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$T_{(\alpha,\beta)}$-SPACES AND THE WALLMAN COMPACTIFICATION

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Some new separation axioms are introduced and studied. We also deal with maps having an extension to a homeomorphism between the Wallman compactifications of their domains and ranges.

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1. Introduction. Among the oldest separation axioms in topology there are three famous ones, T_0, T_1, and T_2.

The T_0-axiom is usually credited to Kolmogoroff and the T_1-axiom to Fréchet or Riesz (and spaces satisfying the axioms are sometimes called Kolmogoroff spaces, Fréchet spaces, or Riesz spaces, accordingly). The T_2-axiom is included in the original list of axioms for a topology given by Hausdorff [10].

We denote by Top the category of topological spaces with continuous maps as morphisms, and by Top_i the full subcategory of Top whose object is T_i-spaces. It is a part of the folklore of topology that Top_{i+1} is a reflective subcategory of Top_i, for $i = -1, 0, 1$, with $\text{Top}_{-1} = \text{Top}$. Thus Top_i is reflective in Top, for each $i = 0, 1, 2$ (see MacLane [17]). In other words, there is a universal T_i-space for every topological space X; we denote it by $\mathbf{T}_i(X)$. The assignment $X \mapsto \mathbf{T}_i(X)$ defines a functor $\mathbf{T}_i$ from Top onto Top_i, which is a left adjoint functor of the inclusion functor $\text{Top}_i \hookrightarrow \text{Top}$.

The first section of this paper is devoted to the characterization of morphisms in Top rendered invertible by the functor $\mathbf{T}_0$.

Let X be a topological space. Then $\mathbf{T}_i(X)$ is a T_i-space; moreover, $\mathbf{T}_i(X)$ may be a T_{i+1}-space. The second section deals with space X such that $\mathbf{T}_i(X)$ is a T_{i+1}-space.

Definition 1.1. Let i, j be two integers such that $0 \le i < j \le 2$. A topological space X is said to be a $T_{(i,j)}$-space if $\mathbf{T}_i(X)$ is a T_j-space (thus there are three new types of separation axioms; namely, $T_{(0,1)}$, $T_{(0,2)}$, and $T_{(1,2)}$).

More generally, one may introduce the following categorical concept.

Definition 1.2. Let $\mathbf{C}$ be a category and $\mathbf{F}$, $\mathbf{G}$ two (covariant) functors from $\mathbf{C}$ to itself.

(1) An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{G})}$-object if $\mathbf{G}(\mathbf{F}(X))$ is isomorphic with $\mathbf{F}(X)$.

(2) Let $\mathbf{P}$ be a topological property. An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{P})}$-object if $\mathbf{F}(X)$ satisfies the property $\mathbf{P}$.

Recall that a topological space X is said to be a T_D-*space* [1] if every one-point set of X is locally closed. For the separation axioms T_0, T_1, T_2, T_D, we classically have the following implications:

$$T_2 \Longrightarrow T_1 \Longrightarrow T_D \Longrightarrow T_0. \tag{1.1}$$

Following Definition 1.2, one may define another new separation axiom; namely, $T_{(0,D)}$. Unfortunately, we have no intrinsic topological characterization of $T_{(1,2)}$-spaces. However, $T_{(0,D)}$-, $T_{(0,1)}$-, and $T_{(0,2)}$-spaces are completely characterized in Section 3.

Section 4 deals with the separation axioms $T_{(0,S)}$, $T_{(S,D)}$, $T_{(S,1)}$, and $T_{(S,2)}$, where S is the functor of soberification from Top to itself (following Definition 1.2, a space X is said to be $T_{(S,D)}$ if $S(X)$ is a T_D-space).

One of the two anonymous referees of this paper has notified that the T_D property is not reflective in Top; the second author has asked Professor H. P. Kunzi (University of Cape Town) for an explanation of this fact. We give this explanation as communicated by Kunzi.

In [5, Remark 4.2, page 408], Brümmer has proved that the countable product of the Sierpinski space is not a T_D-space. On the other hand, according to Herrlich and Strecker [12], if a subcategory **A** is reflective in a category **B**, then for each category **I**, **A** is closed under the formation of **I**-limits in **B** (see [12, Theorem 36.13]). (Taking **I** a discrete category, you see that in particular **A** is closed under products in **B**.) Therefore the full subcategory Top_D of Top whose objects are T_D-spaces is not reflective in Top.

The importance and usefulness of compactness properties in topology and functional analysis is universally recognized. Compactifications of topological spaces have been studied extensively, as well as the associated Stone-Čech compactification.

In [11], Herrlich has stated that it is of interest to determine if the Wallman compactification may be regarded as a functor, especially as an epireflection functor, on a suitable category of spaces. This problem was solved affirmatively by Harris in [9].

Let X, Y be two T_1-topological spaces and $f : X \to Y$ a continuous map. A w-extension of f is a continuous map $w(f) : wX \to wY$ such that $w(f) \circ \omega_X = \omega_Y \circ f$, where wX is the Wallman compactification of X and $\omega_X : X \to wX$ is the canonical embedding of X into its Wallman compactification wX.

In Section 5, we attempt to characterize when Wallman extensions of maps are homeomorphisms.

2. Topologically onto quasihomeomorphisms. Recall that a continuous map $q : Y \to Z$ is said to be a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ defines a bijection $\mathbb{O}(Z) \to \mathbb{O}(Y)$ [8], where $\mathbb{O}(Y)$ is the set of all open subsets of the space Y. A subset S of a topological space X is said to be *strongly dense* in X if S meets every nonempty locally closed subset of X [8]. Thus a subset S of X is strongly dense if and only if the canonical injection $S \hookrightarrow X$ is a quasihomeomorphism. It is well known that a continuous map $q : X \to Y$ is a quasihomeomorphism if and only if the topology of X is the inverse image by q of that of Y and the subset $q(X)$ is strongly dense in Y [8]. The notion of quasihomeomorphism is used in algebraic geometry and it has recently been shown that this notion arises naturally in the theory of some foliations associated to closed connected manifolds (see [3, 4]).

Now, we give some straightforward remarks about quasihomeomorphisms.

Remark 2.1. (1) If $f : X \to Y$, $g : Y \to Z$ are continuous maps and two of the three maps f, g, $g \circ f$ are quasihomeomorphisms, then so is the third one.

(2) Let $q : X \to Y$ be a continuous onto map. Then the following statements are equivalent (see [7, Lemma 1.1]):

(i) q is a quasihomeomorphism;

(ii) q is open and for each open subset U of X, we have $q^{-1}(q(U)) = U$;

(iii) q is closed and for each closed subset C of X, we have $q^{-1}(q(C)) = C$.

We introduce the concept of "topologically onto (resp., one-to-one) maps" as follows.

Definition 2.2. Let $q : X \to Y$ be a continuous map.

(1) It is said that q is topologically onto if, for each $y \in Y$, there exists $x \in X$ such that $\overline{\{y\}} = \overline{\{q(x)\}}$.

(2) q is said to be topologically one-to-one if, for each $y, x \in X$ such that $q(x) = q(y)$, $\overline{\{y\}} = \overline{\{x\}}$.

(3) q is said to be topologically bijective if it is topologically onto and topologically one-to-one.

We recall the T_0-identification of a topological space which is done by Stone [18].

Let X be a topological space and define $\sim$ on X by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Then $\sim$ is an equivalence relation on X and the resulting quotient space $X/\sim$ is a T_0-space. This procedure and the space it produces are referred to as the T_0-identification of X. Clearly, $\mathbf{T}_0(X) = X/\sim$. The canonical onto map from X onto its T_0-identification $\mathbf{T}_0(X)$ will be denoted by μ_X. Of course, μ_X is an onto quasihomeomorphism.

As recalled in the introduction, $\mathbf{T}_0$ defines a (covariant) functor from Top to itself. If $q : X \to Y$ is a continuous map, then the diagram

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\mu_Y"] \\ \mathbf{T}_0(X) \arrow[r, "\mathbf{T}_0(q)"] & \mathbf{T}_0(Y) \end{tikzcd} \tag{2.1}$$

is commutative.

