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ON THE CARLEMAN CLASSES OF VECTORS OF A SCALAR TYPE SPECTRAL OPERATOR

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The Carleman classes of a scalar type spectral operator in a reflexive Banach space are characterized in terms of the operator's resolution of the identity. A theorem of the Paley-Wiener type is considered as an application.

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1. Introduction. As was shown in [8] (see also [9, 10]), under certain conditions, the *Carleman classes* of vectors of a *normal operator* in a complex Hilbert space can be characterized in terms of the operator's *spectral measure* (the *resolution of the identity*).

The purpose of the present paper is to generalize this characterization to the case of a *scalar type spectral operator* in a complex *reflexive* Banach space.

2. Preliminaries

2.1. The Carleman classes of vectors. Let A be a linear operator in a Banach space X with norm $\|\cdot\|$, $\{m_n\}_{n=0}^{\infty}$ a sequence of positive numbers, and

$$C^\infty(A) \overset{\text{def}}{=} \bigcap_{n=0}^{\infty} D(A^n) \tag{2.1}$$

($D(\cdot)$ is the *domain* of an operator).

The sets

$$\begin{aligned} C_{\{m_n\}}(A) &\overset{\text{def}}{=} \{f \in C^\infty(A) \mid \exists\alpha > 0,\ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0,1,2,\dots\}, \\ C_{(m_n)}(A) &\overset{\text{def}}{=} \{f \in C^\infty(A) \mid \forall\alpha > 0\ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0,1,2,\dots\} \end{aligned} \tag{2.2}$$

are called the *Carleman classes* of vectors of the operator A corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$ of *Roumie's* and *Beurling's types*, respectively.

Obviously, the inclusion

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \tag{2.3}$$

holds.

For $m_n := [n!]^\beta$ (or, due to *Stirling's formula*, for $m_n := n^{\beta n}$), $n = 0,1,2,\dots$ ($0 \le \beta < \infty$), we obtain the well-known βth-order *Gevrey classes* of vectors, $\mathscr{E}^{\{\beta\}}(A)$ and $\mathscr{E}^{(\beta)}(A)$, respectively. In particular, $\mathscr{E}^{\{1\}}(A)$ are the *analytic* and $\mathscr{E}^{(1)}(A)$ are the *entire* vectors of the operator A [7, 17].

The sequence $\{m_n\}_{n=0}^{\infty}$ will be subject to the following condition.

(WGR) For any $\alpha > 0$, there exist such a $C = C(\alpha) > 0$ that

$$C\alpha^n \le m_n, \quad n = 0,1,2,\dots. \tag{2.4}$$

Note that the name WGR originates from the words *"weak growth."*

Under this condition, the numerical function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty, \ (0^0 := 1), \tag{2.5}$$

first introduced by Mandelbrojt [15], is well defined.

This function is *nonnegative*, *continuous*, and *increasing*.

As established in [8] (see also [9, 10]), for a *normal operator* A with a *spectral measure* $E_A(\cdot)$ in a complex Hilbert space H with inner product $(\cdot,\cdot)$ and the sequence $\{m_n\}_{n=0}^{\infty}$ satisfying the condition (WGR),

$$\begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)), \end{aligned} \tag{2.6}$$

the normal operators $T(t|A|)$ $(0 < t < \infty)$ being defined in the sense of the spectral *operational calculus* for a normal operator:

$$\begin{aligned} T(t|A|) &:= \int_{\sigma(A)} T(t|\lambda|)\,dE_A, \\ D(T(t|A|)) &:= \left\{ f \in H \mid \int_{\sigma(A)} T^2(t|\lambda|)(dE_A(\lambda)f,f) < \infty \right\}, \end{aligned} \tag{2.7}$$

where the function $T(\cdot)$ can be replaced by any *nonnegative*, *continuous*, and *increasing* function $L(\cdot)$ defined on $[0,\infty)$ such that

$$c_1 L(\gamma_1\lambda) \le T(\lambda) \le c_2 L(\gamma_2\lambda), \quad \lambda > R, \tag{2.8}$$

with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

In particular, $T(\cdot)$ in (2.6) is replaceable by

$$S(\lambda) := m_0 \sup_{n\ge 0} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty, \tag{2.9}$$

or

$$P(\lambda) := m_0\left[\sum_{n=0}^{\infty}\frac{\lambda^{2n}}{m_n^2}\right]^{1/2}, \quad 0 \le \lambda < \infty, \tag{2.10}$$

(see [10]).

