IJMMS 2004:51, 2755–2760
PII. S0161171204402373
http://ijmms.hindawi.com

CLASSIFICATION OF WEAK CONTINUITIES AND DECOMPOSITION OF CONTINUITY

JINGCHENG TONG

Received 24 February 2004

We first introduce 20 weak forms of continuity, which are closely related to 5 known weak forms of continuity. Then we classify them into 9 groups and give 12 decompositions of continuity.

2000 Mathematics Subject Classification: 54C05, 54C08.

1. Introduction. Continuity is one of the most important concepts in mathematics. In order to find deep properties of continuity, many weak forms of continuity were introduced in the literature. For instance, we have Levine's weak continuity [2] and semicontinuity [3], M. K. Singal and A. R. Singal's almost continuity [6], Husain's almost continuity [1], α-continuity of Mashhour et al. [4], and many others. Each of the above forms of continuity is strictly weaker than continuity. Theoretically, for each weak form of continuity, there is another weak form of continuity such that both of them imply continuity. In this connection, there is one result [3] for the general case. A special case is discussed in [7]. In this note, we develop these results. We introduce 20 weak forms of continuity, which are closely related with the above-mentioned weak continuities. Then we classify them into 9 groups and give 12 decompositions of continuity.

2. Preliminaries. We recall some known definitions.

Definition 2.1 [3]**.** A subset S in a topological space X is said to be semiopen if there is an open set O in X such that $O \subset S \subset \mathrm{cl}\, O$, where $\mathrm{cl}\, O$ denotes the closure of O.

Definition 2.2 [3]**.** A mapping $f : X \to Y$ is said to be semicontinuous if for each open set V in Y, $f^{-1}(V)$ is a semiopen set in X.

Definition 2.3 [2]**.** A mapping $f : X \to Y$ is said to be weakly continuous if for each $x \in X$ and each open set V in Y containing $f(x)$, there is an open set U in X containing x such that $f(U) \subset \mathrm{cl}\, V$.

Definition 2.4 [5]**.** A subset S in a topological space is said to be an α-set if $S \subset \mathrm{int}\,\mathrm{cl}\,\mathrm{int}\, S$.

Definition 2.5 [4]**.** A mapping $f : X \to Y$ is said to be α-continuous if for each open set V in Y, $f^{-1}(V)$ is an α-set in X.

There are two different definitions of almost continuous mappings, one is given by Husain [1]; the other one is given by M. K. Singal and A. R. Singal [6]. In this note, following Mashhour et al. [4], we use precontinuity for Husain's almost continuity, and use almost continuity particularly for M. K. Singal and A. R. Singal's.

Definition 2.6 [1]. A mapping $f : X \to Y$ is said to be precontinuous if for each $x \in X$ and each open set V in Y containing $f(x)$, $\operatorname{cl} f^{-1}(V)$ is a neighborhood of x.

Definition 2.7 [6]. A mapping $f : X \to Y$ is said to be almost continuous if for each $x \in X$ and each open set V in Y containing $f(x)$, there is an open set U in X such that $f(U) \subset \operatorname{int}\operatorname{cl} V$.

The relations of the above five weak forms of continuity are as follows [4]:

$$\begin{tikzcd}
& \text{semicontinuity} & \\
\text{continuity} \arrow[r] \arrow[d] & \alpha\text{-continuity} \arrow[r] \arrow[u] & \text{precontinuity} \\
\text{almost continuity} \arrow[r] & \text{weak continuity} &
\end{tikzcd} \tag{2.1}$$

3. Classification of some weak continuities.

The following results are known.

Lemma 3.1. *Let S be a subset in a topological space X. Then*

(i) $\operatorname{int}\operatorname{int} S = \operatorname{int} S$;
(ii) $\operatorname{cl}\operatorname{cl} S = \operatorname{cl} S$;
(iii) $\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} S = \operatorname{int}\operatorname{cl} S$;
(iv) $\operatorname{cl}\operatorname{int}\operatorname{cl}\operatorname{int} S = \operatorname{cl}\operatorname{int} S$.

