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BOUNDS FOR TOTAL ANTIEIGENVALUE OF A NORMAL OPERATOR

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We give an alternative proof of a theorem of Gustafson and Seddighin (1993) following the idea used by Das et al. in an earlier study of antieigenvectors (1998). The result proved here holds for certain classes of normal operators even if the space is infinite dimensional.

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1. Introduction. Let H be a complex Hilbert space and let $B(H)$ denote the Banach algebra of all bounded linear operators on H. The concept of the angle of an operator T was introduced by Gustafson [4, 5, 6] while studying perturbation theory of semigroup generators. From this has developed what we call operator trigonometry, whose theory and applications are still evolving. The properties are intimately associated with the numerical range $W(T)$ of an operator T and the numerical range $W(TT^*)$. Some relevant important results can be found in [3].

The cosine of an operator T in $B(H)$ was originally defined as follows:

$$\cos T = \inf_{Tf\neq 0} \frac{\operatorname{Re}(Tf,f)}{\|Tf\|\|f\|} \tag{1.1}$$

for arbitrary operators in a Banach semi-innerproduct space. Here we will restrict attention primarily to the case of $T \in B(H)$. Clearly, $\operatorname{Cos} T$ is a real cosine defined for the real part of the numerical range of T. The total cosine

$$|\cos|T = \inf_{Tf\neq 0} \frac{|(Tf,f)|}{\|Tf\|\|f\|} \tag{1.2}$$

is also defined.

The expression (1.1) also denoted as the angle $\phi(T)$ measures the maximum (real) turning effect of T.

The quantity $\operatorname{Cos} T$ has another interpretation as the first antieigenvalue of T, where

$$\mu_1(T) = \inf_{Tf\neq 0} \frac{\operatorname{Re}(Tf,f)}{\|Tf\|\|f\|}. \tag{1.3}$$

The terminology "antieigenvalue" and "antieigenvector" was introduced by Gustafson [7] in the year 1972.

Krein [10] and Gustafson [7] have studied $\mu_1(T)$ and indicated how the knowledge of $\mu_1(T)$ can be useful in the study of certain integral operators, initialvalue problems

and some other areas. The upper bound for $\mu_1(T)$ for T, a finite-dimensional, strongly accretive (i.e., $\operatorname{Re} T > 0$) normal operator was obtained by Davis [2] in the year 1980.

The exact value of $\mu_1(T)$ can be found in [2, 7]. Mirman [11] gave a method of estimation of $\mu_n(T)$, the higher antieigenvalues of T, which is defined by Gustafson as follows:

$$\mu_n(T) = \min\left\{\frac{\operatorname{Re}(Tf,f)}{\|Tf\|\|f\|} : f \neq 0,\ f \perp f^1, f^2, \dots, f^{n-1}\right\}, \tag{1.4}$$

where f^k is the kth antieigenvector. In the year 1989 Gustafson and Seddighin [8] proved the following theorem.

Theorem 1.1. *Let T be a normal accretive operator on a finite-dimensional Hilbert space H with eigenvalues*

$$\lambda_k = \beta_k + i\delta_k, \quad k = 1,2,3,\dots,n. \tag{1.5}$$

Let

$$E = \left\{\frac{\beta_i}{|\lambda_i|} : 1 < i < n\right\},$$
$$F = \left\{2\frac{\sqrt{(\beta_j-\beta_i)(\beta_i|\lambda_j|^2+2\beta_j|\lambda_i|^2)}}{|\lambda_j|^2-|\lambda_i|^2} : 0 \le \frac{\beta_j|\lambda_j|^2-2\beta_i|\lambda_j|^2+2\beta_j|\lambda_i|^2}{(|\lambda_i|^2-|\lambda_j|^2)(\beta_i-\beta_j)} \le 1\right\}. \tag{1.6}$$

Then $\mu_1(T)$ is exactly equal to the smallest number in $E \cup F$. Furthermore, if T is diagonal and

$$\mu_1(T) = 2\frac{\sqrt{(\beta_j-\beta_i)(\beta_i|\lambda_j|^2+2\beta_j|\lambda_i|^2)}}{|\lambda_j|^2-|\lambda_i|^2}, \tag{1.7}$$

then, $\mu_1(T) = (T_{z,z})/\|T_z\|$, for some z with

$$\begin{aligned} |z_i|^2 &= \frac{\beta_j|\lambda_j|^2-2\beta_i|\lambda_j|^2+\beta_j|\lambda_i|^2}{(|\lambda_i|^2-|\lambda_j|^2)(\beta_i-\beta_j)}, \\ |z_j|^2 &= \frac{\beta_i|\lambda_i|^2-2\beta_j|\lambda_i|^2+\beta_i|\lambda_j|^2}{(|\lambda_i|^2-|\lambda_j|^2)(\beta_i-\beta_j)}, \end{aligned} \tag{1.8}$$

and $z_k = 0$ for $k \neq i$, $k \neq j$.

