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ON THE MAPPING $xy \to (xy)^n$ IN AN ASSOCIATIVE RING

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We consider the following condition (*) on an associative ring R : (*). There exists a function f from R into R such that f is a group homomorphism of $(R,+)$, f is injective on R^2, and $f(xy) = (xy)^{n(x,y)}$ for some positive integer $n(x,y) > 1$. Commutativity and structure are established for Artinian rings R satisfying (*), and a counterexample is given for non-Artinian rings. The results generalize commutativity theorems found elsewhere. The case $n(x,y) = 2$ is examined in detail.

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Let R be an associative ring, not necessarily with unity, and let R^+ denote the additive group of R. In [3], it was shown that R is commutative if it satisfies the following condition.

(I) For each x and y in R, there exists $n = n(x,y) > 1$ such that $(xy)^n = xy$.

We generalize this result by considering the condition below.

(II) There exists a function f from R into R such that f is a group homomorphism of R^+, f is injective on R^2, and $f(xy) = (xy)^{n(x,y)}$ for some positive integer $n = n(x,y) > 1$ depending on x and y.

An example of a ring satisfying (II) for $n(x,y) = 2$ is given by $R = B \oplus N$, where B is a Boolean ring and N is a zero ring (a ring with trivial product, $xy = 0$ for all x and y). In this case, we may take f to be the identity mapping. It was shown in [2] that a ring which is product-idempotent (i.e., $(xy)^2 = xy$ for every x and y) must be of the form $B \oplus N$. We will see that Artinian rings R for which (II) is true are not far removed from this structure.

In this paper, we give the structure of an Artinian ring R satisfying (II) without invoking the commutativity theorems of Bell [1]. We then exhibit an infinite noncommutative ring for which f is surjective but not injective. Throughout this paper, the notation $J(R)$ denotes the Jacobson radical of the ring R. If r is in R, the symbol $\bar{r}$ denotes the coset $r + J(R)$.

The proposition below states that rings satisfying (II) obey the central-idempotent property.

Proposition 1 (see [3]). *Let R be a ring satisfying (II). If e is an idempotent in R, then e is central.*

Proof. Since $f(yx) = (yx)^{n(y,x)} = y(xy)x\cdots yx$, we have that $xy = 0$ in R implies $yx = 0$, for any x and y in R. Now, for every r in R, $(e^2 - e)r = e(er - r) = 0$. Thus, $(er - r)e = 0$ or $ere = re$. Similarly, $ere = er$. Hence, $er = re$. □

Theorem 2. *Let R be an Artinian ring satisfying (II). If $(xy)^m = 0$ for some positive integer m, then $xy = 0$.*

Proof. Suppose that $(xy)^m = 0$ and $(xy)^{m-1} \neq 0$, $m > 1$. Then, $f[(xy)^{m-1}] = [(xy)^{m-1}]^n = 0$. Since f is injective on R^2, $(xy)^{m-1} = 0$, a contradiction. □

Corollary 3. *If R is an Artinian ring satisfying (II), then $R \cdot J(R) = J(R) \cdot R = (0)$.*

Proof. Since R is Artinian, the ideal $J(R)$ is nilpotent. □

Corollary 4. *For an Artinian ring R satisfying (II), $J(R)$ is a zero ring.*

Corollary 5. *For an Artinian ring R satisfying (II), $R/J(R)$ is commutative.*

Proof. If not, there is a direct summand of $R/J(R)$ isomorphic to a full matrix ring over a division ring. Hence, there exist $\bar{u}$ and $\bar{v}$ in $R/J(R)$ such that $\bar{u}\bar{v} \neq 0$ and $\bar{u}\bar{v}\bar{u} = 0$. It follows that $uv \neq 0$ in R and that uvu is in $J(R)$. But then $f(uv) = (uv)^{n(u,v)} = uv \cdot uv \cdots uv = (uvu)v \cdots uv = 0$. Thus, by the injective property of f on R^2, $uv = 0$, a contradiction.

We now obtain the structure of an Artinian ring R satisfying (II). □

Theorem 6. *If R is an Artinian ring satisfying (II), then R decomposes as a direct sum of rings $eR \oplus N$, where e is an idempotent in R and N is a zero ring.*

Proof. By Corollary 5, the ring $S = R/J(R)$ is a direct sum of fields; hence S has an identity $\bar{t}$, which lifts to a central idempotent e in R such that $e - t$ is in $J(R)$. Let $N = \{r - er : r \in R\}$. It is easy to see that N is an ideal of R, and that the intersection of N with eR is (0). Clearly, $R = eR + N$, and so we may write $R = eR \oplus N$. Now, $e - t$ in $J(R)$ implies that $(e-t)^2 = 0$ or $e = 2et - t^2$. Hence, if r is in R, $(2\bar{e} \cdot \bar{t} - \bar{t}^2)\bar{r} = \bar{e} \cdot \bar{r} = \overline{er}$ or $2\bar{e} \cdot \bar{t} \cdot \bar{r} - \bar{t}^2 \cdot \bar{r} = 2\bar{e} \cdot \bar{r} - \bar{r} = \overline{er}$, since $\bar{t}$ is the identity of S. Thus, $\overline{er} - \bar{r} = 0$ or $r - er$ is in $J(R)$. Therefore, N is a zero subring of $J(R)$. □

