IJMMS 2003:62, 3949–3956
PII. S0161171203211339
http://ijmms.hindawi.com

ON CLEAN IDEALS

HUANYIN CHEN and MIAOSEN CHEN

Received 21 November 2002

We introduce the notion of clean ideal, which is a natural generalization of clean rings. It is shown that every matrix ideal over a clean ideal of a ring is clean. Also we prove that every ideal having stable range one of a regular ring is clean. These generalize the corresponding results for clean rings.

2000 Mathematics Subject Classification: 16E50, 16U99.

1. Introduction. Let R be a unital ring. We say that R is a clean ring in case every element of R is a sum of an idempotent and a unit. It is well known that every endomorphism ring of a countably generated vector space over a division ring is a clean ring (cf. [7, Theorem]). A ring R is said to be unit regular in case for every $x \in R$, there exists a unit $u \in R$ such that $x = xux$. Answering a question of Nicholson, Camillo and Yu [3, Theorem 5] claimed that every unit regular ring is clean. But there was a gap in their proof. In [2, Theorem 1], Camillo and Khurana proved this result by a new route and gave a characterization of unit regular rings.

In this paper, a natural problem asked whether there is a nonclean ring R while some element of R is a sum of an idempotent and a unit. So as to deal with such rings, we will introduce a notion of clean ideals. We show that clean ideals have similar properties to clean rings.

Throughout, all rings are associative with identity and all modules are right modules. We always use $U(R)$ to denote the set of units of R.

Definition 1.1. An ideal I of a unital ring R is clean in case every element in I is a sum of an idempotent and a unit of R.

Clearly, every ideal of clean rings is clean. But there exist nonclean rings which contain some clean ideals. Let R_1 be a clean ring and R_2 not a clean ring. Set $R = R_1 \oplus R_2$. Then R is not a clean ring. Set $I = R_1 \oplus 0$. Given any $(x,0) \in I$, we have an idempotent $e \in R_1$ and a unit $u \in R_1$ such that $x = e+u$ because R_1 is clean. Hence, $(x,0) = (e,1)+(u,-1)$. Clearly, $(e,1) = (e,1)^2 \in R$, and $(u,-1) \in R$ is invertible. Therefore, we conclude that I is a clean ideal of R. Hence, the notion of clean ideals is a nontrivial generalization of clean rings.

Recall that an ideal I of a unital ring R is an exchange ideal provided that for any $x \in I$, there exists an idempotent $e \in I$ such that $e - x \in R(x - x^2)$.

Lemma 1.2. *Every clean ideal of a unital ring is an exchange ideal.*

Proof. Given any $x \in I$, we have an idempotent e and a unit $u \in U(R)$ such that $x = e+u$. Clearly, $u(x-u^{-1}(1-e)u) = x^2-x$. Set $f = u^{-1}(1-e)u$. Then we have $f = f^2 \in R$. Furthermore, $f = x-u^{-1}(x^2-x) = (1-u^{-1}(1-x))x \in Rx$; hence, $f = f^2 \in RxRx \subseteq Ix$. In addition, $1-f = 1-x+u^{-1}(x^2-x) \in R(1-x)$. Therefore, I is an exchange ideal of R by [1, Lemma 2.1]. □

Theorem 1.3. *Let R be a unital ring and I an ideal in which every idempotent is central. Then the following are equivalent:*

(1) *I is a clean ideal,*

(2) *I is an exchange ideal.*

Proof. (1)⇒(2) is clear by Lemma 1.2.

