G|}{|C|}\sum_{\substack{\lambda(\tilde{\tau})\le x\\ [\tilde{\tau}]=C}}\log\left(1-e^{-h\lambda(\tilde{\tau})}\right)^{-1}&=\log\nu+m\log x+m\gamma\\&\quad+\sum_{\chi\neq\chi_0}\chi(g^{-1})\log L(e^{-h},\chi)+o(1)\end{aligned}\tag{3.9}$$

as $x\to\infty$.

A straightforward manipulation of the above equations then gives us the following theorem.

**Theorem 3.5.** *For a conjugacy class $C$ of $\hat{G}$ and $g\in C$,*

$$\prod_{\substack{\lambda(\tilde{\tau})\le x\\ [\tilde{\tau}]=C}}\left(1-\frac{1}{e^{h\lambda(\tilde{\tau})}}\right)\sim D\left(\frac{e^{-m\gamma}}{x^m}\nu\right)^{|C|/|G|}\quad \text{as } x\longrightarrow\infty\tag{3.10}$$

*through the integers, where $D=\left(\prod_{\chi\neq\chi_0}L(e^{-h},\chi)^{\chi(g^{-1})}\right)^{|C|/|G|}$.*

**4. Applications to $(G,\tau)$-extensions.** We now present the main result of this paper which is an asymptotic formula involving the set $A_{\underline{l}}=\{\tilde{\tau}\subset\tilde{T}:\tilde{\tau}$ induces the partition $\underline{l}\}$ for each partition $\underline{l}$ of $G$. Before that, we need the following notion: let $K$ be another subgroup of $\hat{G}$. We can define a left action of $k\in K$ on the coset space $\hat{G}/\hat{H}$ by $k\cdot\hat{g}\hat{H}=k\hat{g}\hat{H}$. Let $K_1,\dots,K_m$ be the distinct orbits of this action and $r_i$, $i=1,\dots,m$ be their respective sizes. This different sizes then form a partition of $|\hat{G}|/|\hat{H}|$. Note that $G$ is identifiable with $\hat{G}/\hat{H}$. In this case, we say that $K$ induces the partition $\underline{r}=(r_1,\dots,r_m)$ of $|G|$ (after reordering if needed). It is easy to see that if $k$ is conjugate to $k'$, then the respective cyclic subgroups generated by them induces the same partition of $|G|$. The proof of the following result can be found in Noorani and Parry [3].

**Theorem 4.1.** *Let $\tilde{\tau}$ be an $\tilde{A}$-closed orbit. Then, $\tilde{\tau}$ induces the partition $\underline{l}$ of $|G|$ if and only if the action of the cylic group generated by some (and hence all) Frobenius element associated with $\tilde{\tau}$ induces the partition $\underline{l}$ on $|G|$.*

Let $C(g)$ denote the conjugacy class containing $g \in \hat{G}$. As an immediate corollary to the above theorem, we have the following corollary.

**Corollary 4.2.** *Let $C_i = \{\tilde{\tau} \subset X : [\tilde{\tau}] = C(g_i)\}$ be the distinct subsets of $\tilde{A}$-closed orbits with Frobenius class $C(g_i)$, such that the cylic subgroup generated by $g_i$, $i = 1,\dots,n$ induces the partition $\underline{l}$. Then,*

$$A_{\underline{l}} = \bigcup_{i=1}^{n} C_i. \tag{4.1}$$

Since the union in the above result is disjoint, a direct application of Theorem 3.5 then gives us the main theorem of this paper.

**Theorem 4.3.** *Let $A$ be a $(G,\tau)$-extension of $\tilde{A}$, and let $\underline{l}$ be a partition of $|G|$ such that $A_{\underline{l}} = \bigcup_{i=1}^{n} C_i$ (as above). Then,*

$$\prod_{\substack{\lambda(\tilde{\tau}) \le x \\ [\tilde{\tau}] \text{ induces } \ell}} \left(1 - \frac{1}{e^{h\lambda(\tilde{\tau})}}\right) \sim \prod_{i=1}^{n} D_i \left(\frac{e^{-m\gamma}}{x^m} \nu\right)^{C(g_i)/|G|} \quad \text{as } x \longrightarrow \infty \tag{4.2}$$

*through the integers, where $D_i = (\prod_{\chi \ne \chi_0} L(e^{-h}, \chi)^{\chi(g^{-1})})^{C(g_i)/|G|}$.*

## References

[1] J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Proceedings of an Instructional Conference Organized by the London Mathematical Society (a NATO Advanced Study Institute) with the Support of the International Mathematical Union, Academic Press, London, 1967.

[2] M. S. M. Noorani, *Mertens theorem and closed orbits of ergodic toral automorphisms*, Bull. Malaysian Math. Soc. (2) **22** (1999), no. 2, 127–133.

[3] M. S. M. Noorani and W. Parry, *A Chebotarev theorem for finite homogeneous extensions of shifts*, Bol. Soc. Brasil. Mat. (N.S.) **23** (1992), no. 1-2, 137–151.

[4] W. Parry and M. Pollicott, *The Chebotarov theorem for Galois coverings of Axiom A flows*, Ergodic Theory Dynam. Systems **6** (1986), no. 1, 133–148.

[5] ______, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque (1990), no. 187-188, 268.

[6] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1977.

[7] R. Sharp, *An analogue of Mertens' theorem for closed orbits of Axiom A flows*, Bol. Soc. Brasil. Mat. (N.S.) **21** (1991), no. 2, 205–229.

[8] S. Waddington, *The prime orbit theorem for quasihyperbolic toral automorphisms*, Monatsh. Math. **112** (1991), no. 3, 235–248.

MOHD. SALMI MD. NOORANI: School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM, Bangi, Selangor Darul Ehsan, Malaysia