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ON THE BANACH ALGEBRA $\mathcal{B}(l_p(\alpha))$

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We give some properties of the Banach algebra of bounded operators $\mathcal{B}(l_p(\alpha))$ for $1 \leq p \leq \infty$, where $l_p(\alpha) = (1/\alpha)^{-1} * l_p$. Then we deal with the continued fractions and give some properties of the operator Δ^h for $h > 0$ or integer greater than or equal to one mapping $l_p(\alpha)$ into itself for $p \geq 1$ real. These results extend, among other things, those concerning the Banach algebra S_α and some results on the continued fractions.

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1. Introduction. In this paper, we are interested in the study of operators represented by infinite matrices. In [10, 15], necessary and sufficient conditions for an operator to map a set of sequences into another set of sequences are given. In this way, in [5, 6, 8, 11, 12], some properties of the set $F(\Delta^h)$ can be found, where h is an integer or a real, F is one of the sets c, c_0, s_α, or s_α°, and Δ is the well-known operator of first difference. Then some characterizations of operators mapping in these sets have been given. Böttcher and Silbermann [1] have put together some results on Toeplitz matrices and have studied some sets of Banach algebras of infinite matrices. In this paper, we are dealing with a particular class of Banach algebras of bounded operators mapping $l_p(\alpha)$ into itself and we are giving some results which extend the previous one.

This paper is organized as follows. In Section 2, we recall some results on the Banach algebra of bounded operators. In Section 3, we deal with the Banach algebra $\mathcal{B}(l_p(\alpha))$, where $1 \leq p \leq \infty$ and $\alpha = (\alpha_n)_n$, with $\alpha_n > 0$ for all n. In Section 4, necessary conditions for A to be invertible in the set $\mathcal{B}(l_p(\alpha))$ are given. In Section 5, we consider applications to infinite tridiagonal matrices and continued fractions. Finally in Section 6 we deal with some new properties of the operators Δ^t and Δ^h, for $h > 0$ integer or real, considered as operators from $l_p(\alpha)$ into itself.

2. Preliminary results. $A = (a_{nm})_{n,m\geq 1}$ being an infinite matrix, we will consider the sequence $X = (x_n)_{n\geq 1}$ as a *column vector* and define the product

$$AX = \begin{pmatrix} a_{11} & \dots & a_{1m} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nm} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}, \tag{2.1}$$

whenever the series $y_n = \sum_{m=1}^{\infty} a_{nm} x_m$ for $n \geq 1$ are convergent. We will denote by s the set of all sequences. For any given subsets E, F of s, we will say that an operator represented by the infinite matrix $A = (a_{nm})_{n,m\geq 1}$ *maps* E *into* F, that is, $A \in (E,F)$ (see [10]) if

(i) for each $n \geq 1$ and for all $X \in E$, *the series defined by* $y_n = \sum_{m=1}^{\infty} a_{nm} x_m$ *is convergent*;

(ii) $AX \in F$ for all $X \in E$.

For any subset E of s, we will write

$$AE = \{Y \in s : Y = AX \text{ for some } X \in E\}. \tag{2.2}$$

If F is a subset of s, we will denote

$$F(A) = F_A = \{X \in s : Y = AX \in F\}. \tag{2.3}$$

Let E be a Banach space with norm $\| \|_E$. We will say that a linear operator $A : E \mapsto E$ *is bounded if*

$$\sup_{X \neq 0} \left(\frac{\|AX\|_E}{\|X\|_E} \right) < \infty. \tag{2.4}$$

It is well known [1] that the *set* $\mathcal{B}(E)$ *of all bounded operators mapping E into itself is a Banach algebra* and we will write

$$\|A\|_{\mathcal{B}(E)} = \sup_{X \neq 0} \left(\frac{\|AX\|_E}{\|X\|_E} \right) < \infty. \tag{2.5}$$

A Banach space E of complex sequences with the norm $\| \cdot \|_E$ *is a BK space if each projection* $P_n : X \mapsto P_n X = x_n$ *is continuous. A BK space* E *is said to have AK if every sequence* $X = (x_n)_{n=1}^{\infty} \in E$ *has a unique representation* $X = \sum_{n=1}^{\infty} x_n e_n$, where e_n is the sequence with 1 in the nth position and 0 otherwise. It is well known [9] that if E has AK, then $\mathcal{B}(E) = (E,E)$.

3. Banach algebra $\mathcal{B}(l_p(\alpha))$ for $1 \leq p \leq \infty$

3.1. Case $1 \leq p < \infty$. Recall that the set

$$l_p = \left\{ X = (x_n)_{n\geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} \tag{3.1}$$

is a Banach space normed by

$$\|X\|_{l_p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}. \tag{3.2}$$

Put now

$$U^{+*} = \{X = (x_n)_{n\geq 1} \in s : x_n > 0 \ \forall n\}. \tag{3.3}$$

Using Wilansky's notations [15], we have, for any given $\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}$ and $p \geq 1$ real,

$$l_p(\alpha) = \left(\frac{1}{\alpha}\right)^{-1} * l_p = \left\{X \in s : \sum_{n=1}^{\infty}\left(\frac{|x_n|}{\alpha_n}\right)^p < \infty\right\}. \tag{3.4}$$

If we define by D_α the diagonal matrix $D_\alpha = (\alpha_n \delta_{nm})_{n,m\geq 1}$ (where $\delta_{nm} = 0$ for all $n \neq m$ and $\delta_{nm} = 1$ otherwise), we see from (2.2) that

$$D_\alpha l_p = l_p(\alpha). \tag{3.5}$$

It is easy to see that $l_p(\alpha)$ is a Banach space with the norm

$$\|X\|_{l_p(\alpha)} = \left\|D_{1/\alpha}X\right\|_{l_p} = \left[\sum_{n=1}^{\infty}\left(\frac{|x_n|}{\alpha_n}\right)^p\right]^{1/p}. \tag{3.6}$$

If $\alpha = (\alpha_n)_{n\geq 1}$, $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$, the condition $\alpha_n = \beta_n O(1)$ $(n \to \infty)$ implies that

$$l_p(\alpha) \subset l_p(\beta). \tag{3.7}$$

We also have $l_p(\alpha) \subset l_{p'}(\alpha)$ for $1 \leq p \leq p'$.