(2)⇒(1). Given any $x \in I$, we have an idempotent $e \in R$ such that $e \in Rx$ and $1-e \in R(1-x)$. Assume that $e = ax$ and $1-e = b(1-x)$. In addition, we may assume that $ea = a$ and $(1-e)b = b$. Hence, $axa = ea = a$, so $ax, xa \in I$ are central idempotents. Thus, we have $ax = axax = (ax)(xa) = x(ax)a = xa$. Likewise, $b(1-x)b = b$, so that $b(1-x)$ and $(1-x)b$, and hence $1-b(1-x)$ and $1-(1-x)b$, are idempotents. But $1-b(1-x) = e \in I$ and $1-(1-x)b = e-bx+xb \in I$; hence, $1-b(1-x)$ and $1-(1-x)b$ are central. Thus, $b(1-x)$ and $(1-x)b$ are central; and arguing as for ax and xa above, we get $b(1-x) = (1-x)b$. One easily checks that $(a-b)(x-(1-e)) = (x-(1-e))(a-b) = 1$. Set $u = x-(1-e)$. Then $x = (1-e)+u$ with $1-e = (1-e)^2 \in R$ and $u \in U(R)$, as required. □

Corollary 1.4. *Every exchange ideal without nonzero nilpotent elements is a clean ideal.*

Proof. Let I be an exchange ideal of a ring R. Assume that every nilpotent element in I is zero. Let $e \in I$ be an idempotent and $x \in R$. We see that $ex(1-e) \in I$ is a nilpotent element; hence, $ex = exe$. Likewise, $xe = exe$. Therefore, every idempotent in I is central and we complete the proof by Theorem 1.3. □

Recall that an ideal I of a ring R is left quasi-duo provided that for any maximal left ideal J of R, $JI \subseteq J$. From Theorem 1.3, it follows that every left quasi-duo exchange ideal of a ring is clean. An ideal I of a unital ring R is π-regular provided that for any $x \in I$, there exists a positive integer $n(x)$ such that $x^{n(x)} = x^{n(x)}yx^{n(x)}$ for a $y \in I$. It follows by Corollary 1.4 that every π-regular ideal without nonzero nilpotent elements is a clean ideal. Recall that a ring R is regular in case for any $x \in R$, there exists a $y \in R$ such that $x = xyx$. Let I be an ideal of a regular ring R. We say that I has stable range one if $aR+bR = R$ with $a \in 1+I$, $b \in R$ implies that $a+by \in U(R)$ for a $y \in R$. It is well known that an ideal I of a regular ring R has stable range one if and only if eRe is unit regular for any idempotent $e \in I$. Let V_1 and V_2 be countably infinitely and finitely generated vector spaces over a division ring D, respectively. Then $\mathrm{End}_D(V_1) \oplus \mathrm{End}_D(V_2)$ is not unit regular, while $0 \oplus \mathrm{End}_D(V_2)$

is its ideal having stable range one. We know that every unit regular ring is clean (cf. [2, Theorem 1]). Now, we extend this result to ideals of regular rings having stable range one.

Lemma 1.5. *Let R be a regular ring and I an ideal of R. Then the following are equivalent:*
(1) *I has stable range one,*
(2) *eRe is unit regular for all idempotents $e \in I$,*
(3) *$I + \mathrm{C}(R)$ is unit regular, where $\mathrm{C}(R)$ is the center of R.*

Proof. See [6, Lemma 1.4]. □

Theorem 1.6. *Let R be a regular ring and I an ideal of R. If I has stable range one, then I is a clean ideal of R.*

Proof. Let $\mathrm{C}(R)$ be the center of R and let $\overline{R} = I + \mathrm{C}(R)$. In view of Lemma 1.5, $\overline{R}$ is unit regular. By [2, Theorem 1], we know that $\overline{R}$ is a clean ring. Given any $x \in I$, we have $x \in \overline{R}$; hence, there exist an idempotent $e \in \overline{R}$ and a unit $u \in \overline{R}$ such that $x = e + u$. We easily check that $e = e^2 \in R$ and $u \in U(R)$. Therefore, I is a clean ideal of R. □

Corollary 1.7. *Let R be a regular, right self-injective ring and let $I = \{x \in R \mid xR$ is directly finite$\}$. Then I is a clean ideal of R.*