Corollary 7. *If R is an Artinian ring satisfying (II), then R is a direct sum $F \oplus N$, where F is a direct sum of fields and N is a zero ring.*

Proof. By Theorem 2, the ring eR in Theorem 6 has no nonzero nilpotent elements, and hence is a direct sum of fields by Corollary 5. □

Corollary 8. *Let R be as in Theorem 2. Then R is commutative.*

Corollary 9. *Let R be as in Theorem 2. Then $J(R)$ consists precisely of the nilpotent elements $\{x : x^2 = 0\}$.*

Remark 10. The function f maps the ideal eR of Theorem 6 into itself, since $f(ex) = (ex)^n = e^n x^n = ex^n$.

Remark 11. The specific fields in the direct sum F of Corollary 7 depend, of course, on the integers $n(x,y)$. A Boolean ring is acceptable for any value of n. The prime field with p elements, p a prime, is acceptable for $n = (p-1)m + 1$, m a positive integer. A finite field of order p^k is acceptable for $n = p$. Of course, an infinite field of characteristic p need not be a pth root field.

We now exhibit an infinite noncommutative ring R for which $f(xy) = (xy)^2$ on R^2.

Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Let R be the free $\mathbb{Z}_4$-module with countable base $A = \{a_i : i = 1,2,3,\ldots\}$. On A, define the multiplication $a_1a_2 = a_3$, $a_2a_1 = -a_3$, $a_ia_j = 0$ otherwise. One may verify that this yields an associative multiplication which extends to a ring multiplication on R considered as an abelian group. Clearly, the ring R is noncommutative. Define $f : A \to A \cup \{0\}$ via $f(a_1) = f(a_3) = 0$ and $f(a_i) = a_{\rho(i)}$, $i \neq 1,3$, where ρ is any bijection of $\{2,4,5,\ldots\}$ onto the set of positive integers. The map f extends to a group homomorphism of R^+. Now, $f(a_ia_j) = f(0) = 0 = (a_ia_j)^2$ for $(i,j) \neq (1,2)$ or $(2,1)$. Moreover, $f(a_1a_2) = f(a_3) = 0 = (a_1a_2)^2 = a_3^2$. Similarly, $f(a_2a_1) = 0 = (a_2a_1)^2$. It is then easy to check that $f(xy) = (xy)^2$ for every x and y in R, since $a_ia_ja_k = 0$ for all a_i, a_j, a_k in A.

The function f above is not injective. We prove the following theorem which insures the commutativity of any ring S, given injectivity of f on the subring S^2 alone.

Theorem 12. *Let f be a function from a ring S into S such that $f(x+y) = f(x) + f(y)$ and $f(xy) = (xy)^2$. Assume further that f is injective on s^2. Then S is commutative.*

Proof. Let x, y, z, and t be arbitrary elements of S. Now, $f(2xy) = 2(xy)^2 = (2xy)^2 = 4(xy)^2$, so $2(xy)^2 = f(2xy) = 0$. Hence, $2xy = 0$ by injectivity. Moreover, if $xy = 0$, then $f(yx) = y(xy)x = 0$ implies $yx = 0$. From $(xy)^2 + (zy)^2 = f(xy) + f(zy) = f((x+z)y) = [(x+z)y]^2 = (xy+zy)^2 = (xy)^2 + xyzy + zyxy + (zy)^2$, we obtain $xyzy = zyxy$. Now, $f(xtyz + yzxt) = f(xtyz) + f(yzxt) = xtyz \cdot xtyz + yzxt \cdot yzxt = (xt)y(zxt)yz + yzxt \cdot yzxt = xtyzy(zxt) + yzxt \cdot yzxt$. Hence, $xtyz(xtyz + yzxt) = 0$. Thus, $(xtyz + yzxt)xtyz = xtyz \cdot xtyz + yzxt \cdot xtyz = xtyz \cdot xtyz + yz \cdot x(t)x(tyz) = xtyz \cdot xtyz + yzx(tyz)xt = f(xtyz + yzxt) = 0$. Therefore, $xtyz + yzxt = 0$ or $(xt)(yz) = (yz)(xt)$. Hence, S^2 is commutative.

Now, $f(xyz) = (xyz)(xyz) = x(yzx)(yz) = x(yz)^2x$. Similarly, $f(yzx) = x(yz)^2x$. So, $xyz = yzx$.

Finally, $f(xy) = (xy)(xy) = x(yxy) = x^2y^2 = y^2x^2 = (yx)(yx) = f(yx)$. Thus, $xy = yx$, and S is commutative. This completes the proof. □

Remark 13. The ring R in the example preceding Theorem 12 does not have a unity. It can be shown that if S is any ring in which every element is a square, and squaring is an endomorphism of S^+, then S is commutative. It follows that a ring R satisfying (II) for $n = 2$ and having a right or left identity is commutative.

In view of Remark 13 and Theorem 12, we make the following conjecture and leave it as a problem.

Conjecture 14. Let S be a ring and $n \geq 2$ a positive integer. If the function $f(x) = x^n$ on S is surjective (injective) and f is a group endomorphism of S^+, then S is commutative.

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