Das et al. [1] also proved the above theorem in a different form which seems to be much simpler. They used the concept of stationary vectors and the result holds even if the space is not finite dimensional for operators with complete orthonormal set of eigenvectors.

Gustafson and Seddighin [9] also obtained the bounds for total antieigenvalues of a normal operator on a finite-dimensional Hilbert space.

We have proved the result following the idea used by Das et al. [1]. The result holds even if the space is infinite dimensional for operators with complete orthonormal set of eigenvectors.

2. Total antieigenvectors. Let A be a strictly accretive operator on H and let $C = A^*A$, and let $|\phi_A(f)| = |(Af,f)|/\|Af\|\|f\|$ represent the modulus of the cosine of largest angle through which an arbitrary nonzero vector f can be rotated by the action of A. Now $|\phi_A(f)|$ is said to have a stationary value at a vector $f \neq 0$ if the function $w_g(t)$ of real variable t defined by

$$w_g(t) = |\phi_A(f+tg)|^2 = \frac{|(A(f+tg),f+tg)|^2}{(C(f+tg),f+tg)(f+tg,f+tg)} \tag{2.1}$$

has a stationary value at $t = 0$ for an arbitrary but fixed vector $g \in H$. In other words we must have $w_g'(0) = 0$ for all $g \in H$. For $\|f\| = 1$, set $A_X = (A+A^*)/2$, $A_Y = (A-A^*)/2i$, $C = A^*A$, $b = (Af,f)$, $b_X = \operatorname{Re}(Af,f)$, $b_Y = \operatorname{Im}(Af,f)$, and $c^2 = (Cf,f)$.

With these notations, we see that $|\phi_A(f)|$ is stationary at f if and only if

$$2c^2\operatorname{Re}(b_XA_Xf+b_YA_Yf,g) - |b|^2\operatorname{Re}(Cf,g) - |b|^2c^2\operatorname{Re}(f,g) = 0. \tag{2.2}$$

Since $g \in H$ is arbitrary, we have the following theorem.

Theorem 2.1. *Let $|\phi_A(f)|$, b, c^2, C, b_X, b_Y, A_X, and A_Y be defined as above. A unit vector f is a stationary vector of $|\phi_A(f)|$ if and only if*

$$2c^2(b_XA_X+b_YA_Y)f - |b|^2Cf - |b|^2c^2f = 0. \tag{2.3}$$

The above equation obviously characterizes the vectors for which $|\phi_A(f)|$ is stationary, in particular, a minimum or a maximum.

We next prove the following theorem.

Theorem 2.2. *If for a stationary vector f, $Af = A^*f$, then f is a linear combination of two eigenvectors of A^*. If further, A is normal, then f is a linear combination of two eigenvectors of A.*

Proof. Suppose f is a stationary vector and $Af = A^*f$. Then, we have by the necessary and sufficient condition for a vector to be stationary

$$2c^2bA^*f - |b|^2A^*Af - |b|^2c^2f = 0, \tag{2.4}$$

that is, (2.4) becomes

$$\begin{aligned}
&2c^2\frac{b}{|b|^2}A^*f - A^*Af - c^2f = 0\\
&\quad\Longrightarrow A^*Af - \frac{c^2}{b}A^*f = \frac{c^2}{b}A^*f - c^2f\\
&\quad\Longrightarrow A^*\left\{Af - \frac{c^2}{b}f \pm \frac{c}{b}\sqrt{c^2-b^2}f\right\} = \left\{\frac{c^2}{b} \pm \frac{c}{b}\sqrt{c^2-b^2}\right\}\left\{Af - \frac{c^2}{b}f \pm \frac{c}{b}\sqrt{c^2-b^2}f\right\}.
\end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} g_1 &= Af - \frac{c^2}{b} f + \frac{c}{b}\sqrt{c^2-b^2} f, \\ \beta_1 &= \frac{c^2}{b} + \frac{c}{b}\sqrt{c^2-b^2}, \\ g_2 &= Af - \frac{c^2}{b} f - \frac{c}{b}\sqrt{c^2-b^2} f, \\ \beta_2 &= \frac{c^2}{b} - \frac{c}{b}\sqrt{c^2-b^2}. \end{aligned} \tag{2.6}$$

Then $A^* g_1 = \beta_1 g_1$ and $A^* g_2 = \beta_2 g_2$. Also, $2(c/b)\sqrt{c^2-b^2} f = g_1 - g_2$. Thus f is a linear combination of the eigenvectors g_1 and g_2 of A^* with corresponding eigenvalues β_1 and β_2.

