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ON FINITELY EQUIVALENT CONTINUA

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For positive integers m and n, relations between (hereditary) m- and n-equivalence are studied, mostly for arc-like continua. Several structural and mapping problems concerning (hereditarily) finitely equivalent continua are formulated.

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A *continuum* means a compact connected metric space. For a positive integer n, a continuum X is said to be *n-equivalent* provided that X contains exactly n topologically distinct subcontinua. A continuum X is said to be *hereditarily n-equivalent* provided that each nondegenerate subcontinuum of X is n-equivalent. If there exists a positive integer n such that X is n-equivalent, then X is said to be *finitely equivalent*. Thus, for $n = 1$, the concepts of "1-equivalent" and "hereditarily 1-equivalent" coincide, and they mean the same as "hereditarily equivalent" in the sense considered, for example, by Cook in [2].

Observe the following statement.

Statement 1. Each subcontinuum of an n-equivalent continuum is m-equivalent for some $m \leq n$. Thus, each finitely equivalent continuum is hereditarily finitely equivalent.

Some structural results concerning finitely equivalent continua are obtained by Nadler Jr. and Pierce in [9]. They have shown that if a continuum X is (a) semi-locally connected at each of its noncut points, then it is finitely equivalent if and only if it is a graph; (b) aposyndetic at each of its noncut points and finitely equivalent, then it is a graph. Furthermore, in both cases (a) and (b), if X is n-equivalent, then each subcontinuum of X is a θ_{n+1}-continuum. Recall that Nadler Jr. and Pierce in [9, page 209] posed the following problem.

Problem 2. Determine which graphs, or at least how many, are n-equivalent for each n.

The arc and the pseudo-arc are the only known 1-equivalent continua. In [10] Whyburn has shown that each planar 1-equivalent continuum is tree-like, and planarity assumption has been deleted after 40 years by Cook [2] who proved tree-likeness of any 1-equivalent continuum. But it is still not known whether or not the arc and the pseudo-arc are the only ones among 1-equivalent continua.

In contrast to 1-equivalent case, 2-equivalent continua need not be hereditarily 2-equivalent, a simple closed curve is 2-equivalent while not hereditarily

2-equivalent. The 2-equivalent continua were studied by Mahavier in [5] who proved that if a 2-equivalent continuum contains an arc, then it is a simple triod, a simple closed curve or irreducible, and that the only locally connected 2-equivalent continua are a simple triod and a simple closed curve. It is also shown that if X is a decomposable, not locally connected, 2-equivalent continuum containing an arc, then X is arc-like and it is the closure of a topological ray R such that the remainder $\mathrm{cl}(R)\setminus R$ is an end continuum of X. Furthermore, two examples of 2-equivalent continua are presented in [5]: the first, [5, Example 1, page 246], is a decomposable continuum X which is the closure of a ray R such that the remainder $\mathrm{cl}(R)\setminus R$ is homeomorphic to X; the second, [5, Example 2, page 247], is an arc-like hereditarily decomposable continuum containing no arc.

Looking for an example of a hereditarily 2-equivalent continuum note that the former example surely is not hereditarily 2-equivalent because it contains an arc. We analyze the latter one.

The continuum M constructed in [5, Example 2, page 247] does not contain any arc, and it contains a continuum N such that each subcontinuum of M is homeomorphic to M or to N, see [5, the paragraph following Lemma 3, page 249]. Further, by its construction, N does contain continua homeomorphic to M (see [5, the final part of the proof, page 251]). Therefore, the following statement is established.

Theorem 3. *The continuum M constructed in* [5, Example 2, page 247] *has the following properties:*
(a) *M is an arc-like;*
(b) *M is hereditarily decomposable;*
(c) *M does not contain any arc;*
(d) *M is hereditarily 2-equivalent.*

In connection with the above theorem, the following problem can be posed.

Problem 4. Determine for what integers $n \geq 3$, there exists a continuum M satisfying conditions (a), (b), and (c) of Theorem 3 and being hereditarily n-equivalent.

The following results are consequences of [1, Theorem, page 35].

Theorem 5. *For each hereditarily n-equivalent continuum X, that does not contain any arc, there exists an $(n+2)$-equivalent continuum Y such that each of its subcontinua is homomorphic either to a subcontinuum of X or to Y, or to an arc.*

Proof. Indeed, a compactification Y of a ray R having the continuum X as the remainder, that is, such that $X = \mathrm{cl}(R)\setminus R$ is such a continuum. □

Since if M is arc-like and hereditarily decomposable, then so is any of compactifications Y of a ray having the continuum X as the remainder, we get the next result as a consequence of Theorem 5.

Corollary 6. *If a continuum M satisfies conditions (a), (b), and (c) of Theorem 3 and is hereditarily n-equivalent, then any of compactifications of a ray having the continuum M as the remainder satisfies conditions (a) and (b) of Theorem 3 and is $(n+2)$-equivalent.*

In [7], an uncountable family $\mathcal{F}$ is constructed of compactifications of the ray with the remainder being the pseudo-arc.

