PARA- f -LIE GROUPS

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Special para- f -structures on Lie groups are studied. It is shown that every para- f -Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup.

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1. Para- f -structures. The notion of a para- f -structure on a differentiable manifold was introduced and studied in [2].

Let M be an n -dimensional differentiable manifold of class C^∞ . The set of all vector fields on M will be denoted by $\chi(M)$ and the tangent space of M at a point $m \in M$ by $T_m M$.

DEFINITION 1.1. Let M be an n -dimensional differentiable manifold. If φ is an endomorphism field of constant rank k on M satisfying

$$\varphi^3 - \varphi = 0, \quad (1.1)$$

then φ is called a *para- f -structure* on M and M is a para- f -manifold.

DEFINITION 1.2. A para- f -structure φ on M is *integrable* if there exists a coordinate system in which φ has constant components

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (1.2)$$

where I is the unit matrix and $p + q = k$.

PROPOSITION 1.3. A para- f -structure φ on M is integrable if and only if its Nijenhuis tensor field N_φ vanishes, that is,

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] = 0, \quad (1.3)$$

where $X, Y \in \chi(M)$.

For a para- f -structure φ on M , let

$$\begin{aligned}\ker \varphi &= \bigcup_{m \in M} (\ker \varphi)_m, \\ \operatorname{im} \varphi &= \bigcup_{m \in M} (\operatorname{im} \varphi)_m\end{aligned}\tag{1.4}$$

be the kernel and image of φ , respectively, where

$$\begin{aligned}(\ker \varphi)_m &= \{X \in T_m M; \varphi_m(X) = 0\}, \\ (\operatorname{im} \varphi)_m &= \{Y \in T_m M; Y = \varphi_m(X) \text{ for some } X \in T_m M\}\end{aligned}\tag{1.5}$$

are the kernel and image of φ at any point $m \in M$, respectively.

PROPOSITION 1.4. *If $(\ker \varphi)_m = \{0\}$ for a para- f -structure φ for all $m \in M$, then φ is an almost product structure on M , that is, $\varphi^2 = \operatorname{Id}$.*

PROPOSITION 1.5. *If $(\operatorname{im} \varphi)_m = \{0\}$ for a para- f -structure φ for all $m \in M$, then φ is the trivial para- f -structure on M , that is, $\varphi = 0$.*

PROPOSITION 1.6. *If φ is a para- f -structure on M , then*

$$\ker \varphi \cap \operatorname{im} \varphi = \{0\}.\tag{1.6}$$

PROOF. If $Z \in \ker \varphi \cap \operatorname{im} \varphi$, then $\varphi(Z) = 0$, and there exists X such that $\varphi(X) = Z$. Hence $\varphi^2(X) = 0$, and from [Definition 1.1](#), we get $0 = \varphi^3(X) = \varphi(X) = Z$. $\square$

DEFINITION 1.7. Let φ_i be a para- f -structure on a para- f -manifold M_i with $i = 1, 2$. A diffeomorphism $h : M_1 \rightarrow M_2$ is called a *para- f -map* if

$$\varphi_2 \circ h_* = h_* \circ \varphi_1,\tag{1.7}$$

where h_* is the differential of h .

2. Para- f -Lie groups. In this section, the notion of a para- f -Lie group is introduced. Some properties of its Lie algebra are established. Finally, its special decomposition in terms of an almost product Lie group and a Lie group with trivial para- f -structure is proved.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. As usual, we define

$$\begin{aligned}L_g : G &\rightarrow G \quad (\text{left multiplication by } g \in G), \\ R_g : G &\rightarrow G \quad (\text{right multiplication by } g \in G), \\ \operatorname{ad}_g : G &\rightarrow G, \quad a \mapsto \operatorname{ad}_g(a) = gag^{-1}, \\ \operatorname{Ad}_X : \mathfrak{g} &\rightarrow \mathfrak{g}, \quad Y \mapsto \operatorname{Ad}_X(Y) = [X, Y].\end{aligned}\tag{2.1}$$

DEFINITION 2.1. Let G be a Lie group with a para- f -structure φ . If both L_g and R_g are para- f -maps, then φ is said to be *bi-invariant*.