If further, A is normal, then proceeding as above, we get

$$A\left\{Af - \frac{c^2}{b} f \pm \frac{c}{b}\sqrt{c^2-b^2} f\right\} = \left\{\frac{c^2}{b} \pm \frac{c}{b}\sqrt{c^2-b^2}\right\}\left\{Af - \frac{c^2}{b} f \pm \frac{c}{b}\sqrt{c^2-b^2} f\right\} \tag{2.7}$$

as before; $A^* g_1 = \beta_1 g_1$, $A^* g_2 = \beta_2 g_2$, and $2(c/b)\sqrt{c^2-b^2} f = g_1 - g_2$.

This completes the proof. □

Theorem 2.3. *A unit vector f is a total antieigenvector of a selfadjoint operator A if and only if there exist two eigenvectors whose appropriate linear combination (in the sense given below) yields f.*

Proof. If f is a stationary vector, in particular, a total antieigenvector, then it satisfies (2.3).

As A is selfadjoint, (2.3) reduces to

$$\begin{aligned} &2\|Af\|^2 (Af,f) Af - |(Af,f)|^2 A^2 f - |(Af,f)|^2 \|Af\|^2 f = 0 \\ &\quad \Longrightarrow 2\|Af\|^2 Af - |(Af,f)| A^2 f - |(Af,f)| \|Af\|^2 f = 0 \\ &\quad \Longrightarrow A^2 f - \frac{\|Af\|}{|\phi_A(f)|} Af = \frac{\|Af\|}{|\phi_A(f)|} Af - \|Af\|^2 f \\ &\quad \Longrightarrow A\left\{Af - \frac{\|Af\|}{|\phi_A(f)|} f \pm \frac{\|h\|}{|\phi_A(f)|} f\right\} = \frac{\|Af\| \pm \|h\|}{|\phi_A(f)|}\left\{Af - \frac{\|Af\|}{|\phi_A(f)|} f \pm \frac{\|h\|}{|\phi_A(f)|} f\right\}, \end{aligned} \tag{2.8}$$

where $h = Af - (Af,f) f$, and $\|h\|^2 = \|Af\|^2 - (Af,f)^2$.

Let

$$\begin{aligned} \mu_1 &= \frac{\|Af\| + \|h\|}{|\phi_A(f)|}, \qquad & \mu_2 &= \frac{\|Af\| - \|h\|}{|\phi_A(f)|}, \\ g_1 &= Af - \mu_2 f, \qquad & g_2 &= Af - \mu_1 f. \end{aligned} \tag{2.9}$$

Then, $Ag_1 = \mu_1 g_1$, $Ag_2 = \mu_2 g_2$, and $f = (1/(\mu_1 - \mu_2))(g_1 - g_2)$, so f is a linear combination of two eigenvectors.

Conversely, let $f = \alpha_i e_i + \alpha_j e_j$ with $|\alpha_i|^2 = \lambda_j/(\lambda_i+\lambda_j)$ and $|\alpha_j|^2 = \lambda_i/(\lambda_i+\lambda_j)$, where e_i, e_j are any two eigenvectors of A corresponding to the eigenvalues λ_j, λ_j.

So $(Af,f) = \lambda_i|\alpha_i|^2+\lambda_j|\alpha_j|^2 = 2\lambda_i\lambda_j/(\lambda_i+\lambda_j)$ and $c^2 = \lambda_i^2|\alpha_i|^2+\lambda_j^2|\alpha_j|^2$. With these values, we can see that (2.8) is satisfied by f. This completes the proof. □

Before we discuss the structure of the stationary vectors in the normal-operator case, we give a few examples to show how the situation arises in this case. In the examples, e_k's are eigenvectors of A corresponding to the eigenvalues λ_k's, the values of which are clear from the context.

Example 2.4. Let $Af = (1+i)(f,e_1)e_1+(1+2i)(f,e_2)e_2$, where $i = \sqrt{-1}$ and $\|f\| = 1$.

So $c^2 = 2+3k$ and $|b|^2 = 2+2k+k^2$, where $k = |(f,e_2)|^2$.

Hence, $|b|^2/c^2 = (2+2k+k^2)/(2+3k)$.