Example 2.3. (1) Every one-to-one continuous map is topologically one-to-one.

(2) Every onto continuous map is topologically onto.

(3) A topologically bijective map need not be one-to-one.

Let X be a topological space which is not T_0. Of course, μ_X is topologically bijective and μ_X is not one-to-one.

For any functor $F : \mathbf{C} \to \mathbf{D}$ between two given categories, the set of all arrows in $\mathbf{C}$ rendered invertible by F has, sometimes, important applications. The following result characterizes morphisms in Top rendered invertible by the functor $\mathbf{T}_0$.

Theorem 2.4. *Let $q : X \to Y$ be a continuous map. Then the following statements are equivalent:*

(i) *q is a topologically onto quasihomeomorphism;*

(ii) *$\mathbf{T}_0(q)$ is a homeomorphism.*

Proof. (i)⇒(ii). The map $\mathbf{T}_0(q) \circ \mu_X$ is a quasihomeomorphism, since $\mathbf{T}_0(q) \circ \mu_X = \mu_Y \circ q$. Hence $\mathbf{T}_0(q)$ is a quasihomeomorphism, by Remark 2.1(1).

$\mathbf{T}_0(q)$ is onto. Let $\mu_Y(y) \in \mathbf{T}_0(Y)$. Then there exists an $x \in X$ such that $\overline{\{y\}} = \overline{\{q(x)\}}$. Hence $\mu_Y(y) = \mu_Y(q(x))$. Thus $\mu_Y(y) = \mathbf{T}_0(q)(\mu_X(x))$.

$\mathbf{T}_0(q)$ is one-to-one. Let $\mu_X(x), \mu_X(x') \in \mathbf{T}_0(X)$ be such that $\mathbf{T}_0(q)(\mu_X(x)) = \mathbf{T}_0(q)(\mu_X(x'))$. Then $\mu_Y(q(x)) = \mu_Y(q(x'))$. Hence $\overline{\{q(x)\}} = \overline{\{q(x')\}}$. We are aiming to prove that $\overline{\{x\}} = \overline{\{x'\}}$; it is sufficient to show that $\overline{\{x\}} \subseteq \overline{\{x'\}}$. Indeed, let U be an open subset of X containing x and V an open subset of Y such that $U = q^{-1}(V)$. Then $q(x) \in V$. It follows that $q(x') \in V$; that is, $x' \in U$.

Therefore, $\mathbf{T}_0(q)$ is a bijective quasihomeomorphism. But one may check easily that bijective quasihomeomorphisms are homeomorphisms.

(ii)⇒(i). The equality $\mathbf{T}_0(q) \circ \mu_X = \mu_Y \circ q$ forces q to be a quasihomeomorphism, by Remark 2.1(1). It remains to prove that q is topologically onto. To do so, let $y \in Y$. Then there exists $x \in X$ such that $\mathbf{T}_0(q)(\mu_X(x)) = \mu_Y(y)$. Thus $\mu_Y(q(x)) = \mu_Y(y)$. Therefore, $\overline{\{y\}} = \overline{\{q(x)\}}$, completing the proof. □

As a direct consequence of [6, Lemma 1.2, Theorem 1.3] and Theorem 2.4, one may give an external characterization of T_0-spaces.

Theorem 2.5. (1) *For any topological space X, the following statements are equivalent:*

(i) *X is a T_0-space;*

(ii) *for each topologically onto quasihomeomorphism $q : Y \to Z$ and each continuous map $f : Y \to X$, there is a unique continuous map $\tilde{f} : Z \to X$ such that $\tilde{f} \circ q = f$.*

(2) *Let $q : Y \to Z$ be a continuous map. Then the following statements are equivalent:*

(i) *q is a topologically onto quasihomeomorphism;*

(ii) *for each T_0-space X and each continuous map $f : Y \to X$, there is a unique continuous map $\tilde{f} : Z \to X$ such that $\tilde{f} \circ q = f$.*

Question 2.6. Give an intrinsic topological characterization of morphisms in Top rendered invertible by the functor F, where $F \in \{\mathbf{T}_1, \mathbf{T}_2\}$.

3. $T_{(0,D)}$-, $T_{(0,1)}$-, and $T_{(0,2)}$-spaces.

We begin by recalling the T_1-reflection. Let X be a topological space and R the intersection of all closed equivalence relations on X (an equivalence relation on X is said to be closed if its equivalence classes are closed in X). The quotient space X/R is homeomorphic to the T_1-reflection of X.

We begin with some straightforward examples and remarks.

Remark 3.1. (1) Since

$$T_2 \Longrightarrow T_1 \Longrightarrow T_D, \tag{3.1}$$

then we get

$$T_{(0,2)} \Longrightarrow T_{(0,1)} \Longrightarrow T_{(0,D)}. \tag{3.2}$$

(2) Each $T_{(0,2)}$-space is $T_{(1,2)}$.

(3) There is a $T_{(1,2)}$-space which is not $T_{(0,1)}$. Let X be the Sierpinski space. Then $\mathbf{T}_1(X)$ is a one-point set, thus X is a $T_{(1,2)}$-space. However, $\mathbf{T}_0(X) = X$; hence X is not a $T_{(0,1)}$-space.

(4) There is a $T_{(0,1)}$-space which is not $T_{(1,2)}$: it suffices to consider a T_1-space which is not T_2.

(5) There is a $T_{(0,1)}$-space which is not $T_{(0,2)}$: take a T_1-space which is not T_2.

(6) There is a $T_{(1,2)}$-space which is not $T_{(0,2)}$: the Sierpinski space does the job.

(7) There is a $T_{(0,D)}$-space which is not $T_{(0,1)}$: it suffices to consider a T_D-space which is not T_1.

(8) There is a $T_{(0,D)}$-space which is not $T_{(1,2)}$. The example in (4) does the job.

(9) There is a $T_{(1,2)}$-space which is not $T_{(0,D)}$: let Y be an infinite set. Let $w \notin Y$ and $X = Y \cup \{w\}$. Equip X with the topology whose closed sets are X and all finite subsets of Y. Clearly, $\mathbf{T}_1(X)$ is a one-point set; then X is a $T_{(1,2)}$-space. But X is not $T_{(0,D)}$-space, since X is T_0 and $\{w\}$ is not locally closed.

Remark 3.2. Let X be a topological space. Then $\mathbf{T}_0(X)$ is a T_1-space if and only if, for each $x \in X$, $\overline{\{x\}} = \mu_X^{-1}(\mu_X(\{x\}))$, where $\mu_X : X \to \mathbf{T}_0(X) = X/\sim$ is the canonical onto map.

For each point x of a space X, we denote by $\gamma(x)$ the set

$$\overline{\{x\}} \setminus \{y \in X : \overline{\{y\}} = \overline{\{x\}}\}. \tag{3.3}$$

With this notation, we have the following.

Theorem 3.3. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,D)}$-space;*

(ii) *for each $x \in X$, $\gamma(x)$ is a closed subset of X.*

The proof needs the following lemma.

Lemma 3.4. *Let $q : X \to Y$ be an onto quasihomeomorphism and C a subset of Y. Then the following statements are equivalent:*

(i) *C is locally closed in Y;*

(ii) *$q^{-1}(C)$ is locally closed in X.*

Proof. Let $\mathscr{LC}(X)$, $\mathscr{LC}(Y)$ be the sets of all locally closed subsets of X and Y, respectively.

It is well known [8] that the map $\mathscr{LC}(Y) \to \mathscr{LC}(X)$ defined by $F \mapsto q^{-1}(F)$ is bijective.

It is sufficient to show (ii)⇒(i).

Indeed, if $q^{-1}(C)$ is locally closed in X, then there is an $L \in \mathscr{LC}(Y)$ such that $q^{-1}(C) = q^{-1}(L)$; and since q is onto, we get $C = L$, proving that C is locally closed. □

Proof of Theorem 3.3. Let $\mu_X : X \to \mathbf{T}_0(X)$ be the canonical map from X to its $\mathbf{T}_0$-reflection $\mathbf{T}_0(X)$.