Our classification is based on Lemma 3.1.

In the following group of weak continuities, (i) is trivial, (ii) and (iii) are given in [4].

Definition 3.2. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is continuous if and only if $f^{-1}(V) \subset \operatorname{int} f^{-1}(V)$;
(ii) f is precontinuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl} f^{-1}(V)$;
(iii) f is α-continuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl}\operatorname{int} f^{-1}(V)$.

It is known [2] that a mapping $f : X \to Y$ is weakly continuous if and only if $f^{-1}(V) \subset \operatorname{int} f^{-1}(\operatorname{cl} V)$. From this we have the following group of definitions.

Definition 3.3. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is weakly continuous if and only if $f^{-1}(V) \subset \operatorname{int} f^{-1}(\operatorname{cl} V)$;
(ii) f is pre-weakly continuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl} f^{-1}(\operatorname{cl} V)$;
(iii) f is α-weakly continuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl}\operatorname{int} f^{-1}(\operatorname{cl} V)$.

In the above definitions, (ii) and (iii) are new.

It is known [6] that a mapping $f: X \to Y$ is almost continuous if and only if $f^{-1}(V) \subset \operatorname{int} f^{-1}(\operatorname{int}\operatorname{cl} V)$ for each open set V in Y. From this we have the following group of definitions.

Definition 3.4. Let $f: X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is almost continuous if and only if $f^{-1}(V) \subset \operatorname{int} f^{-1}(\operatorname{int}\operatorname{cl} V)$;
(ii) f is pre-almost continuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl} f^{-1}(\operatorname{int}\operatorname{cl} V)$;
(iii) f is α-almost continuous if and only if $f^{-1}(V) \subset \operatorname{int}\operatorname{cl}\operatorname{int} f^{-1}(\operatorname{int}\operatorname{cl} V)$.

In the above definitions, (ii) and (iii) are new.

It is easily seen that a mapping $f: X \to Y$ is semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int} f^{-1}(V)$. From this we have the following group of definitions.

Definition 3.5. Let $f: X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int} f^{-1}(V)$;
(ii) f is weak semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int} f^{-1}(\operatorname{cl} V)$;
(iii) f is almost semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int} f^{-1}(\operatorname{int}\operatorname{cl} V)$.

In the above definitions, (ii) and (iii) are new. The following definitions are all new.

Definition 3.6. Let $f: X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is pre-semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int}\operatorname{cl} f^{-1}(V)$;
(ii) f is pre-weak-semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int}\operatorname{cl} f^{-1}(\operatorname{cl} V)$;
(iii) f is pre-almost-semicontinuous if and only if $f^{-1}(V) \subset \operatorname{cl}\operatorname{int}\operatorname{cl} f^{-1}(\operatorname{int}\operatorname{cl} V)$.

The following chart gives the relationships of all the weak forms of continuity in this section:

$$
\begin{tikzcd}
\text{weak continuity} \arrow[r] & \alpha\text{-weak continuity} \arrow[r] & \text{pre-weak continuity} \\
\text{almost continuity} \arrow[u] \arrow[r] & \alpha\text{-almost continuity} \arrow[u] \arrow[r] & \text{pre-almost continuity} \arrow[u] \\
\text{continuity} \arrow[u] \arrow[r] & \alpha\text{-continuity} \arrow[u] \arrow[r] \arrow[d] & \text{precontinuity} \arrow[u] \arrow[d] \\
 & \text{semicontinuity} \arrow[r] \arrow[d] & \text{pre-semicontinuity} \arrow[d] \\
 & \text{almost semicontinuity} \arrow[r] \arrow[d] & \text{pre almost-semicontinuity} \arrow[d] \\
 & \text{weak semicontinuity} \arrow[r] & \text{pre-weak-semicontinuity}
\end{tikzcd}
\tag{3.1}
$$

4. Classification of relative continuities. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then f is continuous if and only if $f^{-1}(V)$ is an open set in X. If we relax the requirement on $f^{-1}(V)$ from being open in X to being open in a subspace, then we can obtain many new weak forms of continuity. For instance, we have the following group of weak continuities corresponding to Definition 3.2.