Statement 7. Each member X of the (uncountable) family $\mathcal{F}$ constructed in [7] is an arc-like 3-equivalent continuum. Any subcontinuum of X is homeomorphic to an arc, to a pseudo-arc, or to the whole X.

A continuum X has the *RNT-property* (retractable onto near trees) provided that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if a tree T is δ-near to X with respect to the Hausdorff distance, then there is an ε-retraction of X onto T, see [6, Definition 0]. It is shown in [6, Theorem 5] that if a continuum X is a compactification of the ray R and X has the RNT-property, then the remainder $\mathrm{cl}(R) \setminus R \subset X = \mathrm{cl}(R)$ is the pseudo-arc. Therefore, Theorem 5 implies the following proposition.

Proposition 8. *Each compactification X of the ray having the RNT-property is a 3-equivalent continuum. Each subcontinuum of X is homeomorphic to an arc, a pseudo-arc, or to the whole X.*

Observe that M of Theorem 3 being an arc-like is hereditarily unicoherent, and being hereditarily decomposable, it is a λ-dendroid (containing no arc). Another (perhaps the first) example of a λ-dendroid, in fact, an arc-like, containing no arc, has been constructed by Janiszewski in 1912, [3] but his description was rather intuitive than precise. It would be interesting to investigate if that old example of Janiszewski is or is not n-equivalent (hereditarily n-equivalent) for some n.

The following problems can be considered as a program of a study in the area rather than particular questions.

Problems 9. For each positive integer n, characterize continua which are (a) n-equivalent; (b) hereditarily n-equivalent.

Problem 10. Characterize continua which are finitely equivalent.

Sometimes a characterization of a class of spaces (or of spaces having a certain property) can be expressed in terms of containing some particular spaces. A classical illustration of this is a well-known characterization of nonplanar graphs by containing the two Kuratowski's graphs: K_5 and $K_{3,3}$, see, for example, [8, Theorem 9.36, page 159]. To be more precise, recall the following concept. Let $\mathcal{A}$ be a class of spaces and let $\mathcal{P}$ be a property. Then $\mathcal{P}$ is said to be *finite (or countable) in the class $\mathcal{A}$* provided that there is a finite (or countable, respectively) set $\mathcal{S}$ of members of $\mathcal{A}$ such that a member X has the property $\mathcal{P}$ if and only if X contains a homeomorphic copy of some member of $\mathcal{S}$. The result of [7] mentioned above in Statement 7 shows that this is not the way of characterizing 3-equivalent continua. Namely, the existence of the family $\mathcal{F}$ shows the following theorem.

Theorem 11. *The property of being* 3*-equivalent is neither finite nor countable in the class of (a) all continua; (b) arc-like continua.*

A mapping $f : X \to Y$ between continua X and Y is said to be
- (i) *atomic* provided that for each subcontinuum K of X, either $f(K)$ is degenerate or $f^{-1}(f(K)) = K$;
- (ii) *monotone* provided that the inverse image of each subcontinuum of Y is connected;
- (iii) *hereditarily monotone* provided that for each subcontinuum K of X, the partial mapping $f|K : K \to f(K)$ is monotone.

It is known that each atomic mapping is hereditarily monotone, see, for example, [4, (4.14), page 17]. Since each arcwise connected 2-equivalent continuum is either a simple closed curve or a simple triod, see [5, Theorem 2, page 244], each semilocally connected 3-equivalent continuum is either a simple 4-od [8, Definition 9.8, page 143] (i.e., a letter X) or a letter H, see [9, page 209]. And since these continua are preserved under atomic mappings (as it is easy to see), we conclude that atomic mappings preserve the property of being 2-equivalent and being 3-equivalent for locally connected continua. However, this is not an interesting result, because each atomic mapping of an arcwise connected continuum onto a nondegenerate continuum is a homeomorphism, see [4, (6.3), page 51]. But the result cannot be extended to hereditarily monotone mappings, because a mapping that shrinks one arm of a simple triod to a point is hereditarily monotone and not atomic, and it maps a 2-equivalent continuum onto an arc that is 1-equivalent. On the other hand, if X is the 2-equivalent continuum which is the closure of a ray R as described in [5, Example 1, page 246], then the mapping $f : X \to [0,1]$, that shrinks the remainder $\mathrm{cl}(R) \setminus R$ to a point (and is a homeomorphism on R), is atomic and it maps 2-equivalent continuum X onto the 1-equivalent continuum $[0,1]$. Therefore, atomic mappings do not preserve the property of being a 2-equivalent continuum. In connection with these examples, the following question can be asked.

Question 12. Let a continuum X be n-equivalent and let a mapping $f : X \to Y$ be an atomic surjection. Must then Y be m-equivalent for some $m \le n$?

In general, we can pose the following problems.

Problems 13. What kinds of mappings between continua preserve the property of being: (a) n-equivalent? (b) hereditarily n-equivalent? (c) finitely equivalent?

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