For a minimum or a maximum, we must have $k = (-2+\sqrt{10})/3$ or $(-2-\sqrt{10})/3$. The case $(-2-\sqrt{10})/3$ must be ruled out as $k<0$. For $k = (-2+\sqrt{10})/3$, $|b|^2 = (20+2\sqrt{10})/9$, and $c^2 = \sqrt{10}$. Hence, $|\phi_A(f)| = (20+2\sqrt{10})/9\sqrt{10}$. Here, both of (f,e_1) and (f,e_2) are not zero. Thus, an antieigenvector may be a linear combination of two eigenvectors.

Example 2.5. This is the most important example in this section. It shows that a linear combination of more than two eigenvectors may exist for the attainment of the minimum of $|\phi_A(f)|$. We consider the normal operator A such that

$$Af = (1+2i)(f,e_1)e_1+(1-i)(f,e_2)e_2+(1-i)(f,e_3)e_3 \tag{2.10}$$

with $\|f\| = 1$, where $i = \sqrt{-1}$. Clearly,

$$\begin{aligned}(Af,f) &= |(f,e_1)|^2+|(f,e_2)|^2+|(f,e_3)|^2\\ &\quad +i2|(f,e_1)|^2-|(f,e_2)|^2-|(f,e_3)|^2\\ &= 1+i(2-3k),\end{aligned} \tag{2.11}$$

where $k = |(f,e_2)|^2+|(f,e_3)|^2$, and

$$c^2 = 5|(f,e_1)|^2+2|(f,e_2)|^2+2|(f,e_3)|^2 = (5-3k). \tag{2.12}$$

Hence,

$$\frac{|b|^2}{c^2} = \frac{5-12k+9k^2}{5-3k}. \tag{2.13}$$

For a minimum or a maximum, we must have $k = (5+\sqrt{10})/3$ or $(5-\sqrt{10})/3$.

The case $k = (5+\sqrt{10})/3$ must be ruled out as in that case $k>1$.

For $k = (5-\sqrt{10})/3$, we have $|\phi_A(f)|^2 = (20-6\sqrt{10})/\sqrt{10}$. Let $|(f,e_1)|^2 = (-2+\sqrt{10})/3$, $|(f,e_2)|^2 = 1/3$, and $|(f,e_3)|^2 = (4-\sqrt{10})/3$, so that the unit vector $f = \sqrt{(-2+\sqrt{10})/3}\,e_1 + \sqrt{1/3}\,e_2 + \sqrt{(4-\sqrt{10})/3}\,e_3$ will be the first antieigenvector of A. However, it is possible to have a combination of only two eigenvectors corresponding to two eigenvalues for which the minimum, in question, is attained. Set $|(f,e_1)|^2 = \sqrt{2}/(\sqrt{2}+\sqrt{5})$, $|(f,e_2)|^2 = \sqrt{5}/(\sqrt{2}+\sqrt{5})$, and $|(f,e_3)|^2 = 0$. Clearly, $f = \sqrt{\sqrt{2}/(\sqrt{2}+\sqrt{5})}e_1 + \sqrt{\sqrt{5}/(\sqrt{2}+\sqrt{5})}e_2$ will be the required vector.

We now prove the main theorem.

Theorem 2.6. *Let A be a normal operator on an infinite-dimensional Hilbert space H with a complete orthonormal set of eigenvectors e_k and the corresponding eigenvalues $\lambda_k = \beta_k + i\delta_k$ such that for any unit vector $f \in H$, $Af = \sum \lambda_k (f,e_k)e_k$. If $|\phi_A(f)|$ is stationary at f and f is not an eigenvector of A, then either f is a linear combination of two eigenvectors, or there exists a suitable linear combination g of two eigenvectors corresponding to two distinct eigenvalues such that $|\phi_A(f)| = |\phi_A(g)|$ and $|\phi_A|$ is stationary at g. Further, the relation*

$$|\phi_A(f)| = \left\{ \frac{(\beta_k|\lambda_j| + \beta_j|\lambda_k|)^2 + (\delta_k|\lambda_j| + \delta_j|\lambda_k|)^2}{(|\lambda k| + |\lambda_j|)^2 |\lambda_k||\lambda_j|} \right\} \tag{2.14}$$

holds if $\lambda_k = \beta_k + i\delta_k$ and $\lambda_j = \beta_j + i\delta_j$ are the distinct eigenvalues referred to as above.