According to Lemma 3.4, X is a $T_{(0,D)}$-space if and only if $\mu_X^{-1}(\mu_X(\{x\}))$ is locally closed in X, for each $x \in X$.

One may check easily that $\overline{\mu_X^{-1}(\mu_X(\{x\}))} \setminus \mu_X^{-1}(\mu_X(\{x\})) = y(x)$. But it is well known that a subset S of a space X is locally closed if and only if $\overline{S} \setminus S$ is closed, completing the proof. $\square$

The following result gives a characterization of $T_{(0,1)}$-spaces.

Theorem 3.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,1)}$-space;*

(ii) *for each $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there is a neighborhood of x not containing y;*

(iii) *for each $x \in X$ and each closed subset C of X such that $\overline{\{x\}} \cap C \neq \emptyset$, $x \in C$;*

(iv) *for each open subset U of X and each $x \in U$, $\overline{\{x\}} \subseteq U$;*

(v) *for each $x \in X$, $\bigcap\{U : U \in \mathbb{O}(x)\} = \bigcap\{U : U \in \mathcal{V}(x)\} = \overline{\{x\}}$, where $\mathbb{O}(x)$ is the set of all open subsets of X containing x and $\mathcal{V}(x)$ is the set of all neighborhoods of X.*

Proof. Of course, for any topological space X and any $x \in X$, we have $\bigcap\{U : U \in \mathbb{O}(x)\} = \bigcap\{U : U \in \mathcal{V}(x)\}$.

We show the implications (i)⇔(ii), (i)⇒(iii)⇒(iv)⇒(i), and (i)⇒(v)⇒(iv).

(i)⇒(ii). Let $x, y \in X$ be such that $\overline{\{x\}} \neq \overline{\{y\}}$; then $\mu_X(x) \neq \mu_X(y)$. Since $\mathbf{T}_0(X)$ is a T_1-space, there exists a neighborhood U' of $\mu_X(x)$ not containing $\mu_X(y)$. Hence $\mu_X^{-1}(U')$ is a neighborhood of x not containing y.

(ii)⇒(i). Let $\mu_X(x)$, $\mu_X(y)$ be two distinct points in $\mathbf{T}_0(X)$. Then $\overline{\{x\}} \neq \overline{\{y\}}$. Hence there is a neighborhood U of x not containing y. Since μ_X is a quasihomeomorphism, there exists a neighborhood U' in $\mathbf{T}_0(X)$ such that $U = \mu_X^{-1}(U')$ and thus U' is a neighborhood of $\mu_X(x)$ not containing $\mu_X(y)$. Therefore $\mathbf{T}_0(X)$ is a T_1-space.

(i)⇒(iii). Let C be a closed subset of X and $x \in X$ such that $\overline{\{x\}} \cap C \neq \emptyset$. Since μ_X is a quasihomeomorphism, there exists a closed subset C_0 of $\mathbf{T}_0(X)$ such that $C = \mu_X^{-1}(C_0)$. Hence $\mu_X^{-1}(C_0 \cap \{\mu_X(x)\}) \neq \emptyset$; and thus $C_0 \cap \{\mu_X(x)\} \neq \emptyset$. Therefore, $\mu_X(x) \in C_0$; that is, $x \in C$.

(iii)⇒(iv). Let U be an open subset of X and, $x \in U$. Then $\overline{\{x\}} \cap X \setminus U = \emptyset$, and consequently, $\overline{\{x\}} \subseteq U$.

(iv)⇒(i). It is easily seen that $\mu_X^{-1}(\mu_X(\{x\})) \subseteq \overline{\{x\}}$. Conversely, let $y \in \overline{\{x\}}$. For each open subset U of X containing x, we have $\overline{\{x\}} \subseteq U$. Thus $y \in U$; so that $\overline{\{x\}} \subseteq \overline{\{y\}}$. Hence $\overline{\{x\}} = \overline{\{y\}}$; that is, $\mu_X(x) = \mu_X(y)$. This yields $\overline{\{x\}} \subseteq \mu_X^{-1}(\mu_X(\{x\}))$. Thus $\mu_X^{-1}(\mu_X(\{x\})) = \overline{\{x\}}$. Therefore, $\mathbf{T}_0(X)$ is a T_1-space, by Remark 3.2.

(i)⇒(v). Since $\mathbf{T}_0(X)$ is a T_1-space, we have $\overline{\{x\}} = \mu_X^{-1}(\{\mu_X(x)\})$ and $\{\mu_X(x)\} = \cap\{V : V \in \mathbb{O}(\mu_X(x))\}$, for each $x \in X$. On the other hand, since μ_X is a continuous surjective open map, we have $\mathbb{O}(\mu_X(x)) = \{\mu_X(U) : U \in \mathbb{O}(x)\}$. Therefore,

$$\overline{\{x\}} = \mu_X^{-1}\Big(\bigcap\{\mu_X(U) : U \in \mathbb{O}(x)\}\Big) = \bigcap\{\mu_X^{-1}(\mu_X(U)) : U \in \mathbb{O}(x)\}. \tag{3.4}$$

Moreover, since μ_X is a quasihomeomorphism, we have $\mu_X^{-1}(\mu_X(U)) = U$, for each open subset U of X (see Remark 2.1). It follows that $\overline{\{x\}} = \bigcap\{U : U \in \mathbb{O}(x)\}$.

(v)⇒(iv). Straightforward. □

Corollary 3.6. *The $T_{(0,1)}$-property is a productive and hereditary property: any product of $T_{(0,1)}$-spaces is $T_{(0,1)}$ and any subspace of a $T_{(0,1)}$-space is $T_{(0,1)}$.*

A subspace Y of X is called *irreducible* if each nonempty open subset of Y is dense in Y (equivalently, if C_1 and C_2 are two closed subsets of X such that $Y \subseteq C_1 \cup C_2$, then $Y \subseteq C_1$ or $Y \subseteq C_2$). Let C be a closed subset of a space X. We say that C *has a generic point* if there exists $x \in C$ such that $C = \overline{\{x\}}$.

Recall that a topological space X is said to be quasisober [14] (resp., sober [8]) if any nonempty irreducible closed subset of X has a generic point (resp., a unique generic point).

Lemma 3.7. *Let $q : X \to Y$ be a quasihomeomorphism.*

(1) *If X is a T_0-space, then q is one-to-one.*
(2) *If Y is a T_D-space, then q is onto.*
(3) *If Y is a T_D-space and X is a T_0-space, then q is a homeomorphism.*
(4) *If X is sober and Y is a T_0-space, then q is a homeomorphism.*

Proof. (1) Let x_1, x_2 be two points of X with $q(x_1) = q(x_2)$. Suppose that $x_1 \neq x_2$. Then there exists an open subset U of X such that, for example, $x_1 \in U$ and $x_2 \notin U$. Since there exists an open subset V of Y satisfying $q^{-1}(V) = U$, we get $q(x_1) \in V$ and $q(x_2) \notin V$, which is impossible. It follows that q is one-to-one.

(2) Let $y \in Y$. Then $\{y\}$ is a locally closed subset of Y. Hence $\{y\} \cap q(X) \neq \emptyset$, since $q(X)$ is strongly dense in Y. Thus $y \in q(X)$, proving that q is onto.

(3) One may check easily that bijective quasihomeomorphisms are homeomorphisms.

(4) By (1), q is one-to-one.

Now, observe that if S is a closed subset of Y, then S is irreducible if and only if so is $q^{-1}(S)$.