Since $l_p(\alpha)$ has AK, we have $\mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha))$ (see [9]) so $A \in \mathcal{B}(l_p(\alpha))$ if and only if $A \in (l_p(\alpha), l_p(\alpha))$ and

$$\|A\|_{\mathcal{B}(l_p(\alpha))} = \sup_{X\neq 0}\left(\frac{\|AX\|_{l_p(\alpha)}}{\|X\|_{l_p(\alpha)}}\right) < \infty. \tag{3.8}$$

The set $\mathcal{B}(l_p(\alpha))$ is a Banach algebra with identity; see [1]. So we get

$$\|AX\|_{l_p(\alpha)} \leq \|A\|_{\mathcal{B}(l_p(\alpha))}\|X\|_{l_p(\alpha)} \quad \forall X \in l_p(\alpha). \tag{3.9}$$

We have $l_p = l_p(e)$, where $e = (1,\dots,1,\dots)$ and for all $A \in \mathcal{B}(l_p(\alpha))$,

$$\left\|D_{1/\alpha}AD_\alpha\right\|_{\mathcal{B}(l_p)} = \|A\|_{\mathcal{B}(l_p(\alpha))}. \tag{3.10}$$

Indeed, writing $D_\alpha X = Y$, we get

$$\sup_{X\neq 0}\left(\frac{\left\|(D_{1/\alpha}AD_\alpha)X\right\|_{l_p}}{\|X\|_{l_p}}\right) = \sup_{Y\neq 0}\left(\frac{\left\|D_{1/\alpha}AY\right\|_{l_p}}{\left\|D_{1/\alpha}Y\right\|_{l_p}}\right) = \|A\|_{\mathcal{B}(l_p(\alpha))}. \tag{3.11}$$

So we can write that

$$A \in \mathcal{B}(l_p(\alpha)) \Longleftrightarrow D_{1/\alpha}AD_\alpha \in \mathcal{B}(l_p). \tag{3.12}$$

When $\alpha = (r^n)_{n\geq 1}$, for a given real $r > 0$, $l_p(\alpha)$ is denoted by $l_p(r)$. When $p = \infty$, we obtain the next results.

3.2. Case $p=\infty$. The Banach algebra S_α. Let $\alpha=(\alpha_n)_{n\geq 1}\in U^{+*}$. Using Wilansky's notation, we have

$$l_\infty(\alpha)=\left(\frac{1}{\alpha}\right)^{-1}*l_\infty=D_\alpha l_\infty. \tag{3.13}$$

We will write $s_\alpha=l_\infty(\alpha)=\{X\in s: x_n/\alpha_n=O(1)\}$; see [2, 3, 4, 5, 6, 7]. The set s_α is a Banach space with the norm

$$\|X\|_{s_\alpha}=\sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right). \tag{3.14}$$

The set

$$S_\alpha=\left\{A=(a_{nm})_{n,m\geq 1}:\sup_{n\geq 1}\left(\sum_{m=1}^{\infty}|a_{nm}|\frac{\alpha_m}{\alpha_n}\right)<\infty\right\} \tag{3.15}$$

is *Banach algebra with identity* normed by

$$\|A\|_{S_\alpha}=\sup_{n\geq 1}\left(\sum_{m=1}^{\infty}|a_{nm}|\frac{\alpha_m}{\alpha_n}\right). \tag{3.16}$$

Putting $B(s_\alpha)=\mathcal{B}(s_\alpha)\bigcap(s_\alpha,s_\alpha)$, we can state a first elementary result.

Lemma 3.1. *Let $\alpha\in U^{+*}$. Then*

$$B(s_\alpha)=S_\alpha=(s_\alpha,s_\alpha). \tag{3.17}$$

Proof. First, for any given infinite matrix A, the condition $A\in S_\alpha$ is equivalent to $D_{1/\alpha}AD_\alpha\in S_e$ and since $S_e=(l_\infty,l_\infty)$ (see [8, 10]) we conclude that $S_\alpha=(s_\alpha,s_\alpha)$. Now we show that $B(s_\alpha)=S_\alpha$. It can be easily seen that $B(s_\alpha)\subset(s_\alpha,s_\alpha)$. Conversely, assume that $A\in S_\alpha$. Then for every $X\in s_\alpha$,

$$\|AX\|_{s_\alpha}=\sup_{n\geq 1}\left(\frac{\left|\sum_{m=1}^{\infty}a_{nm}x_m\right|}{\alpha_n}\right)\leq\|A\|_{S_\alpha}\|X\|_{s_\alpha}. \tag{3.18}$$

So A is bounded and belongs to $B(s_\alpha)$. $\square$

As we have seen above when $\alpha=(r^n)_{n\geq 1}$, $r>0$, S_α and s_α are denoted by S_r and s_r. Note that for $r=1$ we get $s_1=l_\infty$.

4. Inverse of an infinite matrix in $\mathcal{B}(l_p(\alpha))$. In this section, we are interested in the case when an operator $A\in(l_p(\alpha),l_p(\alpha))$ is bijective. For this, we need to explicitly show the set $\mathcal{B}(l_p(\alpha))$.

We put

$$N_{p,\alpha}(A)=\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\left(|a_{nm}|\frac{\alpha_m}{\alpha_n}\right)^q\right)^{p-1}\right]^{1/p}, \tag{4.1}$$

for $1<p<\infty$ and $q=p/(p-1)$. In order to state the next results recall the following lemma (see [10]).

Lemma 4.1. *$A \in (l_1, l_1)$ if and only if $A^t \in S_1$.*

We get the following results.