Definition 4.1. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is continuous$^{\#}$ if and only if $f^{-1}(V)$ is an open set in the subspace $f^{-1}(V)$;
(ii) f is pre$^{\#}$-continuous if and only if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\, f^{-1}(V)$;
(iii) f is $\alpha^{\#}$-continuous if and only if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\, f^{-1}(V)$.

It is easily seen that any mapping is a continuous$^{\#}$ mapping. We have the following group of definitions corresponding to Definition 3.3.

Definition 4.2. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is weak$^{\#}$ continuous if and only if $f^{-1}(V)$ is an open set in the subspace $f^{-1}(\mathrm{cl}\, V)$;
(ii) f is pre-weak$^{\#}$ continuous if and only if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\, f^{-1}(\mathrm{cl}\, V)$;
(iii) f is α-weak$^{\#}$ continuous if and only if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\, f^{-1}(\mathrm{cl}\, V)$.

We have the following group of definitions corresponding to Definition 3.4.

Definition 4.3. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is almost$^{\#}$ continuous if $f^{-1}(V)$ is an open set in the subspace $f^{-1}(\mathrm{int}\,\mathrm{cl}\, V)$;
(ii) f is pre-almost$^{\#}$ continuous if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\, f^{-1}(\mathrm{int}\,\mathrm{cl}\, V)$;
(iii) f is α-almost$^{\#}$ continuous if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\, f^{-1}(\mathrm{int}\,\mathrm{cl}\, V)$.

Now we go to the last group of definitions.

Definition 4.4. Let $f : X \to Y$ be a mapping and let V be an arbitrary open set in Y. Then

(i) f is pre-semi$^{\#}$-continuous if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\, f^{-1}(V)$;
(ii) f is pre-weak-semi$^{\#}$- continuous if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\, f^{-1}(\mathrm{cl}\, V)$;
(iii) f is pre-almost-semi$^{\#}$-continuous if $f^{-1}(V)$ is an open set in the subspace $\mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\, f^{-1}(\mathrm{int}\,\mathrm{cl}\, V)$.

The following chart gives the relationships of all the weak continuities in this section:

$$
\begin{tikzcd}
\text{weak}^{\#}\text{ continuity} \arrow[d] & \alpha\text{-weak}^{\#}\text{ continuity} \arrow[l] \arrow[d] & \text{pre-weak}^{\#}\text{ continuity} \arrow[l] \arrow[d] \\
\text{almost}^{\#}\text{ continuity} & \alpha\text{-almost}^{\#}\text{ continuity} \arrow[l] \arrow[d] & \text{pre-almost}^{\#}\text{ continuity} \arrow[l] \arrow[d] \\
 & \alpha^{\#}\text{-continuity} & \text{pre}^{\#}\text{-continuity} \arrow[l] \\
 & & \text{pre-semi}^{\#}\text{-continuity} \arrow[u] \arrow[uull] \arrow[uu, bend right] \\
 & & \text{pre-almost-semi}^{\#}\text{-continuity} \arrow[u] \\
 & & \text{pre-weak-semi}^{\#}\text{-continuity} \arrow[u] \arrow[uuuuull, bend left] \arrow[uuuuu, bend right]
\end{tikzcd}
\tag{4.1}
$$

5. Decompositions of continuity. We need an important lemma.

Lemma 5.1. *Let $\alpha : 2^X \to 2^X$ be a mapping with $\alpha(A\cap B) \subset \alpha A \cap \alpha B$ and let $\beta : 2^X \to 2^X$ be another mapping with $V \subset \beta V$ for each open set V in X. Let $f : X \to Y$ be a mapping such that for each open set V in Y,*

(i) $f^{-1}(V) \subset \operatorname{int}\alpha f^{-1}(\beta V)$;
(ii) *there is an open set O in X such that $f^{-1}(V) = \alpha f^{-1}(\beta V)\cap O$.*

Then f is continuous.