We prove that q is onto. To this end, let $y \in Y$. According to the above observation, $q^{-1}(\overline{\{y\}})$ is a nonempty irreducible closed subset of X. Hence $q^{-1}(\overline{\{y\}})$ has a generic point x. Thus we have the containments

$$\overline{\{x\}} \subseteq q^{-1}(\overline{\{q(x)\}}) \subseteq q^{-1}(\overline{\{y\}}) = \overline{\{x\}}. \tag{3.5}$$

So that $q^{-1}(\overline{\{q(x)\}}) = q^{-1}(\overline{\{y\}})$. It follows from the fact that q is a quasihomeomorphism that $\overline{\{q(x)\}} = \overline{\{y\}}$. Since Y is a T_0-space, we get $q(x) = y$. This proves that q is onto, and thus q is bijective. But a bijective quasihomeomorphism is a homeomorphism. □

Proposition 3.8. *Let $q : X \to Y$ be a quasihomeomorphism. If Y is a $T_{(0,1)}$-space, then so is X.*

Proof. Clearly, $\mathbf{T}_0(q) : \mathbf{T}_0(X) \to \mathbf{T}_0(Y)$ is a quasihomeomorphism. Hence, since $\mathbf{T}_0(X)$ is a T_0-space and $\mathbf{T}_0(Y)$ is a T_1-space, $\mathbf{T}_0(q)$ is a homeomorphism, by Lemma 3.7. Thus $\mathbf{T}_0(X)$ is a T_1-space, proving that X is a $T_{(0,1)}$-space. □

Example 3.9. A quasihomeomorphism $q : Y \to X$ such that Y is a $T_{(0,1)}$-space but X is not.

Take Y and X as in Remark 3.1(9). Then each nonempty locally closed subset of X meets Y. Hence, the canonical embedding $q : Y \to X$ is a quasihomeomorphism.

Of course, Y is a $T_{(0,1)}$-space. However, X is not a $T_{(0,1)}$-space. Indeed, for each $x \in X \setminus \{w\}$, $\overline{\{x\}} = \{x\}$ and $\overline{\{w\}} = X$; hence X is a T_0-space which is not T_1. Therefore, X is not $T_{(0,1)}$.

The following proposition follows immediately from Theorem 2.4.

Proposition 3.10. *Let $q : X \to Y$ be a topologically onto quasihomeomorphism. Then the following statements are equivalent:*

(i) *X is a $T_{(0,1)}$-space;*
(ii) *Y is a $T_{(0,1)}$-space.*

It is well known that a space X is a T_2-space if and only if, for each $x \in X$, $\bigcap\{\overline{U} : U \in \mathcal{V}(x)\} = \{x\}$, where $\mathcal{V}(x)$ is the set of all neighborhoods of x.

Before giving a characterization of $T_{(0,2)}$-spaces, we need a technical lemma.

Lemma 3.11. *Let $q : X \to Y$ be an onto quasihomeomorphism. Then the following properties hold.*

(1) *For each subset B of Y, $q^{-1}(\overline{B}) = \overline{q^{-1}(B)}$.*

(2) *For each $x \in X$, $\bigcap\{\overline{q^{-1}(V)} : V \in \mathcal{V}(q(x))\} = q^{-1}(\bigcap\{\overline{V} : V \in \mathcal{V}(q(x))\}) = \bigcap\{\overline{U} : U \in \mathcal{V}(x)\}$.*

Proof. (1) We observe that a continuous map $q : X \to Y$ is open if and only if, for each subset B of Y, we have $q^{-1}(\overline{B}) = \overline{q^{-1}(B)}$ (see [8, Chapter 0, (2.10.1)]). Now by Remark 2.1(2) an onto quasihomeomorphism is open, so that (1) follows immediately.

(2) Straightforward. □

Theorem 3.12. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,2)}$-space;*
(ii) *for each $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there are two disjoint open sets U and V in X with $x \in U$ and $y \in V$;*
(iii) *for each $x \in X$, $\bigcap\{\overline{U} : U \in \mathcal{V}(x)\} = \overline{\{x\}}$.*

Proof. (i)⇒(ii). Let $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, then $\mu_X(x) \neq \mu_X(y)$. Since $\mathbf{T}_0(X)$ is a T_2-space, there exist two disjoint open sets U' and V' in $\mathbf{T}_0(X)$ with $\mu_X(x) \in U'$ and $\mu_X(y) \in V'$. Therefore, $U = \mu_X^{-1}(U')$ and $V = \mu_X^{-1}(V')$ are two disjoint open sets in X with $x \in U$ and $y \in V$.

(ii)⇒(i). Let $\mu_X(x)$, $\mu_X(y)$ be two distinct points in $\mathbf{T}_0(X)$. Then $\overline{\{x\}} \neq \overline{\{y\}}$ and, by (ii), there are disjoint open sets U and V of X with $x \in U$ and $y \in V$. Since μ_X is a quasihomeomorphism, there exist two disjoint open sets U', V' of $\mathbf{T}_0(X)$ with $\mu_X(x) \in U'$ and $\mu_X(y) \in V'$ and such that $U = \mu_X^{-1}(U')$ and $V = \mu_X^{-1}(V')$, so that $\mathbf{T}_0(X)$ is a T_2-space.

(i)⇒(iii). Let μ_X be the canonical map from X onto its T_0-reflection. Then for each $x \in X$, we have $\bigcap\{\overline{V} : V \in \mathcal{V}(\mu_X(x))\} = \{\mu_X(x)\}$, since $\mathbf{T}_0(X)$ is Hausdorff.

According to Remark 3.2, $\mu_X^{-1}(\{\mu_X(x)\}) = \overline{\{x\}}$. Hence

$$\overline{\{x\}} = \bigcap\{\mu_X^{-1}(\overline{V}) : V \in \mathcal{V}(\mu_X(x))\}. \tag{3.6}$$

Thus, for each $x \in X$, we have $\bigcap\{\overline{U} : U \in \mathcal{V}(x)\} = \overline{\{x\}}$, by Lemma 3.11(2).

(iii)⇒(i). Let $x \in X$. First, we prove that

$$\bigcap\{U : U \in \mathcal{V}(x)\} = \bigcap\{\overline{U} : U \in \mathcal{V}(x)\} = \overline{\{x\}}. \tag{3.7}$$

Clearly, $\bigcap\{U : U \in \mathcal{V}(x)\} \subseteq \bigcap\{\overline{U} : U \in \mathcal{V}(x)\}$. Conversely, let $y \in \overline{\{x\}}$. Then $x \in V$, for each $V \in \mathcal{V}(y)$. Hence

$$x \in \bigcap\{\overline{V} : V \in \mathcal{V}(y)\} = \overline{\{y\}}. \tag{3.8}$$

Thus $y \in \bigcap\{U : U \in \mathcal{V}(x)\}$; so that (3.7) holds for each $x \in X$. Therefore, X is a $T_{(0,1)}$-space, by Theorem 3.5. Thus, according to Remark 3.2, $\overline{\{x\}} = \mu_X^{-1}(\{\mu_X(x)\})$. Applying Lemma 3.11, we have

$$\{\mu_X(x)\} = \bigcap\{\overline{V} : V \in \mathcal{V}(\mu_X(x))\}, \tag{3.9}$$

proving that $\mathbf{T}_0(X)$ is a T_2-space and thus X is a $T_{(0,2)}$-space. □

Corollary 3.13. *It is clear that the $T_{(0,2)}$-property is a productive and hereditary property.*

Proposition 3.14. *Let $q : X \to Y$ be a quasihomeomorphism. Then the following statements are equivalent:*

(i) *X is a $T_{(0,2)}$-space;*

(ii) *Y is a $T_{(0,2)}$-space.*

Proof. (i)⇒(ii). Clearly, $\mathbf{T}_0(q) : \mathbf{T}_0(X) \to \mathbf{T}_0(Y)$ is a quasihomeomorphism. On the other hand, since $\mathbf{T}_0(X)$ is a T_2-space, it is a sober space; and since in addition $\mathbf{T}_0(Y)$ is a T_0-space, then $\mathbf{T}_0(q)$ is a homeomorphism, by Lemma 3.7. Therefore, $\mathbf{T}_0(Y)$ is a T_2-space. This means that Y is a $T_{(0,2)}$-space.

(ii)⇒(i). Again $\mathbf{T}_0(q) : \mathbf{T}_0(X) \to \mathbf{T}_0(Y)$ is a quasihomeomorphism. Now, since $\mathbf{T}_0(X)$ is a T_0-space and $\mathbf{T}_0(Y)$ is a T_1-space, then $\mathbf{T}_0(q)$ is a homeomorphism, by Lemma 3.7. Therefore, $\mathbf{T}_0(X)$ is a T_2-space. This means that X is a $T_{(0,2)}$-space. □

Example 3.15. A quasihomeomorphism $q : X \to Y$ such that X is a T_2-space and Y is not a T_2-space.