Theorem 4.2. *Let $\alpha \in U^{+*}$.*

(i) (a) *Then*

$$\mathcal{B}(l_1(\alpha)) = \{A = (a_{nm})_{n,m\geq 1} : A^t \in S_{1/\alpha}\}, \qquad (l_\infty(\alpha), l_\infty(\alpha)) \subset S_\alpha. \tag{4.2}$$

(b) *If*

$$\widehat{\mathcal{B}}_p(\alpha) = \{A = (a_{nm})_{n,m\geq 1} : N_{p,\alpha}(A) < \infty\} \text{ for } 1 < p < \infty, \tag{4.3}$$

then

$$\widehat{\mathcal{B}}_p(\alpha) \subset \mathcal{B}(l_p(\alpha)). \tag{4.4}$$

(ii) (a) *For every $A \in \mathcal{B}(l_1(\alpha))$,*

$$\|A\|_{\mathcal{B}(l_1(\alpha))} \leq \|A^t\|_{S_{1/\alpha}} = \sup_{m\geq 1}\left(\sum_{n=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\alpha_n}\right); \tag{4.5}$$

(b) *when $1 < p < \infty$, for every $A \in \mathcal{B}(l_p(\alpha))$,*

$$\|A\|_{\mathcal{B}(l_p(\alpha))} \leq N_{p,\alpha}(A); \tag{4.6}$$

(c) *for every $A \in \mathcal{B}(l_\infty(\alpha))$,*

$$\|A\|_{\mathcal{B}(l_\infty(\alpha))} \leq \|A\|_{S_\alpha} = \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\alpha_n}\right). \tag{4.7}$$

(iii) *The identity*

$$A(A'X) = (AA')X, \quad \forall X \in l_p(\alpha), \tag{4.8}$$

holds in the following cases:

(a) *$A, A' \in \mathcal{B}(l_1(\alpha))$ when $p = 1$;*
(b) *$A, A' \in \widehat{\mathcal{B}}_p(\alpha)$ when $1 < p < \infty$;*
(c) *$A, A' \in S_\alpha$ when $p = \infty$.*

Proof. (i)(a) Since $l_1(\alpha)$ has AK, then $\mathcal{B}(l_1(\alpha)) = (l_1(\alpha), l_1(\alpha))$. So $A \in \mathcal{B}(l_1(\alpha))$ if and only if $D_{1/\alpha} A D_\alpha \in (l_1, l_1)$. From the characterization of (l_1, l_1), the condition $D_{1/\alpha} A D_\alpha \in (l_1, l_1)$ is equivalent to $(D_{1/\alpha} A D_\alpha)^t = D_\alpha A^t D_{1/\alpha} \in S_1$, that is, $A^t \in S_{1/\alpha}$, and we have shown that $\mathcal{B}(l_1(\alpha)) = \widehat{\mathcal{B}}(\alpha)$. The identity $(l_\infty(\alpha), l_\infty(\alpha)) = S_\alpha$ comes from Lemma 3.1.

(i)(b) Take any $X \in l_p$. We have

$$\begin{aligned}\|AX\|_{l_p}^p &= \left\|\left(\sum_{m=1}^{\infty} a_{nm}x_m\right)_{n\geq 1}\right\|_{l_p}^p = \sum_{n=1}^{\infty}\left|\sum_{m=1}^{\infty} a_{nm}x_m\right|^p \\ &\leq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}x_m|\right)^p,\end{aligned} \tag{4.9}$$

and from Hölder's inequality, we get, for every n,

$$\begin{aligned}\sum_{m=1}^{\infty}|a_{nm}x_m| &\leq \left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{1/q}\left(\sum_{m=1}^{\infty}|x_m|^p\right)^{1/p} \\ &= \left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{1/q}\|X\|_{l_p},\end{aligned} \tag{4.10}$$

with $q = p/(p-1)$. We deduce

$$\|AX\|_{l_p}^p \leq \sum_{n=1}^{\infty}\left[\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{1/q}\|X\|_{l_p}\right]^p = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{p/q}\|X\|_{l_p}^p; \tag{4.11}$$

and since $p/q = p-1$, we have

$$\|AX\|_{l_p} \leq \left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{p-1}\right]^{1/p}\|X\|_{l_p}, \tag{4.12}$$

$$\|A\|_{\mathcal{B}(l_p)} = \sup_{X\neq 0}\left(\frac{\|AX\|_{l_p}}{\|X\|_{l_p}}\right) \leq \left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{p-1}\right]^{1/p}. \tag{4.13}$$

We have proved that if $A \in \widehat{\mathcal{B}}_p(e)$, then $A \in \mathcal{B}(l_p)$. So if $A \in \widehat{\mathcal{B}}_p(\alpha)$, then $D_{1/\alpha}AD_\alpha \in \widehat{\mathcal{B}}_p(e)$, so from the equivalence given by (3.12), $D_{1/\alpha}AD_\alpha \in \mathcal{B}(l_p)$ and A belongs to $\mathcal{B}(l_p(\alpha))$. This concludes the proof.

(ii)(a) Now take $A \in \widehat{\mathcal{B}}(\alpha)$. For any fixed $X \in l_1(\alpha)$ and for any integers N_1 and $N_2 \geq 1$, we have

$$\begin{aligned}\sum_{n=1}^{N_1}\left(\left|\sum_{m=1}^{N_2} a_{nm}x_m\right|\frac{1}{\alpha_n}\right) &\leq \sum_{n=1}^{N_1}\left(\sum_{m=1}^{N_2}|a_{nm}x_m|\frac{1}{\alpha_n}\right) \\ &\leq \sum_{m=1}^{N_2}\left(\sum_{n=1}^{N_1}|a_{nm}|\frac{\alpha_m}{\alpha_n}\frac{|x_m|}{\alpha_m}\right) \\ &\leq \left(\sum_{m=1}^{N_2}\frac{|x_m|}{\alpha_m}\right)\sup_{m\geq 1}\left(\sum_{n=1}^{N_1}|a_{nm}|\frac{\alpha_m}{\alpha_n}\right) \\ &\leq \|X\|_{l_1(\alpha)}\left\|D_\alpha A^t D_{1/\alpha}\right\|_{S_1}.\end{aligned} \tag{4.14}$$

Letting N_1 and $N_2 \to \infty$, we obtain

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}|a_{nm}x_m|\frac{1}{\alpha_n}\right) < \infty. \tag{4.15}$$

Then the series $\sum_{n=1}^{\infty}((1/\alpha_n)|\sum_{m=1}^{\infty}a_{nm}x_m|)$ is also convergent and

$$\sum_{n=1}^{\infty}\left(\left|\sum_{m=1}^{\infty}a_{nm}x_m\right|\frac{1}{\alpha_n}\right) = \|AX\|_{l_1(\alpha)} \le \|A^t\|_{S_{1/\alpha}}\|X\|_{l_1(\alpha)}. \tag{4.16}$$

So A is bounded and

$$\|A\|_{\mathcal{B}(l_1(\alpha))} \le \|A^t\|_{S_{1/\alpha}} \tag{4.17}$$

is a direct consequence of (i).