Proof. By [4, Corollary 9.21], I is an ideal of R. Given any idempotent $e \in I$, we have that eR is directly finite; hence, $eRe \cong \mathrm{End}_R(eR)$ is a directly finite ring. Using [4, Theorem 9.17], eRe is unit regular. It follows by Lemma 1.5 that I has stable range one. Therefore, we complete the proof by Theorem 1.6. □

Let R be a regular, right self-injective ring and xR directly finite. Then x is a sum of an idempotent and a unit by Corollary 1.7. Recall that an ideal I of a ring R is of bounded index if there is a positive integer n such that $x^n = 0$ for any nilpotent x of I.

Corollary 1.8. *Let I be an ideal of a regular ring R. If I is of bounded index, then I is a clean ideal of R.*

Proof. Given any idempotent $e \in I$, we have $eRe \subseteq I$. Assume now that the nilpotent bounded index of I is n. If $(ere)^m = 0$ in eRe, then we have $(ere)^n = 0$. Hence, eRe is a regular ring of bounded index. According to [4, Corollary 7.11], eRe is unit regular. It follows by Lemma 1.5 that I has stable range one. Using Theorem 1.6, we get that I is a clean ideal of R. □

It is well known that every exchange ring of bounded index is a clean ring. We ask a natural question: let I be an ideal of an exchange ring R, if I is of bounded index, is it a clean ideal of R? In [5], Han and Nicholson proved that every matrix ring over a clean ring is a clean ring. We extend this result to clean ideals of a ring.

Theorem 1.9. *Let I be a clean ideal of a unital ring R. Then $\mathrm{M}_n(I)$ is a clean ideal of $\mathrm{M}_n(R)$.*

Proof. Clearly, the result holds for $n=1$. Assume now that the result holds for $n=k-1$ $(k\geq 2)$. Suppose that $A\in \mathrm{M}_k(I)$, write $A=\left(\begin{smallmatrix} a & q\\ p & B\end{smallmatrix}\right)$, where $a\in I$, $B\in \mathrm{M}_{k-1}(I)$. Since I is a clean ideal of R, we have $e=e^2\in R$ and $u\in U(R)$ such that $a=e+u$. Since $B-pu^{-1}q\in \mathrm{M}_{k-1}(I)$, there exist $F=F^2\in \mathrm{M}_{k-1}(R)$ and $V\in \mathrm{GL}_{k-1}(R)$ such that $B-pu^{-1}q=F+V$. Set

$$E=\begin{pmatrix} e & 0\\ 0 & F\end{pmatrix},\qquad U=\begin{pmatrix} u & q\\ p & V+pu^{-1}q\end{pmatrix}. \tag{1.1}$$

It is easy to verify that $E=E^2$ and

$$\begin{aligned} U&\begin{pmatrix} u^{-1}+u^{-1}qV^{-1}pu^{-1} & -u^{-1}qV^{-1}\\ -V^{-1}pu^{-1} & V^{-1}\end{pmatrix}\\ &=\begin{pmatrix} u^{-1}+u^{-1}qV^{-1}pu^{-1} & -u^{-1}qV^{-1}\\ -V^{-1}pu^{-1} & V^{-1}\end{pmatrix}U \\ &=\begin{pmatrix} 1 & 0\\ 0 & I_{k-1}\end{pmatrix}\quad \text{in } \mathrm{M}_k(R).\end{aligned} \tag{1.2}$$

Hence, $U\in \mathrm{GL}_k(R)$. Clearly, $A=E+U$. Therefore, $\mathrm{M}_k(I)$ is a clean ideal of $\mathrm{M}_n(R)$. By induction, we complete the proof. □

Corollary 1.10. *Let R be a unital ring and I an exchange ideal in which every idempotent is central. Then $\mathrm{M}_n(I)$ is a clean ideal of $\mathrm{M}_n(R)$.*