Let $Y = \{0,1,2\}$ equipped with the topology $\{\emptyset, Y, \{1,2\}, \{0\}\}$ and let $X = \{0,1\}$ be provided with a discrete topology. Then X is a T_2-space and Y is not a T_2-space. The canonical embedding of X into Y does the job.

4. $T_{(0,S)}$-, $T_{(S,D)}$-, $T_{(S,1)}$-, and $T_{(S,2)}$-spaces. Let X be a topological space and $\mathbf{S}(X)$ the set of all nonempty irreducible closed subset of X [8]. Let U be an open subset of X; set $\widetilde{U} = \{C \in \mathbf{S}(X) : U \cap C \neq \emptyset\}$; then the collection $\{\widetilde{U} : U$ is an open subset of $X\}$ provides a topology on $\mathbf{S}(X)$ and the following properties hold [8].

(i) The map $\eta_X : X \to \mathbf{S}(X)$ which carries $x \in X$ to $\eta_X(x) = \overline{\{x\}}$ is a quasihomeomorphism.

(ii) $\mathbf{S}(X)$ is a sober space.

(iii) The topological space $\mathbf{S}(X)$ is called the soberification of X, and the assignment $\mathbf{S}(X)$ defines a functor from the category Top to itself.

(iv) Let $q : X \to Y$ be a continuous map, then the diagram

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\eta_Y"] \\ \mathbf{S}(X) \arrow[r, "\mathbf{S}(q)"] & \mathbf{S}(Y) \end{tikzcd} \tag{4.1}$$

is commutative.

In [14, Proposition 2.2], Hong has proved that a space is quasisober if and only if its T_0-reflection is sober. The following result makes [14, Proposition 2.2] more precise.

Theorem 4.1. *Let $q : X \to Y$ be a quasihomeomorphism. Then the following properties hold.*

(1) *If X is a $T_{(0,S)}$-space, then so is Y.*

(2) *Suppose that Y is a $T_{(0,S)}$-space. Then the following statements are equivalent:*

  (i) *X is a $T_{(0,S)}$-space;*

  (ii) *q is topologically onto.*

(3) *If q is topologically onto, then the following statements are equivalent:*

  (i) *X is a $T_{(0,S)}$-space;*

  (ii) *Y is a $T_{(0,S)}$-space.*

Proof. (1) Since $\mathbf{T}_0(q)$ is a quasihomeomorphism and $\mathbf{T}_0(X)$ is sober, we deduce that $\mathbf{T}_0(q)$ is a homeomorphism, by Lemma 3.7(4). Hence $\mathbf{T}_0(Y)$ is sober, proving that Y is a $T_{(0,S)}$-space.

(2) (i)⇒(ii). According to Lemma 3.7(4), $\mathbf{T}_0(q)$ is a homeomorphism. Thus q is topologically onto, by Theorem 2.4.

(ii)⇒(i). Again, according to Theorem 2.4, $\mathbf{T}_0(q)$ is a homeomorphism. Hence $\mathbf{T}_0(X)$ is sober, since $\mathbf{T}_0(Y)$ is.

(3) Combine (1) and (2). □

Proposition 4.2 [14, Proposition 2.2]**.** *A topological space is quasisober if and only if its T_0-reflection is sober.*

Proof. The canonical map $\mu_X : X \to \mathbf{T}_0(X)$ is an onto quasihomeomorphism. Then, applying Theorem 4.1(3), the proof is complete. □

Now, the result follows from the following simple facts: if $q : X \to Y$ is a quasihomeomorphism and X is quasisober, then Y is quasisober; if q is an onto quasihomeomorphism and Y is quasisober, then X is quasisober.

Remark 4.3. (1) Each $T_{(S,2)}$-space is a $T_{(S,1)}$-space.

(2) Each $T_{(S,1)}$-space is a $T_{(S,D)}$-space.

(3) Each $T_{(0,2)}$-space is a $T_{(0,S)}$-space.

(4) There is a $T_{(0,1)}$-space which is not a $T_{(0,S)}$-space. It suffices to consider a T_1-space which is not sober.

(5) There is a $T_{(S,1)}$-space which is not a $T_{(S,2)}$-space. It suffices to consider a sober T_1-space which is not T_2 (see [19, pages 675-676] and [13, pages 12-13]).

(6) There is a $T_{(0,S)}$-space which is not $T_{(S,1)}$-space.

Take X as in Remark 3.1(9). Note that a nonempty closed subset C of X is irreducible if and only if $C = X$ or $C = \{y\}$, where $y \in Y$, and thus X is a sober space. Therefore X is a $T_{(0,S)}$-space which is not a $T_{(S,1)}$-space.

Example 4.4. A quasihomeomorphism $q : Y \to X$ such that X is a $T_{(0,S)}$-space, but Y is not.

Take X and Y as in Remark 3.1(9). Then $\mathbf{T}_0(X) = X$ is a sober space, but $\mathbf{T}_0(Y) = Y$ is not because Y is a nonempty irreducible closed subset without a generic point. Thus the canonical embedding $Y \hookrightarrow X$ does the job.

We next give a characterization of $T_{(S,D)}$-spaces.

Theorem 4.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(S,D)}$-space;*

(ii) *X is a quasisober $T_{(0,D)}$-space.*

Proof. (i)⇒(ii). Let $\mu_X : X \to \mathbf{T}_0(X)$ (resp., $\eta_X : X \to \mathbf{S}(X)$) be the canonical map from X onto its T_0-reflection $\mathbf{T}_0(X)$ (resp., soberification $\mathbf{S}(X)$). Then the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbf{S}(X) \arrow[d, equal, "\mu_{\mathbf{S}(X)}"] \\ \mathbf{T}_0(X) \arrow[r, "\mathbf{T}_0(\eta_X)"] & \mathbf{T}_0(\mathbf{S}(X)) \end{tikzcd} \tag{4.2}$$

is commutative.

Hence $\mathbf{T}_0(\eta_X)$ is a quasihomeomorphism. Thus $\mathbf{T}_0(X)$ is homeomorphic to the subspace $\mathbf{T}_0(\eta_X)(\mathbf{T}_0(X))$, since $\mathbf{T}_0(X)$ is a T_0-space.

According to [13, Theorem 2.2], every subspace of a sober T_D-space is sober. Thus $\mathbf{T}_0(X)$ is sober and, consequently, $\mathbf{T}_0(\eta_X)$ is a homeomorphism, by Lemma 3.7. Therefore, $\mathbf{T}_0(X)$ is a sober T_D-space; this means that X is a $T_{(0,S)}$- and a $T_{(0,D)}$-space; but the axiom $T_{(0,S)}$ is equivalent to quasisober, by Proposition 4.2.

(ii)⇒(i). By Proposition 4.2, $\mathbf{T}_0(X)$ is a sober space. On the other hand, X is quasihomeomorphic to $\mathbf{T}_0(X)$; then $\mathbf{S}(X)$ is homeomorphic to $\mathbf{S}(\mathbf{T}_0(X))$, by [2, Theorem 2.2]. Thus $\mathbf{S}(X)$ is homeomorphic to $\mathbf{T}_0(X)$ so that $\mathbf{S}(X)$ is a T_D-space, proving that X is a $T_{(S,D)}$-space. □

Now, we give a characterization of $T_{(S,1)}$-spaces.

Theorem 4.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(S,1)}$-space;*

(ii) *whenever F and G are distinct nonempty irreducible closed subsets of X, there is an open subset U of X such that $U \cap F \neq \emptyset$ and $U \cap G = \emptyset$;*

(iii) *for each nonempty irreducible closed subsets F and G of X, $F \subseteq G$ if and only if $F = G$.*

Proof. (i)⇒(ii). Let F and G be two distinct nonempty irreducible closed subsets of X. Since $\mathbf{S}(X)$ is a T_1-space and $F \neq G$, there exists an open set $\tilde{U}$ of $\mathbf{S}(X)$ such that $F \in \tilde{U}$ and $G \notin \tilde{U}$. Thus U is an open subset of X with $U \cap F \neq \emptyset$ and $U \cap G = \emptyset$.