(ii)(b) comes from inequality (4.13) and equivalence (3.12).

(ii)(c) comes from the preliminary results.

(iii) is an immediate consequence of the fact that $\mathcal{B}(l_1(\alpha))$, $\mathcal{B}(l_p(\alpha))$, and S_α are Banach algebras of operators represented by infinite matrices and $\widehat{\mathcal{B}}_p(\alpha) \subset \mathcal{B}(l_p(\alpha))$. □

We deduce the following corollary, in which we put $a = (a_{nn})_{n\ge1}$ and $|a| = (|a_{nn}|)_{n\ge1}$.

Corollary 4.3. *The operator represented by A is bijective from $l_p(\alpha)$ into $l_p(\alpha|a|)$ in each of the following cases:*

(i) *for $p = 1$,*

$$\|I - A^tD_{1/a}\|_{S_{1/\alpha}} = \sup_{m\ge1}\left(\sum_{\substack{n=1\\ n\ne m}}^{\infty}\left|\frac{a_{nm}}{a_{nn}}\right|\frac{\alpha_m}{\alpha_n}\right) < 1; \tag{4.18}$$

(ii) *for $1 < p < \infty$,*

$$N_{p,\alpha}^{p}(I - D_{1/a}A) = \sum_{n=1}^{\infty}\left(\sum_{\substack{m=1\\ m\ne n}}^{\infty}\left(\left|\frac{a_{nm}}{a_{nn}}\right|\frac{\alpha_m}{\alpha_n}\right)^q\right)^{p-1} < 1; \tag{4.19}$$

(iii) *for $p = \infty$,*

$$\|I - D_{1/a}A\|_{S_\alpha} = \sup_{n\ge1}\left(\sum_{\substack{m=1\\ m\ne n}}^{\infty}\left|\frac{a_{nm}}{a_{nn}}\right|\frac{\alpha_m}{\alpha_n}\right) < 1. \tag{4.20}$$

Proof. (i) First, from Theorem 4.2(ii)(a), we have

$$\|I - D_{1/a}A\|_{\mathcal{B}(l_1(\alpha))} \le \|I - A^tD_{1/a}\|_{S_{1/\alpha}} < 1. \tag{4.21}$$

For any given B such that $D_{1/a}B \in l_1(\alpha)$, consider the solutions of the equation $AX = B$ belonging to $l_1(\alpha)$. The previous equation is equivalent to

$$(D_{1/a}A)X = D_{1/a}B. \tag{4.22}$$

Since $D_{1/a}A$ is invertible in $\mathcal{B}(l_1(\alpha))$, we deduce from Theorem 4.2(iii)(a) that, for every $X \in l_1(\alpha)$,

$$(D_{1/a}A)^{-1}[(D_{1/a}A)X] = X = (D_{1/a}A)^{-1}D_{1/a}B. \tag{4.23}$$

We conclude that the equation $AX = B$ admits a unique solution in $l_1(\alpha)$ since $D_{1/a}B \in l_1(\alpha)$ and $(D_{1/a}A)^{-1} \in \mathcal{B}(l_1(\alpha))$.

(ii) Since $p > 0$, we have $N_{p,\alpha}^{p}(I - D_{1/a}A) < 1$ if and only if $N_{p,\alpha}(I - D_{1/a}A) < 1$ and we conclude reasoning as in (i).

(iii) can also be obtained reasoning as in (i). □

Remark 4.4. Let $r > 0$. If $\alpha = (r^n)_n$, it is obvious that for $1 < p < \infty$, the condition

$$N_{p,r}^{p}(I - D_{1/a}A) = \sum_{n=1}^{\infty}\left(\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left(\left|\frac{a_{nm}}{a_{nn}}\right| r^{(m-n)}\right)^{q}\right)^{p-1} < 1 \tag{4.24}$$

implies that the operator A is bijective from $l_p(r)$ into $l_p((r^n|a_{nn}|)_{n\geq 1})$.

We deduce the following application.

Proposition 4.5. *Let $A = (a_{nm})_{n,m\geq 1}$ be an infinite matrix. Consider $\tau > 0$, $0 < \rho < 1$, and $1 < p < \infty$, satisfying the inequality*

$$\frac{\rho^p}{(1-\rho^q)^{p-1}} < \tau p - 1, \quad \textit{with } q = \frac{p-1}{p}, \tag{4.25}$$

and assume that

$$\begin{aligned} |a_{nm}| &\leq \frac{1}{n^{\tau}} && \textit{for } 1 \leq m < n-1;\\ a_{nn} &= 1, && \textit{for } n \geq 1;\\ a_{nm} &= 0, && \textit{otherwise.} \end{aligned} \tag{4.26}$$

Then A is bijective from $l_p(1/\rho)$ into itself.