Proof. The proof is clear by Theorems 1.3 and 1.9. □

A Morita context denoted by (A,B,M,N,ψ,ϕ) consists of two rings A, B, two bimodules ${}_AN_B$, ${}_BM_A$, and a pair of bimodule homomorphisms (called pairings) $\psi: N\otimes_B M\to A$ and $\phi: M\otimes_A N\to B$ which satisfy the following associativity: $\psi(n\otimes m)n'=n\phi(m\otimes n')$, $\phi(m\otimes n)m'=m\psi(n\otimes m')$, for any $m,m'\in M$, $n,n'\in N$. These conditions insure that the set T of generalized matrices $\left(\begin{smallmatrix} a & n\\ m & b\end{smallmatrix}\right)$, $a\in A$, $b\in B$, $m\in M$, and $n\in N$, forms a ring, called the ring of the context. Haghany studied hopficity and co-hopficity for Morita contexts with zero pairings. Now, we investigate clean Morita contexts with zero pairings.

Lemma 1.11. *Let T be the ring of a Morita context (A,B,M,N,ψ,ϕ) with zero pairings. If I and J are clean ideals of A and B, respectively, then $\left(\begin{smallmatrix} I & N\\ M & J\end{smallmatrix}\right)$ is a clean ideal of T.*

Proof. Since T is the ring of a Morita context (A,B,M,N,ψ,ϕ) with zero pairings, we check that $\left(\begin{smallmatrix} I & N\\ M & J\end{smallmatrix}\right)$ is an ideal of T. Let $A=\left(\begin{smallmatrix} a & n\\ m & b\end{smallmatrix}\right)\in\left(\begin{smallmatrix} I & N\\ M & J\end{smallmatrix}\right)$, where

$a \in I$, $b \in J$, $m \in M$, and $n \in N$. As I is a clean ideal of A, we have $e = e^2 \in A$ and $u \in U(A)$ such that $a = e + u$. Inasmuch as $b \in J$, there exist $f = f^2 \in B$ and $v \in U(B)$ such that $b = f + v$. Set

$$E = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}, \qquad U = \begin{pmatrix} u & n \\ m & v \end{pmatrix}. \tag{1.3}$$

It is easy to verify that $E = E^2 \in T$ and

$$\begin{aligned} U\begin{pmatrix} u^{-1} & -u^{-1}nv^{-1} \\ -v^{-1}mu^{-1} & v^{-1} \end{pmatrix} &= \begin{pmatrix} u^{-1} & -u^{-1}nv^{-1} \\ -v^{-1}mu^{-1} & v^{-1} \end{pmatrix} U \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in } T. \end{aligned} \tag{1.4}$$

Hence, $U \in U(T)$. Obviously, we have $A = E + U$. Therefore, we get the result. □

Let A_1, A_2, and A_3 be associative rings with identities and M_{21}, M_{31}, and M_{32} be (A_2, A_1)-, (A_3, A_1)-, and (A_3, A_2)-bimodules, respectively. Let $\phi : M_{32} \otimes_{A_2} M_{21} \to M_{31}$ be an (A_3, A_1)-homomorphism and let $T = \left(\begin{smallmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{smallmatrix}\right)$ with usual matrix operations.

Theorem 1.12. *The following are equivalent:*

(1) *I, J, and K are clean ideals of A_1, A_2, and A_3, respectively,*

(2) *the formal triangular matrix ideal $\left(\begin{smallmatrix} I & 0 & 0 \\ M_{21} & J & 0 \\ M_{31} & M_{32} & K \end{smallmatrix}\right)$ is a clean ideal of $\left(\begin{smallmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{smallmatrix}\right)$.*