Proof. If $|\phi_A(f)|$ is stationary at f, then we have (2.3), where $A_X = (A+A^*)/2$, $A_Y = (A-A^*)/2i$, $C = A^*A$, $b = (Af,f)$, $b_X = \operatorname{Re}(Af,f)$, $b_Y = \operatorname{Im}(Af,f)$, and $c^2 = (Cf,f)$. Substituting $f = \sum \lambda_k (f,e_k)e_k$ in (2.3), we get

$$2c^2 b_X \operatorname{Re}\lambda_k + 2c^2 b_Y \operatorname{Im}\lambda_k - |b|^2|\lambda_k|^2 - |b|^2 c^2 = 0 \tag{2.15}$$

if $(f,e_k) \neq 0$.

Now it is easy to show that (2.15) is satisfied by $Af = \lambda f$, $b_X = \operatorname{Re}\lambda$, $b_Y = \operatorname{Im}\lambda$, $b = \lambda$, and $c^2 = |\lambda|^2$.

If f is not an eigenvector, then f may be a linear combination of two eigenvectors corresponding to two eigenvalues λ_k which satisfies (2.3). If, however, f is a linear combination of more than two eigenvectors, then we show that there always exists a linear combination g of two eigenvectors corresponding to two eigenvalues such that $|\phi_A(f)| = |\phi_A(g)|$ and $|\phi_A|$ is stationary at g.

Let $Ae_k = \lambda_k e_k$ and $Ae_j = \lambda_j e_j$, where λ_k, λ_j satisfy (2.15). We find α_k and α_j such that $g = \alpha_k e_k + \alpha_j e_j$, $|\alpha_k|^2 + |\alpha_j|^2 = 1$, $|(Af,f)| = |(Ag,g)|$, and $\|Af\| = \|Ag\|$. We now show that $|\phi_A|$ is stationary at g.

Choose $|\alpha_i|^2 = |\lambda_j|/(|\lambda_i| + |\lambda_j|)$ and $|\alpha_j|^2 = |\lambda_i|/(|\lambda_i| + |\lambda_j|)$ such that $|(Af,f)| = |(Ag,g)|$ and $\|Af\| = \|Ag\|$. We first show that g is a stationary vector.

We have

$$\begin{aligned} b_X &= \frac{(\beta_k|\lambda_j|+\beta_j|\lambda_k|)}{(|\lambda_k|+|\lambda_j|)}, \\ b_Y &= \frac{(\delta_k|\lambda_j|+\delta_j|\lambda_k|)}{(|\lambda_k|+|\lambda_j|)}, \\ c^2 &= |\lambda_k||\lambda_j|, \end{aligned} \tag{2.16}$$

and so

$$\begin{aligned} &2c^2 b_X \operatorname{Re}\lambda_k + 2c^2 b_Y \operatorname{Im}\lambda_k - |b|^2|\lambda_k|^2 - |b|^2c^2 \\ &\quad = 2c^2\beta_k \frac{(\beta_k|\lambda_j|+\beta_j|\lambda_k|)}{(|\lambda_k|+|\lambda_j|)} \\ &\qquad + 2c^2\delta_k \frac{(\delta_k|\lambda_j|+\delta_j|\lambda_k|)}{(|\lambda_k|+|\lambda_j|)} - |b|^2|\lambda_k|^2 - |b|^2c^2 \\ &\quad = \frac{2|\lambda_k|^2|\lambda_j|}{(|\lambda_k|+|\lambda_j|)}(|\lambda_k||\lambda_j|+\beta_k\beta_j+\delta_k\delta_j) \\ &\qquad - |b|^2(|\lambda_k|^2+|\lambda_k||\lambda_j|) = 0, \end{aligned} \tag{2.17}$$

where

$$|b|^2 = b_X^2 + b_Y^2 = \frac{2|\lambda_k|^2|\lambda_j|}{(|\lambda_k|+|\lambda_j|)^2}(|\lambda_k||\lambda_j|+\beta_k\beta_j+\delta_k\delta_j), \tag{2.18}$$

and so

$$|b|^2(|\lambda_k|^2+|\lambda_k||\lambda_j|) = \frac{2|\lambda_k|^2|\lambda_j|}{(|\lambda_k|+|\lambda_j|)}(|\lambda_k||\lambda_j|+\beta_k\beta_j+\delta_k\delta_j). \tag{2.19}$$

So, g is a stationary vector as it satisfies the necessary and sufficient condition.

Also we have,

$$|\phi_A(f)|^2 = \frac{|b|^2}{c^2} = \left\{\frac{(\beta_k|\lambda_j|+\beta_j|\lambda_k|)^2+(\delta_k|\lambda_j|+\delta_j|\lambda_k|)^2}{(|\lambda_k|+|\lambda_j|)^2|\lambda_k||\lambda_j|}\right\}. \tag{2.20}$$

Hence, the proof of the theorem is complete. □

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