Proof. We have

$$\sigma_n = \sum_{\substack{m=1\\ m\neq n}}^{\infty} |a_{nm}|^q \left(\frac{1}{\rho}\right)^{(m-n)q} \leq \frac{1}{n^{\tau q}} \sum_{m=1}^{n-1}\left(\frac{1}{\rho}\right)^{(m-n)q} \leq \frac{1}{n^{\tau q}}\frac{\rho^q}{1-\rho^q}. \tag{4.27}$$

Then

$$\sum_{n=1}^{\infty}\sigma_n^{p-1} \leq \frac{\rho^p}{(1-\rho^q)^{p-1}}\sum_{n=2}^{\infty}\frac{1}{n^{\tau p}}, \tag{4.28}$$

since $p-1=p/q$ and $\tau p-1>0$ in (4.25). Now from the inequality

$$\sum_{n=2}^{\infty}\frac{1}{n^{\tau p}}\le\int_1^{\infty}\frac{dx}{x^{\tau p}}=\frac{1}{\tau p-1}, \tag{4.29}$$

we conclude using (4.25) that

$$[N_{p,1/\rho}(I-A)]^p=\sum_{n=1}^{\infty}\sigma_n^{p-1}\le\frac{\rho^p}{(1-\rho^q)^{p-1}}\frac{1}{\tau p-1}<1. \tag{4.30}$$

So A is invertible in $\mathcal{B}(l_p(1/\rho))$ and A is bijective from $l_p(1/\rho)$ to itself. □

Remark 4.6. If we put $\rho=1/2$, $p=q=2$, and $\tau>2/3$, then (4.25) holds and A is bijective from $l_2(2)$ to itself.

5. Application to the infinite tridiagonal systems.

In this section, we deal with operators represented by infinite tridiagonal matrix A. We will deduce that under some condition A is bijective from $l_p(\alpha)$ into itself.

5.1. First properties.

For simplification we will write

$$A=\begin{pmatrix} d_1 & b_1 & & & & \\ a_2 & d_2 & b_2 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & a_n & d_n & b_n & \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5.1}$$

where $a=(a_n)_n$, $b=(b_n)_n$, and $d=(d_n)_n$ are three given sequences. We immediately obtain

$$D_{1/d}A-I=\begin{pmatrix} 0 & b_1' & & & & \\ a_2' & 0 & b_2' & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & a_n' & 0 & b_n' & \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5.2}$$

with $a_n'=a_n/d_n$ and $b_n'=b_n/d_n$. Putting $K_1=\sup_{n\ge1}(|a_n'|)<\infty$ and $K_2=\sup_{n\ge1}(|b_n'|)<\infty$, we get the following result, in which we do the convention $\alpha_0=0$.

Theorem 5.1. (i) *Let $p\ge1$ be a real. If*

$$\widetilde{K}_p=K_1^p+K_2^p<\frac{1}{2^{p-1}}, \tag{5.3}$$

then A is bijective from l_p into $l_p(|d|)$.

(ii) *The operator A is bijective from $l_p(\alpha)$ into $l_p(\alpha|d|)$ in the following cases:*
(a) *for $1 \le p < \infty$,*

$$\widetilde{K}_{p,\alpha} = \left[\sup_{n\ge1}\left(\left|\frac{a_n}{d_n}\frac{\alpha_{n-1}}{\alpha_n}\right|\right)\right]^p + \left[\sup_{n\ge1}\left(\left|\frac{b_n}{d_n}\frac{\alpha_{n+1}}{\alpha_n}\right|\right)\right]^p < \frac{1}{2^{p-1}}, \tag{5.4}$$

(b) *when $p = \infty$,*

$$\widetilde{K}'_\alpha = \sup_{n\ge1}\left(\left|\frac{a_n}{d_n}\frac{\alpha_{n-1}}{\alpha_n}\right| + \left|\frac{b_n}{d_n}\frac{\alpha_{n+1}}{\alpha_n}\right|\right) < 1, \tag{5.5}$$

with the convention $\alpha_0 = 0$.

Proof. Using the convention $x_0 = 0$, we get, from (5.2),

$$\|(I - D_{1/d}A)X\|_{l_p}^p = \sum_{n=1}^{\infty} |a'_n x_{n-1} + b'_n x_{n+1}|^p \le \sum_{n=1}^{\infty} (K_1|x_{n-1}| + K_2|x_{n+1}|)^p; \tag{5.6}$$

and since

$$\begin{aligned}\sum_{n=1}^{\infty} (K_1|x_{n-1}| + K_2|x_{n+1}|)^p &\le 2^{p-1}\left(\sum_{n=1}^{\infty} K_1^p|x_{n-1}|^p + \sum_{n=1}^{\infty} K_2^p|x_{n+1}|^p\right)\\ &\le 2^{p-1}(K_1^p + K_2^p)\|X\|_{l_p}^p,\end{aligned} \tag{5.7}$$

we conclude that

$$\|I - D_{1/d}A\|_{\mathcal{B}(l_p)} = \sup_{X\ne0}\left(\frac{\|(I - D_{1/d}A)X\|_{l_p}}{\|X\|_{l_p}}\right) \le \left[2^{p-1}(K_1^p + K_2^p)\right]^{1/p} < 1. \tag{5.8}$$

So $D_{1/d}A$ is invertible in $\mathcal{B}(l_p)$ and $A = D_d(D_{1/d}A)$ is bijective from l_p into $l_p(|d|)$. (ii)(a) is obtained replacing A by $D_{1/\alpha}AD_\alpha = (a_{nm}\alpha_m/\alpha_n)_{n,m\ge1}$. (ii)(b) comes from the fact that $\|I - D_{1/d}A\|_{\mathcal{B}(l_\infty(\alpha))} \le \|I - D_{1/d}A\|_{S_\alpha} = K'_\alpha$. ☐

We deduce the next corollaries.

Corollary 5.2. *If $\widetilde{K}_{1,\alpha} < 1$, then A is bijective from $l_1(\alpha)$ to $l_1(\alpha|d|)$ and bijective from s_α to $s_{\alpha|d|}$.*

Proof. First taking $p = 1$ in Theorem 5.1(ii), we deduce that A is bijective from $l_1(\alpha)$ to $l_1(\alpha|d|)$. Then from

$$\|I - D_{1/d}A\|_{S_\alpha} = \widetilde{K'_\alpha} \le \widetilde{K}_{1,\alpha} < 1, \tag{5.9}$$

we deduce, reasoning as in Theorem 5.1(i), that A is bijective from s_α to $s_{\alpha|d|}$. ☐

Corollary 5.3. *Let $p \ge 1$ be a real. If $K = \sup(K_1, K_2) < 1/2$, then A is bijective from l_p into $l_p(|d|)$.*