(ii)⇒(iii). Straightforward.

(iii)⇒(i). First, note that if H is a closed subset of X, then $\{G \in \mathbf{S}(X) : G \subseteq H\}$ is a closed subset of $\mathbf{S}(X)$. Now, let F be a nonempty irreducible closed subset of X. Then, by (iii), $\{G \in \mathbf{S}(X) : G \subseteq F\} = \{F\}$ is a closed subset of $\mathbf{S}(X)$. Hence each point of $\mathbf{S}(X)$ is closed. Therefore, X is a $T_{(S,1)}$-space. □

We finish this section by characterizing $T_{(S,2)}$-spaces.

Theorem 4.7. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(S,2)}$-space;*

(ii) *whenever F and G are distinct nonempty irreducible closed subsets of X, there are disjoint open subsets U and V of X such that $F \cap U \neq \emptyset$ and $G \cap V \neq \emptyset$.*

Proof. (i)⇒(ii). Let F and G be two distinct nonempty irreducible closed subsets of X. Since $\mathbf{S}(X)$ is a T_2-space and $F \neq G$, then there exist two disjoint open subsets $\tilde{U}$ and $\tilde{V}$ in $\mathbf{S}(X)$ such that $F \in \tilde{U}$ and $G \in \tilde{V}$, so that the two open sets U and V satisfy (ii).

(ii)⇒(i). Let F and G be two distinct points in $\mathbf{S}(X)$ and let U and V be as in (ii). Then it is clear that $F \in \tilde{U}$ and $G \in \tilde{V}$. Clearly $\tilde{U} \cap \tilde{V} = \widetilde{U \cap V} = \tilde{\emptyset} = \emptyset$; therefore, X is a $T_{(S,2)}$-space. □

Proposition 4.8. *Let $q : X \to Y$ be a quasihomeomorphism. Then the following statements are equivalent:*

(i) *X is a $T_{(S,1)}$- (resp., $T_{(S,D)}$-, resp., $T_{(S,2)}$-)space;*

(ii) *Y is a $T_{(S,1)}$- (resp., $T_{(S,D)}$-, resp., $T_{(S,2)}$-)space.*

Proof. This follows easily from the fact that $\mathbf{S}(q) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is a homeomorphism, by [2, Theorem 2.2]. □

5. The Wallman compactification. The Wallman compactification of a T_1-space is introduced and studied by Wallman [20] as follows.

Let X be a T_1-space, and let wX be the collection of all closed ultrafilters on X. For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{U} \in wX : D \in \mathcal{U}\}$ if $D \neq \emptyset$ and $\emptyset^* = \emptyset$. Then $\{D^* : D$ is a closed subset of $X\}$ is a base for the closed sets of a topology on wX. Let U be an open subset of X; we define $U^* \subseteq wX$ to be the set $U^* = \{\mathcal{U} \in wX : A \subseteq U$ for some A in $\mathcal{U}\}$. The class $\{U^* : U$ is an open subset of $X\}$ is a base for the open sets of the topology of wX.

The following properties are well known and may be found in any standard textbook on general topology (see, e.g., Kelley [16]).

Properties 5.1. Let X be a T_1-space. Consider the map $\omega_X : X \to wX$ which takes $x \in X$ to $\omega_X(x) = \{A : A$ is a closed subset of X and $x \in A\}$. Then the following properties hold.

(1) If D is closed in X, then $\overline{\omega_X(D)} = D^*$. In particular, $\omega_X(X)$ is dense in wX.
(2) ω_X is continuous and it is an embedding of X in wX if and only if X is a T_1-space.
(3) If A and B are closed subsets of X, then $\overline{\omega_X(A\cap B)} = \overline{\omega_X(A)} \cap \overline{\omega_X(B)}$.
(4) wX is a compact T_1-space.
(5) Every continuous map on X to a compact Hausdorff space K can be extended to wX.

For a $T_{(0,1)}$-space X, we define $WX = w(\mathbf{T}_0(X))$ and we call it the Wallman compactification of X. The notation wX is reserved only for T_1-spaces so that it is better to use some other notation for $T_{(0,1)}$-spaces; the same for ω_X: ω_X is reserved for T_1-spaces; for $T_{(0,1)}$-spaces, we define $w_X = \omega_{\mathbf{T}_0(X)} \circ \mu_X$.

Since μ_X is an onto quasihomeomorphism, one obtains immediately that WX can be described exactly as wX is for T_1-spaces. Properties 5.1 are also true for $T_{(0,1)}$-spaces.

Remark 5.2. Let X be a $T_{(0,1)}$-space. Then the following properties hold:

(1) For each open subset U of X, we have $w_X(U) \subseteq U^*$.
(2) For each closed subset D of X, we have $w_X(D) \subseteq D^*$.
(3) Let U be open and D closed in a $T_{(0,1)}$-space. Then $U \cap D \neq \emptyset$ if and only if $U^* \cap D^* \neq \emptyset$.

Now we give some new observations about Wallman compactifications.

Proposition 5.3. *Let X be a $T_{(0,1)}$-space and U an open or closed subset of X. If U is compact, then $U^* = w_X(U)$.*

Proof. Suppose that U is open in X. Let $\mathcal{U} \in U^*$. Then there exists $F \in \mathcal{U}$ such that $F \subseteq U$. F is compact, by the compactness of U. Thus $\bigcap\{H \bigcap F : H \in \mathcal{U}\} \neq \emptyset$. Let $x \in \bigcap\{H \cap F : H \in \mathcal{U}\}$; then $\mathcal{U} = w_X(x)$. Hence, $\mathcal{U} \in w_X(U)$.

On the other hand, according to Remark 5.2, $w_X(U) \subseteq U^*$. Therefore, $w_X(U) = U^*$.

Now suppose that U is closed in X. Let $\mathcal{U} \in U^*$; then $U \in \mathcal{U}$. Since $\bigcap\{H : H \in \mathcal{U}\} \neq \emptyset$, pick an $x \in \bigcap\{H : H \in \mathcal{U}\}$. It is easily seen that $\mathcal{U} = w_X(x)$. Therefore, according to Remark 5.2, $w_X(U) = U^*$. □

Corollary 5.4. *Let U be a compact open subset of X. Then U^* is compact.*

Remark 5.5. The converse of Corollary 5.4 does not hold. Let X be a noncompact $T_{(0,1)}$-space. Thus $X^* = WX$ is compact; however, X is not compact.

Question 5.6. Let U be an open subset of a $T_{(0,1)}$-space with $U \neq \emptyset$ and $U \neq X$. If we suppose that U^* is compact, is U compact?

Proposition 5.7. *Let D be a closed subset of a $T_{(0,1)}$-space. Then D is irreducible if and only if D^* is irreducible in WX.*

Proof. Let D be an irreducible closed subset of X. Since $D^* = \overline{w_X(D)}$, D^* is irreducible.

Conversely, suppose that D^* is irreducible. Let U, V be two open subsets of X such that $U \cap D \neq \emptyset$ and $V \cap D \neq \emptyset$. Hence $U^* \cap D^* \neq \emptyset$ and $V^* \cap D^* \neq \emptyset$, by Remark 5.2. Thus $U^* \cap V^* \cap D^* \neq \emptyset$, so that $(U \cap V)^* \cap D^* \neq \emptyset$. Therefore, $U \cap V \cap D \neq \emptyset$, by Remark 5.2. □

We need to introduce new concepts.

Definition 5.8. (1) A subset C of a topological space X is said to be closedly dense if C meets each nonempty closed subset of X.

(2) A subset S of a topological space X is said to be sufficiently dense if S meets each nonempty closed subset and each nonempty open subset of X.

(3) By an almost-homeomorphism (a-homeomorphism, for short), we mean a continuous map $q : X \to Y$ such that $q(X)$ is sufficiently dense in Y and the topology of X is the inverse image of that of Y by q.

(4) By a Wallman morphism (W-morphism, for short), we mean a continuous map $q : X \to Y$ such that $q(X)$ is closedly dense in Y and the topology of X is the inverse image of that of Y by q.