Proof. (1)⇒(2). Clearly, $\left(\begin{smallmatrix} I & 0 & 0 \\ M_{21} & J & 0 \\ M_{31} & M_{32} & K \end{smallmatrix}\right)$ is an ideal of $\left(\begin{smallmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{smallmatrix}\right)$. Let $B = \left(\begin{smallmatrix} A_2 & 0 \\ M_{32} & A_3 \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} M_{21} \\ M_{31} \end{smallmatrix}\right)$. Since J and K are clean ideals of A_2 and A_3, respectively, by Lemma 1.11, we see that $\left(\begin{smallmatrix} J & 0 \\ M_{32} & K \end{smallmatrix}\right)$ is a clean ideal of B. By Lemma 1.11 again, $\left(\begin{smallmatrix} I & 0 & 0 \\ M_{21} & J & 0 \\ M_{31} & M_{32} & K \end{smallmatrix}\right)$ is a clean ideal of $\left(\begin{smallmatrix} A_1 & 0 \\ M & B \end{smallmatrix}\right)$, as required.

(2)⇒(1). Inasmuch as $\left(\begin{smallmatrix} I & 0 & 0 \\ M_{21} & J & 0 \\ M_{31} & M_{32} & K \end{smallmatrix}\right)$ is an ideal of $\left(\begin{smallmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{smallmatrix}\right)$, we show that I, J, and K are ideals of A_1, A_2, and A_3, respectively. For any $x \in J$, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} I & 0 & 0 \\ M_{21} & J & 0 \\ M_{31} & M_{32} & K \end{pmatrix}. \tag{1.5}$$

Thus, we have idempotent $\begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} \in T$ and a unit $\begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix} \in T$ such that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} + \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix}. \tag{1.6}$$

Clearly, $e_2 = e_2^2$ and $u_2 \in U(A_2)$. Furthermore, we have $x = e_2 + u_2$. Therefore, J is a clean ideal of A_2. Likewise, we claim that I and K are clean ideals of A_1 and A_3, respectively. □

Corollary 1.13. *Let R be a unital ring and I an ideal of R. Then the following are equivalent:*

(1) *I is a clean ideal of R,*

(2) *triangular matrix ideal*

$$\begin{pmatrix} I & 0 & \cdots & 0 \\ R & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & I \end{pmatrix}_{n\times n} \tag{1.7}$$

is a clean ideal of the ring of all $n\times n$ lower triangular matrices over R.

Proof. The proof is an immediate consequence of Theorem 1.12 and the induction. □

Analogously, we can derive the following proposition.

Proposition 1.14. *An ideal I of a ring R is clean if and only if the ideal of all $n\times n$ lower triangular matrices over I is a clean ideal of the ring of all $n\times n$ lower triangular matrices over R.*

Similarly, we deduce that an ideal I of a ring R is clean if and only if the ideal of all $n\times n$ upper triangular matrices over I is a clean ideal of the ring of all $n\times n$ upper triangular matrices over R. We say that an ideal I of a ring R is strongly π-regular if for any $x \in I$, there exists a positive integer $n(x)$ such that $x^{n(x)} = x^{n(x)+1}y$ for a $y \in I$. It is proved that an ideal I of a ring R is strongly π-regular if and only if there exists a positive integer $n(x)$ such that $x^{n(x)} = x^{n(x)+1}y$ and $xy = yx$, for a $y \in I$. So every strongly π-regular ideal of a ring is left-right symmetric. Hence all strongly π-regular ideals of a ring are clean ideals. Thus, we see that the ideal of all $n\times n$ upper (lower) triangular matrices over a strongly π-regular ideal of a ring is a clean ideal of the ring of all $n\times n$ lower triangular matrices over R.

A finite orthogonal set of idempotents $e_1,\dots,e_n$ in a ring R is said to be complete in case $e_1+\cdots+e_n = 1 \in R$. Using the method in Theorem 1.9, we now observe the following fact.