Proof. In fact, $K_1^p + K_2^p \le 2K^p < 1/2^{p-1}$ if and only if $K^p < 1/2^p$ and $K < 1/2$. ☐

5.2. Application to the continued fractions. We consider the system of linear equations

$$\begin{aligned} (\beta_1+z)x_1-c_1x_2 &= b_1,\\ -c_1x_1+(\beta_2+z)x_2-c_2x_3 &= b_2,\\ -c_2x_2+(\beta_3+z)x_3-c_3x_4 &= b_3,\\ &\vdots \end{aligned} \tag{5.10}$$

If ${}^tB = e_1' = (1,0,0,\dots)$, it is well known that we may write the linear equations in the form

$$\begin{gathered} x_1=\frac{1}{\beta_1+z-(c_1x_2/x_1)}, \qquad \frac{c_1x_2}{x_1}=\frac{c_1^2}{\beta_2+z-(c_2x_3/x_2)},\\ \frac{c_2x_3}{x_2}=\frac{c_2^2}{\beta_3+z-(c_3x_4/x_3)},\dots. \end{gathered} \tag{5.11}$$

If we substitute in succession from each into the preceding, we obtain *the formal expansion of* x_1 *into a continued fraction, also denoted by the* A *fraction, that is,*

$$x_1=\cfrac{1}{\beta_1+z-\cfrac{c_1^2}{\beta_2+z-\cfrac{c_2^2}{\beta_3+z-\cdots}}}. \tag{5.12}$$

The system defined by (5.10) is equivalent to the matrix equation $A_zX=B$; see [14]. The infinite tridiagonal matrix A_z admits infinitely many right inverses and x_1 can be written, as above, in a continued fraction when A_z admits an inverse $A_z'=(a_{nm}'(z))_{n,m\ge1}$. Then we have $x_1=c_{11}'(z)$; see [2, 14]. Recall the following well-known result [15].

Lemma 5.4. *Let $p>1$ be a real. $A\in(l_p,l_\infty)$ if and only if*

$$\sup_{n\ge1}\left(\sum_{m=1}^{\infty}|a_{nm}|^{p/(p-1)}\right)<\infty. \tag{5.13}$$

We deduce from the preceding the following.

Proposition 5.5. *Let $p>1$ be a real. If $A_z\in(l_p,l_p)$, there is a real $K_1>0$ such that A_z is bijective from l_p to l_p and from l_∞ to l_∞ for $|z|>K_1$.*

Proof. First we see that $A_z\in(l_p,l_p)$ implies $A_z\in(l_p,l_\infty)$. So, from Lemma 5.4, we get

$$\sup_{n\ge2}\left(|c_{n-1}|^q+|\beta_n+z|^q+|c_n|^q\right)<\infty, \tag{5.14}$$

with $q = p/(p-1)$, and the sequences $(\beta_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ are bounded. Putting $K = \sup_{n\geq 1}(|c_n|, |\beta_n|)$, we deduce that for $|z| > 3K$,

$$\sup_{n\geq 1}\left(\left|\frac{c_{n-1}}{\beta_n + z}\right|, \left|\frac{c_n}{\beta_n + z}\right|\right) \leq \frac{K}{|z| - K} < \frac{1}{2}, \tag{5.15}$$

with the convention $c_0 = 0$. By Corollary 5.3, we conclude that if $d = (\beta_n + z)_{n\geq 1}$, A is bijective from l_p to $l_p(|d|)$. Now, since there exist $k_1 = 2K$ and $k_2 = K + |z| > 0$ such that $k_1 \leq |d_n| \leq k_2$ for all n, $l_p(|d|) = l_p$. From Theorem 5.1 we see that A_z is bijective from l_∞ to l_∞. This gives the conclusion. □

6. Some properties of the operator Δ^h mapping $l_p(\alpha)$ into itself. In this section, we deal with necessary and sufficient conditions for Δ and Δ^+ to be bijective from $l_p(\alpha)$ to $l_p(\alpha)$.

Recall that Δ is the *first difference operator mapping s into itself*, defined for each $X = (x_n)_{n\geq 1}$ by $[\Delta X]_1 = x_1$, and $[\Delta X]_n = x_n - x_{n-1}$ for $n \geq 2$. The operator $\Sigma \in (s, s)$, defined by $\Sigma X = (\sum_{k=1}^{n} x_k)_{n\geq 1}$, satisfies

$$\Sigma(\Delta X) = \Delta(\Sigma X) = X \quad \forall X \in s. \tag{6.1}$$

We will denote by Δ^+ the operator Δ^t and by Σ^+ the operator Σ^t. We note that these operators are well known and have been used, for instance, in [2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 15]. Recall that for any integer $h > 0$, the sets $l_\infty(\Delta^h)$, $c(\Delta^h)$, and $c_0(\Delta^h)$ are *BK spaces* (recall that a BK space is a Banach space with continuous coordinates), with respect to their natural norms

$$\|X\|_{l^\infty(\Delta^h)} = \sup_{n\geq 1}\left([\,|\Delta^h X|\,]_n\right) = \sup_{n\geq 1}\left(\left|\sum_{j=0}^{h}(-1)^j\binom{h}{j}\right|x_{n-j}\right), \tag{6.2}$$

and $c(\Delta^h)$ and $c_0(\Delta^h)$ are closed subspaces of $l_\infty(\Delta^h)$ (see [13]). We can also state that the set s_α is a BK space *with respect to the norm* $\|\,\|_{s_\alpha}$; see [8]. In order to express the next results, we need to define the following sets:

$$\begin{aligned}
\widehat{C_1} &= \left\{\alpha \in U^{+*} : \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1)\ (n \longrightarrow \infty)\right\},\\
\widehat{C_1^+} &= \left\{\alpha \in U^{+*} \bigcap cs : \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = O(1)\ (n \longrightarrow \infty)\right\},\\
\Gamma &= \left\{\alpha \in U^{+*} : \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\},\\
\Gamma^+ &= \left\{\alpha \in U^{+*} : \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.
\end{aligned} \tag{6.3}$$