DEFINITION 2.2. If G is a Lie group with an integrable bi-invariant para- f -structure φ , then G is called a *para- f -Lie group*.

PROPOSITION 2.3. If φ is a bi-invariant para- f -structure on a Lie group G , then

$$\varphi[X, Y] = [\varphi(X), Y] \quad (2.2)$$

for all $X, Y \in \mathfrak{g}$.

PROOF. Since $\varphi \circ (L_g)_* = (L_g)_* \circ \varphi$ and $\varphi \circ (R_g)_* = (R_g)_* \circ \varphi$, we have $\varphi \circ (\text{ad}_g)_* = (\text{ad}_g)_* \circ \varphi$ for all $g \in G$. If $g = \exp(tX)$, where $t \in \mathbb{R}$, then $\varphi \circ (\text{ad}_{\exp(tX)})_* = (\text{ad}_{\exp(tX)})_* \circ \varphi$. Hence, by a standard result in Lie groups,

$$\varphi \circ e^{\text{Ad}_{tX}} = e^{\text{Ad}_{tX}} \circ \varphi, \quad (2.3)$$

or, for any $Y \in \mathfrak{g}$,

$$\begin{aligned} \varphi\left(Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \cdots\right) \\ = \varphi(Y) + t[X, \varphi(Y)] + \frac{t^2}{2!}[X, [X, \varphi(Y)]] + \cdots. \end{aligned} \quad (2.4)$$

Hence,

$$\varphi[X, Y] + \frac{t}{2!}[X, [X, Y]] + \cdots = [X, \varphi(Y)] + \frac{t}{2!}[X, [X, \varphi(Y)]] + \cdots. \quad (2.5)$$

Letting $t \rightarrow 0$ in (2.5) gives us the desired result. $\square$

PROPOSITION 2.4. A bi-invariant para- f -structure φ on a Lie group G is integrable.

PROOF. From Proposition 2.3, the Nijenhuis tensor of a bi-invariant para- f -structure φ must vanish at the unity e of G . $\square$

COROLLARY 2.5. A Lie group G with a bi-invariant para- f -structure φ is a para- f -Lie group.

EXAMPLE 2.6. Let $G = \text{GL}(n, \mathbb{R})$ be the group of all real nonsingular $n \times n$ matrices. Let $\varphi : G \rightarrow G$, $X \mapsto \varphi(X) = X - (1/n)\text{trace}(X)I$, where I is the unit matrix. Then φ is a bi-invariant para- f -structure on G .

PROPOSITION 2.7. Let G be a para- f -Lie group with a para- f -structure φ . Then its Lie algebra $\mathfrak{g}$ is expressed as

$$\mathfrak{g} = V_k \oplus V_i, \quad (2.6)$$

the direct sum (as a Lie algebra), where $V_k = (\ker \varphi)_e$ and $V_i = (\text{im } \varphi)_e$ are subalgebras of $\mathfrak{g}$, and $e \in G$ is the unity of G .

PROOF. From Proposition 1.6, $V_k \cap V_i = \{0\}$. Therefore, $\mathfrak{g}$ is the direct sum (as a vector space) of V_k and V_i . It is clear, from Proposition 2.3, that both V_k and V_i are Lie subalgebras of $\mathfrak{g}$. Furthermore, if $X = \varphi(Z) \in V_i$ and $Y \in V_k$, then, again applying Proposition 2.3, $[X, Y] = \varphi[Z, Y] = [Z, \varphi(Y)] = 0$. Hence, $\mathfrak{g} = V_k \oplus V_i$ as a Lie algebra. $\square$

THEOREM 2.8. *Every para- f -Lie group G is the quotient of the product of an almost product Lie group and a Lie group with trivial para- f -structure by a discrete subgroup.*

PROOF. Let V_k and V_i be subalgebras (defined in Proposition 2.7) of the Lie algebra $\mathfrak{g}$ of a para- f -Lie group G . From Proposition 2.7, $\mathfrak{g}$ is the Lie algebra direct sum of V_k and V_i . Using Propositions 1.4 and 1.5, we obtain the theorem from [4]. $\square$

REMARK 2.9. Since a para- f -structure with parallelizable kernel [2] is an almost r -paracontact structure [1], some examples of almost r -paracontact structures are used in [3] to illustrate para- f -Lie groups.

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