Proposition 1.15. *Let R be a unital ring and I an ideal of R. Then the following are equivalent:*

(1) *I is a clean ideal of R,*

(2) *there exists a complete set $\{e_1,\dots,e_n\}$ of idempotents such that e_iIe_i is a clean ideal of e_iRe_i for all i.*

Proof. (1)⇒(2) is clear by choosing $n=1$.

(2)⇒(1). Suppose that $\{e_1,\dots,e_n\}$ is a complete set of idempotents such that e_iIe_i is a clean ideal of e_iRe_i for all i. It suffices to show that the result holds for $n=2$. Clearly, $I \cong \left(\begin{smallmatrix} e_1Ie_1 & e_1Ie_2 \\ e_2Ie_1 & e_2Ie_2 \end{smallmatrix}\right)$ and $R \cong \left(\begin{smallmatrix} e_1Re_1 & e_1Re_2 \\ e_2Re_1 & e_2Re_2 \end{smallmatrix}\right)$. Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} e_1Ie_1 & e_1Ie_2 \\ e_2Ie_1 & e_2Ie_2 \end{smallmatrix}\right)$. As e_1Ie_1 is a clean ideal of e_1Re_1, we have $e=e^2 \in e_1Re_1$ and $u \in U(e_1Re_1)$ such that $a_{11}=e+u$. Inasmuch as $a_{22}-a_{21}u^{-1}a_{12} \in e_2Re_2$, there exist $f=f^2 \in e_2Re_2$ and $v \in U(e_2Re_2)$ such that $a_{22}-a_{21}u^{-1}a_{12}=f+v$. Set

$$E=\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}, \qquad U=\begin{pmatrix} u & a_{12} \\ a_{21} & v+a_{21}u^{-1}a_{12} \end{pmatrix}. \tag{1.8}$$

It is easy to verify that $E=E^2$ and

$$\begin{aligned} &U\begin{pmatrix} u^{-1}+u^{-1}a_{12}v^{-1}a_{21}u^{-1} & -u^{-1}a_{12}v^{-1} \\ -v^{-1}a_{21}u^{-1} & v^{-1} \end{pmatrix} \\ &\quad=\begin{pmatrix} u^{-1}+u^{-1}a_{21}v^{-1}a_{12}u^{-1} & -u^{-1}a_{21}v^{-1} \\ -v^{-1}a_{12}u^{-1} & v^{-1} \end{pmatrix}U \\ &\quad=\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}. \end{aligned} \tag{1.9}$$

Hence, U is invertible. Clearly, $A=E+U$, so we get the result. □

Acknowledgments. The authors are grateful to the referee for useful suggestions which led to the new version of Theorem 1.3. These helped us to improve the presentation considerably. This work was supported by the Natural Science Foundation (NSF) of Zhejiang Province.

References

[1] P. Ara, *Extensions of exchange rings*, J. Algebra **197** (1997), no. 2, 409–423.

[2] V. P. Camillo and D. Khurana, *A characterization of unit regular rings*, Comm. Algebra **29** (2001), no. 5, 2293–2295.

[3] V. P. Camillo and H.-P. Yu, *Exchange rings, units and idempotents*, Comm. Algebra **22** (1994), no. 12, 4737–4749.

[4] K. R. Goodearl, *Von Neumann Regular Rings*, Monographs and Studies in Mathematics, vol. 4, Pitman, Massachusetts, 1979.

[5] J. Han and W. K. Nicholson, *Extensions of clean rings*, Comm. Algebra **29** (2001), no. 6, 2589–2595 .

[6] P. Menal and J. Moncasi, *Lifting units in self-injective rings and an index theory for Rickart C^*-algebras*, Pacific J. Math. **126** (1987), no. 2, 295–329.

[7] W. K. Nicholson and K. Varadarajan, *Countable linear transformations are clean*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 61–64.

Huanyin Chen: Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

E-mail address: chyzxl@sparc2.hunnu.edu.cn

Miaosen Chen: Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

E-mail address: miaosen@mail.jhptt.zj.cn