Note that $\alpha \in \Gamma^+$ if and only if $1/\alpha \in \Gamma$. We will see in Proposition 6.1 that if $\alpha \in \widehat{C_1}$, then $\alpha_n \to \infty$ $(n \to \infty)$. Furthermore, $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n\geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1. \tag{6.4}$$

We obtain the following results in which we put

$$[C(\alpha)\alpha]_n = \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right). \tag{6.5}$$

Proposition 6.1. *Let $\alpha \in U^{+*}$. Then*

(i) *if $\alpha \in \widehat{C_1}$, there are $K > 0$ and $\gamma > 1$ such that*

$$\alpha_n \geq K\gamma^n, \quad \forall n; \tag{6.6}$$

(ii) *the condition $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C_1}$ and there exists a real $b > 0$ such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\gamma_q(\alpha)} + b\left[\gamma_q(\alpha)\right]^n \quad \text{for } n \geq q+1; \tag{6.7}$$

(iii) *the condition $\alpha \in \Gamma^+$ implies $\alpha \in \widehat{C_1^+}$.*

Proof. (i) and (ii) were proved in [6, Proposition 2.1, pages 1786–1788]. (iii) If $\alpha \in \Gamma^+$, there are $\chi' \in]0,1[$ and an integer $q' \geq 1$ such that

$$\frac{\alpha_k}{\alpha_{k-1}} \leq \chi' \quad \text{for } k \geq q'. \tag{6.8}$$

Then for every $n \geq q'$, we have

$$\begin{aligned}\frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) &= \sum_{k=n}^{\infty}\left(\frac{\alpha_k}{\alpha_n}\right) \leq 1 + \sum_{k=n+1}^{\infty}\prod_{i=0}^{k-n-1}\left(\frac{\alpha_{k-i}}{\alpha_{k-i-1}}\right)\\ &\leq \sum_{k=n}^{\infty}\chi'^{k-n} = O(1) \quad (n \longrightarrow \infty).\end{aligned} \tag{6.9}$$

This gives the conclusion. □

Remark 6.2. Note that as a direct consequence of Proposition 6.1, we have $\widehat{C_1} \cap \widehat{C_1^+} = \Gamma \cap \Gamma^+ = \phi$.

Remark 6.3. It is proved in [4] that $\alpha \in \widehat{C_1}$ does not imply $\alpha \in \Gamma$.

Before stating the following, note that $l_1(\Delta) \neq l_1$. Indeed, putting $e_1 = (1,0,\dots)$, we get $\Sigma e_1 = e \notin l_1$ and $\Delta^{-1} \notin (l_1, l_1)$. We will see that when l_1 is replaced by $l_1(\alpha)$ we can make explicit the set $l_1(\alpha)(\Delta^h)$ for $h > 0$. We have the next result.

Proposition 6.4. (i) *The condition $\Delta^+ \in (l_1(\alpha), l_1(\alpha))$ is equivalent to*

$$\frac{\alpha_n}{\alpha_{n-1}} = O(1) \quad (n \longrightarrow \infty); \tag{6.10}$$

(ii) *the condition $1/\alpha \in \widehat{C}_1$ is equivalent to $\Sigma^+ \in (l_1(\alpha), l_1(\alpha))$;*

(iii) *if $1/\alpha \in \widehat{C}_1$, then Δ^+ is bijective from $l_1(\alpha)$ into itself.*

Proof. $\Delta^+ \in (l_1(\alpha), l_1(\alpha))$ if and only if $\Delta \in S_{1/\alpha}$, that is, $\alpha_n/\alpha_{n-1} = O(1)$ $(n \to \infty)$. (ii) comes from the fact that $\Sigma^+ \in (l_1(\alpha), l_1(\alpha))$ if and only if

$$(\Sigma^+)^t = \Sigma \in S_{1/\alpha}. \tag{6.11}$$

(iii) The condition $1/\alpha \in \widehat{C}_1$ implies

$$\frac{\alpha_n}{\alpha_{n-1}} \le \alpha_n \left(\sum_{k=1}^{n} \frac{1}{\alpha_k} \right) = O(1) \quad (n \longrightarrow \infty), \tag{6.12}$$

so $\Delta^+ \in (l_1(\alpha), l_1(\alpha))$. Consider now the equation

$$\Delta^+ X = B \tag{6.13}$$

for any given $B \in l_1(\alpha)$. We will prove that (6.13) admits a unique solution in the set $l_1(\alpha)$. First, from Proposition 6.1(i), the condition $1/\alpha \in \widehat{C}_1$ implies that there are $K > 0$ and $\gamma > 1$ such that $\alpha_n \le 1/K\gamma^n$ for all n. So $\alpha \in c_0$ and $X \in l_1(\alpha)$ imply together $x_n/\alpha_n = o(1)$ $(n \to \infty)$ and $x_n = \alpha_n o(1) = o(1)$ $(n \to \infty)$, that is, $X \in c_0$. Then for every $X = (x_n)_{n\ge1} \in l_1(\alpha)$,

$$\Sigma^+(\Delta^+ X) = \left(\sum_{k=n}^{\infty} (x_k - x_{k+1}) \right)_{n\ge1} = X, \tag{6.14}$$

and from (ii) we deduce that (6.13) admits in $l_1(\alpha)$ the unique solution $X = \Sigma^+ B$. So we have proved that Δ^+ is bijective from $l_1(\alpha)$ into itself. □

Define now the operator Δ^h for h real by

$$(\Delta^h)_{nm} = \begin{cases} \begin{pmatrix} -h+n-m-1 \\ n-m \end{pmatrix} & \text{if } m \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

See [3, 7]. Here we will see that when l_1 is replaced by $l_1(\alpha)$, we can study the set $l_1(\alpha)(\Delta^h)$. So we obtain the following.