Thus we have the following implications:

$$\begin{array}{ccccc} \text{Strongly dense} & \Rightarrow & \text{Sufficiently dense} & \Rightarrow & \text{closedly dense} \\ & & \Downarrow & & \\ & & \text{Dense} & & \end{array}$$

$$\text{Homeomorphism} \Rightarrow \text{quasihomeomorphism} \Rightarrow a\text{-homeomorphism} \Rightarrow W\text{-morphism}. \tag{5.1}$$

The converses fail, as shown by the following examples.

Example 5.9. Consider $X = [0,\omega]$ to be the set of all ordinal numbers less than or equal to the first limit ordinal ω. We equip X with the natural order $\leq$.

The discrete Alexandroff topology on X associated to the reverse order is $\mathbb{O}(X) = \{\emptyset, X, [0,\omega[\} \cup \{(\downarrow x) : x \in X\}$, where $(\downarrow x) = \{y \in X : y \leq x\}$. Set $D = [0,\omega[$, $C = X - \{0\}$ and $K = \{0,\omega\}$.

Note that usually the Alexandroff topology is defined by the upper sets (see Johnstone [15]); here the topology used is associated with the reverse order.

(a) Since $\{\omega\}$ is a closed subset of X and $\{\omega\} \cap D = \emptyset$, D is not closedly dense in X. However, D is a dense subset of X.

(b) Since $\{0\}$ is open and $\{0\}\cap C = \emptyset$, C is not dense in X. However, C is closedly dense in X.

(c) The subset K is sufficiently dense but not strongly dense in X.

(d) It is easily seen that the canonical embedding $i_C : C \to X$ is a W-morphism which is not an a-homeomorphism. The canonical embedding $i_K : K \to X$ is an a-homeomorphism which is not a quasihomeomorphism.

(e) It is well known that a quasihomeomorphism need not be a homeomorphism.

Remark 5.10. An onto W-morphism $q : X \to Y$ is a quasihomeomorphism. Indeed it suffices to show that if F and G are two closed subsets of Y such that $q^{-1}(F) = q^{-1}(G)$, then $F = G$ which is clear since q is an onto map.

Proposition 5.11. (1) *The composite of two a-homeomorphisms (resp., W-morphisms) is an a-homeomorphism (resp., W-morphism).*

(2) *If $q : X \to Y$ is a W-morphism and X is T_0, then q is one-to-one.*

(3) *If $q : X \to Y$ is a W-morphism and Y is T_1, then q is an onto quasihomeomorphism.*

(4) *If q is a W-morphism and if X is T_0 and Y is T_1, then q is a homeomorphism.*

Proof. (1) We show (1) for two a-homeomorphisms. Let $p : X \to Y$ and $q : Y \to Z$ be two a-homeomorphisms. Clearly, the topology of X is the inverse image of that of Z by $q \circ p$.

Let A be a closed (resp., an open) subset of Z. Since $q^{-1}(A)$ is closed (resp., open) in Y, then $p(X)\cap q^{-1}(A) \neq \emptyset$, so that $A\cap q(p(X)) \neq \emptyset$. Hence $q \circ p$ is an a-homeomorphism.

(2) and (3) have the same proof as Lemma 3.7.

(4) follows immediately from (2), (3), and Remark 5.10. □

Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps. One may check easily that if two among the three maps $g \circ f$, f, g are quasihomeomorphisms, then so is the third one (see Remark 2.1).

For a-homeomorphisms and W-morphisms, we get the following result which has a straightforward proof.

Proposition 5.12. *Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps.*

(1) *Suppose that $g \circ f$ and g are a-homeomorphisms (resp., W-morphisms). Then f is an a-homeomorphism (resp., a W-morphism).*

(2) *Suppose that $g \circ f$ is an a-homeomorphism (resp., a W-morphism) and f is a quasihomeomorphism. Then g is an a-homeomorphism (resp., a W-morphism).*

The following example shows that Remark 2.1(1) fails to be true for a-homeomorphisms or W-morphisms.

Example 5.13. Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps such that $g \circ f$ and f are a-homeomorphisms (resp., W-morphisms). Then g is not necessarily an a-homeomorphism (resp., a W-morphism).

In fact, let $X = [0,\omega]$ be equipped with the discrete Alexandroff topology as in Example 5.9, $Y = \{0,\omega\}$ and $Z = \{0,1,\omega\}$ considered as subspaces of X.

Let $f:\{0,\omega\}\to\{0,1,\omega\}$ be the canonical embedding and let $g:\{0,1,\omega\}\to X$ be defined by $g(0)=0$, $g(1)=0$, $g(\omega)=\omega$. Clearly, $g\circ f$ and f are a-homeomorphisms. However, since the closed subset $\{1,\omega\}$ of $\{0,1,\omega\}$ is not an inverse image by g of a closed subset of X, g is not a W-morphism.

Recall from [9] that a continuous map $q:X\to Y$ between T_1-spaces is said to be a *w-extension* if there is a continuous map $w(q):wX\to wY$ such that $\omega_Y\circ q = w(q)\circ\omega_X$. In an analogous manner, one may define *W-extensions* for $T_{(0,1)}$-spaces.

The following gives a class of morphisms $q:X\to Y$ which yield a W-extension $W(q):WX\to WY$ that is a homeomorphism.

Proposition 5.14. *Let X, Y be two $T_{(0,1)}$-spaces and $q:X\to Y$ a W-morphism. Then q has a W-extension which is a homeomorphism.*

Proof. We first remark that diagram (2.1) commutes. Hence $\mathbf{T}_0(q)\circ\mu_X=\mu_Y\circ q$. Thus $\mathbf{T}_0(q)\circ\mu_X$ is a W-morphism. Now, since μ_X is a quasihomeomorphism, $\mathbf{T}_0(q)$ is a W-morphism, by Proposition 5.12(2). Therefore, $\mathbf{T}_0(q)$ is a homeomorphism, by Proposition 5.11(4). It follows that $\mathbf{T}_0(q)$ has a canonical w-extension $w(\mathbf{T}_0(q))$ which is a homeomorphism. Thus the diagram

$$\begin{tikzcd}
\mathbf{T}_0(X) \arrow[r, "\mathbf{T}_0(q)"] \arrow[d, "\omega_{\mathbf{T}_0(X)}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbf{T}_0(Y) \arrow[d, "\omega_{\mathbf{T}_0(Y)}"] \\
w(\mathbf{T}_0(X))=WY \arrow[r, "\omega(\mathbf{T}_0(q))"] & w(\mathbf{T}_0(Y))=WY
\end{tikzcd} \tag{5.2}$$

commutes.

If we denote $W(q)=w(\mathbf{T}_0(q))$, then the above diagrams indicate clearly that $W(q)$ is a W-extension of q which is a homeomorphism. □

It is well known that the Wallman compactification of a T_1-space X is Hausdorff if and only if X is normal and in this case $wX=\beta(X)$ (the Stone-Čech compactification of X) (see, e.g., Wallman [20]).

Corollary 5.15. *WX is Hausdorff if and only if $\mathbf{T}_0(X)$ is a normal space. In this case $WX=\beta(\mathbf{T}_0(X))$.*

Remark 5.16. If a continuous map $q:X\to Y$ has a W-extension which is a homeomorphism, then q need not be a homeomorphism. To see this it suffices to take a noncompact T_1-space X. Of course, 1_{wX} is a w-extension of ω_X; however, ω_X is not a homeomorphism.

Definition 5.17. Let X be a $T_{(0,1)}$-space and Y a subspace of X.

(1) It is said that Y is a Wallman generator (W-generator, for short) of X, if WY is homeomorphic to WX.

(2) Y is called a *strong Wallman generator* (*sW-generator*, for short) of X if the canonical embedding $i: Y \hookrightarrow X$ has a W-extension $W(i)$ which is a homeomorphism.

Clearly, each sW-generator is a W-generator.

As an immediate consequence of Proposition 5.14, we get the following examples.

Example 5.18. Let X be a $T_{(0,1)}$-space.

(1) $w_X(X)$ is an sW-generator of WX.