Proof. Since $f^{-1}(V) = \alpha f^{-1}(\beta V)\cap O$, hence $f^{-1}(V) \subset O$. Therefore

$$
\begin{aligned}
\operatorname{int} f^{-1}(V) &= \operatorname{int}\alpha f^{-1}(\beta V)\cap \operatorname{int} O \\
&= \operatorname{int}\alpha f^{-1}(\beta V)\cap O \\
&\supset f^{-1}(V)\cap f^{-1}(V) \\
&= f^{-1}(V).
\end{aligned}
\tag{5.1}
$$

We have proved that $f^{-1}(V)$ is an open set, hence f is continuous. □

Now we turn to the decomposition of continuity. Because $\operatorname{int}(A\cap B) = \operatorname{int}A\cap\operatorname{int}B$ and $\operatorname{cl}(A\cap B) \subset \operatorname{cl}A\cap\operatorname{cl}B$, we know that $\operatorname{cl}\operatorname{int}(A\cap B)\subset \operatorname{cl}\operatorname{int}A\cap\operatorname{cl}\operatorname{int}B$. Therefore we have the following theorem.

Theorem 5.2. *Let $f : X \to Y$ be a mapping. Then f is continuous if and only if*

(i) *f is continuous and continuous$^{\#}$;*
(ii) *f is precontinuous and pre$^{\#}$-continuous;*
(iii) *f is α-continuous and $\alpha^{\#}$-continuous;*
(iv) *f is weakly continuous and weak$^{\#}$ continuous;*
(v) *f is pre-weakly continuous and pre-weak$^{\#}$ continuous;*
(vi) *f is α-weakly continuous and α-weak$^{\#}$ continuous;*

(vii) f *is almost continuous and almost*$^\#$ *continuous;*
(viii) f *is pre-almost continuous and pre-almost*$^\#$ *continuous;*
(ix) f *is* α*-almost continuous and* α*-almost*$^\#$ *continuous.*

In the above decompositions, (i) is trivial and the other eight are all new.

Since $\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} f^{-1}(\beta V) = \operatorname{int}\operatorname{cl} f^{-1}(\beta V)$ and $\operatorname{cl}\operatorname{int}\operatorname{cl}(A\cap B) = \operatorname{cl}\operatorname{int}\operatorname{cl} A \cap \operatorname{cl}\operatorname{int}\operatorname{cl} B$, we have the following decompositions.

Theorem 5.3. *Let* $f : X \to Y$ *be a mapping. Then* f *is continuous if and only if*
(x) f *is pre-continuous and pre-semi*$^\#$*-continuous;*
(xi) f *is pre-weakly continuous and pre-weak-semi*$^\#$*-continuous;*
(xii) f *is pre-almost continuous and pre-almost-semi*$^\#$*-continuous;*
(xiii) f *is pre-continuous and* $\alpha^\#$*-continuous.*

In the above twelve nontrivial decompositions, if we choose a proper operator β other than identity mapping, cl or int cl, we can have infinitely many decompositions. For instance, we may let $\beta A = A \cup E$, where E is a subset of X such that $A \cap E \neq \phi$.

References

[1] T. Husain, *Almost continuous mappings*, Prace Mat. **10** (1966), 1–7.
[2] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46.
[3] ______, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
[4] A. S. Mashhour, I. A. Hasanein, and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar. **41** (1983), no. 3-4, 213–218.
[5] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.
[6] M. K. Singal and A. R. Singal, *Almost-continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.
[7] B. D. Smith, *An alternate characterization of continuity*, Proc. Amer. Math. Soc. **39** (1973), 318–320.

Jingcheng Tong: Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

E-mail address: jtong@unf.edu