Theorem 6.5. *Let $\alpha \in U^{+*}$. Then*

(i) *for any given real $h > 0$, $l_1(\alpha)(\Delta^h) = l_1(\alpha)$ if and only if*

$$\alpha_n \left(\sum_{m=n}^{\infty} \begin{pmatrix} h+m-n-1 \\ m-n \end{pmatrix} \frac{1}{\alpha_m} \right) = O(1) \quad (n \longrightarrow \infty); \tag{6.16}$$

(ii) *let $h \ge 1$ be an integer and $p \ge 1$ a real. If $\alpha \in \Gamma$, then*

$$l_p(\alpha)(\Delta^h) = l_p(\alpha). \tag{6.17}$$

Proof. First $\Delta^h \in (l_1(\alpha), l_1(\alpha))$ if and only if $(\Delta^h)^t = (\Delta^+)^h \in S_{1/\alpha}$. So

$$\alpha_n \left(\sum_{m=n+1}^{\infty} \left| \begin{pmatrix} -h+m-n-1 \\ m-n \end{pmatrix} \right| \frac{1}{\alpha_m} \right) = O(1) \quad (n \longrightarrow \infty). \tag{6.18}$$

On the other hand, $\Sigma^h \in (l_1(\alpha), l_1(\alpha))$ if and only if $(\Delta^{-h})^t = (\Delta^+)^{-h} \in S_{1/\alpha}$. This means that

$$\alpha_n \left(\sum_{m=n+1}^{\infty} \binom{h+m-n-1}{m-n} \frac{1}{\alpha_m} \right) = O(1) \quad (n \longrightarrow \infty). \tag{6.19}$$

It can be easily seen that (6.19) implies (6.18), since for $m > n$

$$\begin{aligned} \left| \binom{-h+m-n-1}{m-n} \right| &= \frac{1}{(m-n)!} \left| (-h+m-n-1)(-h+m-n-2)\cdots(-h) \right| \\ &\le \frac{1}{(m-n)!} (h+m-n-1)(h+m-n-2)\cdots(h). \end{aligned} \tag{6.20}$$

This gives the conclusion.

Assertion (ii). It is enough to show (ii) for $h = 1$, because if Δ is a one-to-one mapping from $l_p(\alpha)$ to $l_p(\alpha)$, it is the same for Δ^h, h being an integer. If we put

$$l = \overline{\lim_{n\to\infty}} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1 \tag{6.21}$$

for given ε_0, such that $0 < \varepsilon_0 < 1 - l$, there exists N_0 such that

$$\sup_{n \ge N_0+1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) \le l + \varepsilon_0 < 1. \tag{6.22}$$

Consider now the infinite matrix

$$\Sigma_\alpha^{(N_0)} = \begin{pmatrix} \left[\Delta_\alpha^{(N_0)}\right]^{-1} & & & 0 \\ & 1 & & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix}, \tag{6.23}$$

$\Delta_\alpha^{(N_0)}$ being the finite matrix whose entries are those of $\Delta_\alpha = D_{1/\alpha} \Delta D_\alpha$ for $n, m \le N_0$. We get

$$Q = \Sigma_\alpha^{(N_0)} \Delta_\alpha = (q_{nm})_{n,m \ge 1}, \tag{6.24}$$

with

$$q_{nm} = \begin{cases} 1 & \text{for } m = n, \\ -\dfrac{\alpha_m}{\alpha_{m+1}} & \text{for } m = n-1 \ge N_0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.25}$$

For every $X \in l_p$ we get

$$(I - Q)X = \left(0, \ldots, 0, \frac{\alpha_{N_0}}{\alpha_{N_0+1}} x_{N_0}, \ldots, \frac{\alpha_{n-1}}{\alpha_n} x_{n-1}, \ldots \right)^t, \tag{6.26}$$

where $\alpha_{N_0}x_{N_0}/\alpha_{N_0+1}$ is in the (N_0+1)th position. So

$$\begin{aligned}\|(I-Q)X\|_{l_p}^p &= \sum_{n=N_0+1}^{\infty}\left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p|x_{n-1}|^p \le \sup_{n\ge N_0+1}\left[\left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p\right]\left(\sum_{n=N_0}^{\infty}|x_n|^p\right),\\ \|I-Q\|_{\mathcal{B}(l_p)} &= \sup_{X\neq 0}\left(\frac{\|(I-Q)X\|_{l_p}}{\|X\|_{l_p}}\right) \le \left[\sup_{n\ge N_0+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p\right]^{1/p}.\end{aligned} \tag{6.27}$$

Since $\alpha_{n-1}/\alpha_n \le l+\varepsilon_0 < 1$ for all $n \ge N_0+1$, we deduce

$$\sup_{n\ge N_0+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p < 1. \tag{6.28}$$

Hence $\|I-Q\|_{\mathcal{B}(l_p)} < 1$. We have shown that Q is invertible in $\mathcal{B}(l_p)$. Now let $B \in l_p$. The equations $\Delta_\alpha X = B$ and $QX = \Sigma^{(N_0)}Y$ are equivalent in l_p. Since $Q^{-1} \in \mathcal{B}(l_p)$, we deduce from Theorem 4.2(iii) that for every $X \in l_p$, $Q^{-1}(QX) = (Q^{-1}Q)X = X = (\Delta_\alpha)^{-1}B$. This proves that the map Δ_α is bijective from l_p to l_p and Δ is bijective from $l_p(\alpha)$ to $l_p(\alpha)$. □

Remark 6.6. Note that we also have $1/\alpha \in \widehat{C}_1^+$ if and only if $l_1(\alpha)(\Delta) = l_1(\alpha)$. Indeed the conditions $\Delta \in (l_1(\alpha), l_1(\alpha))$ and $\Sigma \in (l_1(\alpha), l_1(\alpha))$ are equivalent to $\Delta^+ \in S_{1/\alpha}$ and $\Sigma^+ \in S_{1/\alpha}$, that is,

$$\frac{\alpha_n}{\alpha_{n-1}} = O(1), \quad \alpha_n\left(\sum_{k=1}^{n}\frac{1}{\alpha_k}\right) = O(1) \quad (n \longrightarrow \infty). \tag{6.29}$$

From the inequality $\alpha_n/\alpha_{n-1} \le \alpha_n(\sum_{k=1}^n 1/\alpha_k)$ for all n, we conclude that $1/\alpha \in \widehat{C}_1^+$ if and only if $l_1(\alpha)(\Delta) = l_1(\alpha)$.

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