(2) Each closedly dense subset of X is an sW-generator of X.

We think that it is of interest to answer the following questions.

Questions 5.19. Let X be a $T_{(0,1)}$-space.

(1) Characterize W-generators of X.

(2) Characterize sW-generators of X.

(3) Is there a W-generator which is not an sW-generator?

Now we are in a position to give a characterization of continuous maps between $T_{(0,1)}$-spaces having an extension to Wallman compactifications that is a homeomorphism.

Theorem 5.20. *Let X, Y be two $T_{(0,1)}$-spaces and $q: X \to Y$ a continuous map. Then the following statements are equivalent:*

(i) *q has a W-extension which is a homeomorphism;*

(ii) *$q(X)$ is an sW-generator of Y and the topology of X is the inverse image of that of Y by q.*

Proof. (i)⇒(ii). (a) *The topology of X is the inverse image of that of Y by q.* Let C be a closed subset of X. Since $W(q)$ is a homeomorphism, $W(q)(C^*) = K$ is a closed subset of WY.

Set $H = w_Y^{-1}(K)$. We prove that $C = q^{-1}(H)$.

(1) Let $x \in C$. Then $w_X(x) \in w_X(C) \subseteq C^*$. Hence $W(q)(w_X(x)) \in W(q)(C^*) = K$, which gives $w_Y(q(x)) \in K$. It follows that $q(x) \in w_Y^{-1}(K) = H$. Therefore, $x \in q^{-1}(H)$.

(2) Conversely, let $x \in q^{-1}(H)$. Then $q(x) \in H = w_X^{-1}(K)$; this means that $(w_Y \circ q)(x) \in K$, so that $W(q)(w_X(x)) \in K = W(q)(C^*)$. Since $W(q)$ is bijective, $w_X(x) \in C^*$. Hence $x \in w_X^{-1}(C^*) = C$. We have thus proved that $C = q^{-1}(H)$. In other words, the topology of X is the inverse image of that of Y by q.

(b) *$q(X)$ is an sW-generator of Y.* According to (a), the induced map $q_1: X \to q(X)$ by q is a W-morphism. Hence q_1 has a W-extension $W(q_1)$ which is a homeomorphism, by Proposition 5.14. Thus the diagrams

$$\begin{tikzcd} X \arrow[r, "q_1"] \arrow[d, "w_X"'] \arrow[dr, phantom, "\circlearrowleft"] & q(X) \arrow[d, "w_{q(X)}"] \\ WX \arrow[r, "W(q_1)"] & Wq(X) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q"] \arrow[d, "w_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "w_Y"] \\ WX \arrow[r, "W(q)"] & WY \end{tikzcd} \tag{5.3}$$

commute.

Let $j : q(X) \hookrightarrow Y$ be the canonical embedding. Clearly, the diagram

$$\begin{tikzcd} q(X) \arrow[r, "j"] \arrow[d, "w_{q(X)}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "w_Y"] \\ wq(X) \arrow[r, "W(q)\circ(W(q_1))^{-1}"] & WY \end{tikzcd} \tag{5.4}$$

commutes. Therefore, j has $W(q)\circ(W(q_1))^{-1}$ as a W-extension which is a homeomorphism. This means that $q(X)$ is an sW-generator of Y.

(ii)⇒(i). Under the assumptions of (ii), the induced map $q_1 : X \to q(X)$ by q is a W-morphism. Thus, according to Proposition 5.14, q_1 has a W-extension $W(q_1)$ which is a homeomorphism. On the other hand, the canonical embedding $j : q(X) \hookrightarrow Y$ has a W-extension which is a homeomorphism, by Proposition 5.14. It follows that the two diagrams

$$\begin{tikzcd} X \arrow[r, "q_1"] \arrow[d, "w_X"'] \arrow[dr, phantom, "\circlearrowleft"] & q(X) \arrow[r, "j"] \arrow[d, "w_{q(X)}"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "w_Y"] \\ WX \arrow[r, "W(q_1)"] & Wq(X) \arrow[r, "W(j)"] & WY \end{tikzcd} \tag{5.5}$$

commute. Therefore, $W(j)\circ W(q_1)$ is a W-extension of $q : X \to Y$ which is a homeomorphism. □

Question 5.21. Is it possible to replace the word "sW-generator" in Theorem 5.20 by "W-generator"?

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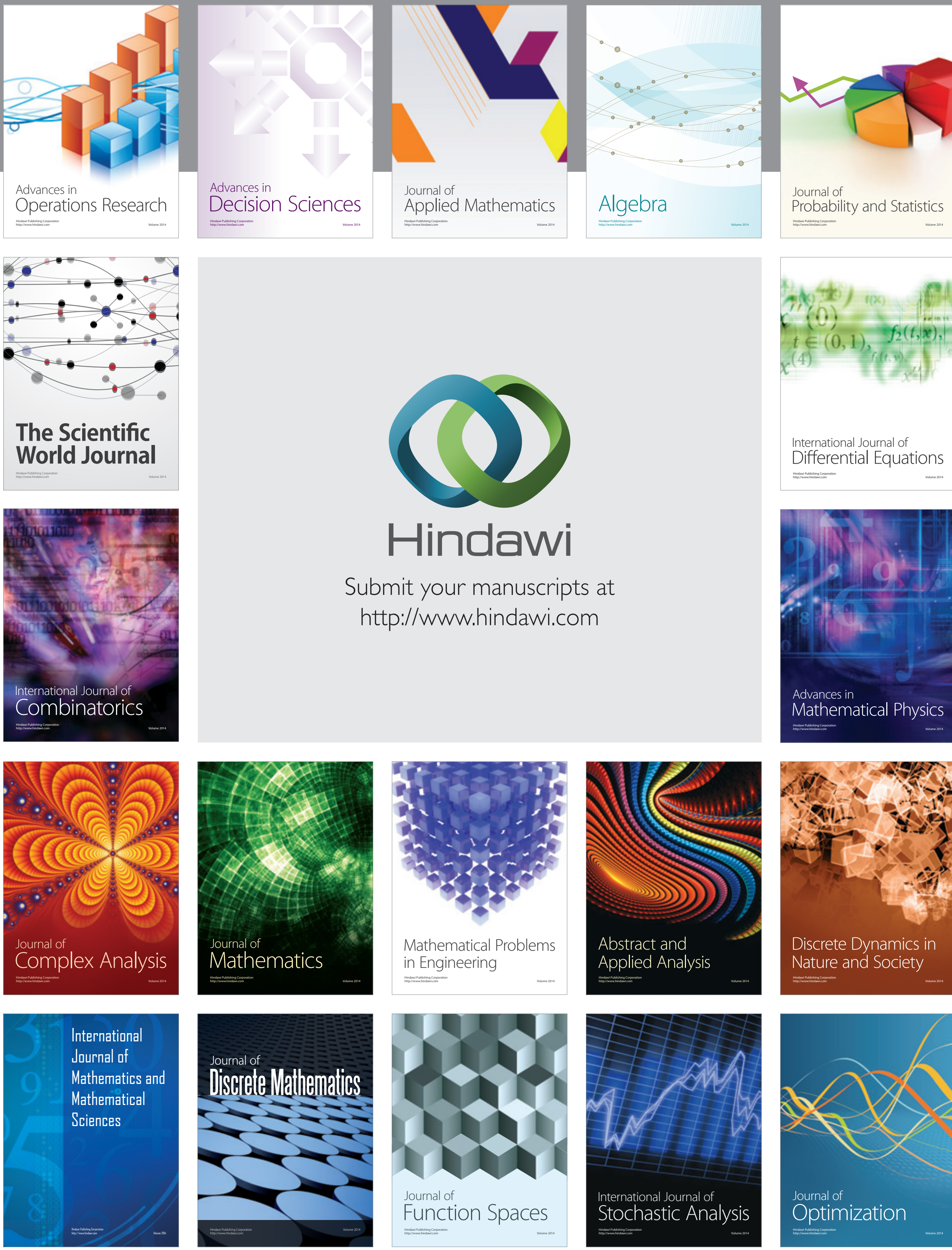
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