2.2. Carleman ultradifferentiability. Let I be an interval of the real axis, $\mathbb{C}^\infty(I)$ the set of all complex-valued functions strongly infinite differentiable on I, and $\{m_n\}_{n=0}^\infty$ a sequence of positive numbers.

$$C_{\{m_n\}}(I) \stackrel{\text{def}}{=} \begin{cases} \Big\{f(\cdot)\in C^\infty(I) \mid \forall [a,b]\subseteq I,\ \exists\alpha>0,\ \exists c>0: \\ \max_{a\le x\le b}\|f^{(n)}(x)\| \le c\alpha^n m_n,\ n=0,1,2,\dots\Big\}, \\ \Big\{f(\cdot)\in C^\infty(I) \mid \forall [a,b]\subseteq I,\ \forall\alpha>0,\ \exists c>0: \\ \max_{a\le x\le b}\|f^{(n)}(x)\| \le c\alpha^n m_n,\ n=0,1,2,\dots\Big\} \end{cases} \tag{2.11}$$

are the *Carleman classes of ultradifferentiable functions* of *Roumie's* and *Beurling's types*, respectively, [1, 12, 13, 14].

In particular, for $m_n := [n!]^\beta$ (or, due to *Stirling's formula*, for $m_n := n^{\beta n}$), $n = 0,1,2,\dots$ $(0 \le \beta < \infty)$, these are the well-known βth-order *Gevrey classes*, $\mathcal{E}^{\{\beta\}}(I)$ and $\mathcal{E}^{(\beta)}(I)$, respectively, [6, 12, 13, 14].

Observe that $\mathcal{E}^{\{1\}}(I)$ is the class of the *real analytic* on I functions and $\mathcal{E}^{(1)}(I)$ is the class of *entire* functions, that is, the restrictions to I of *analytic* and *entire* functions, correspondingly, [15].

Note that condition (WGR), in particular, implies that $\lim_{n\to\infty} m_n = \infty$. Since, as is easily seen, the *Carleman classes* of vectors and functions coincide for the sequence $\{m_n\}_{n=1}^\infty$ and the sequence $\{dm_n\}_{n=1}^\infty$ for any $d > 0$, without loss of generality, we can regard that

$$\inf_{n\ge 0} m_n \ge 1. \tag{2.12}$$

2.3. Scalar type spectral operators. Henceforth, unless specified otherwise, A is a *scalar type spectral operator* in a complex Banach space X with norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (the *resolution of the identity*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [2, 5].

Note that, in a Hilbert space, the *scalar type spectral operators* are those similar to the *normal* ones [21].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on $\mathbb{C}$ (on $\sigma(A)$) [2, 5], $F(\cdot)$ being such a function; a new *scalar type spectral operator*

$$F(A) = \int_{\mathbb{C}} F(\lambda)\,dE_A(\lambda) = \int_{\sigma(A)} F(\lambda)\,dE_A(\lambda) \tag{2.13}$$

is defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \mid \lim_{n\to\infty} F_n(A)f \text{ exists} \right\} \tag{2.14}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A) \mid |F(\lambda)|\le n\}}(\cdot), \quad n = 1,2,\ldots, \tag{2.15}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1,2,\ldots, \tag{2.16}$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar type spectral operators* on X defined in the same manner as for *normal operators* (see, e.g., [4, 19]).

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying the entire subsequent argument are exhaustively delineated in [2, 5]. We just observe here that, due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [3], that is, there is an $M > 0$ such that, for any Borel set δ,

$$\|E_A(\delta)\| \le M. \tag{2.17}$$

Observe that, in (2.17), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on X. We will adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the dual space X^* as well.

Due to (2.17), for any $f \in X$ and $g^* \in X^*$ (X^* is the *dual space*), the total variation $v(f,g^*,\cdot)$ of the complex-valued measure $\langle E_A(\cdot)f,g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space X and its dual, X^*) is *bounded*. Indeed, δ being an arbitrary Borel subset of $\sigma(A)$, [3],

$$\begin{aligned} &v(f,g^*,\sigma(A)) \\ &\quad\le 4 \sup_{\delta\subseteq\sigma(A)} |\langle E_A(\delta)f,g^*\rangle| \le 4 \sup_{\delta\subseteq\sigma(A)} \|E_A(\delta)\|\|f\|\|g^*\| \quad \text{(by (2.17))} \\ &\quad\le 4M\|f\|\|g^*\|. \end{aligned} \tag{2.18}$$

For the reader's convenience, we reformulate here [16, Proposition 3.1], heavily relied upon in what follows, which allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive measures (see [16] for a complete proof).

On account of compactness, the terms *spectral measure* and *operational calculus* for scalar type spectral operators, frequently referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

Proposition 2.1. *Let A be a scalar type spectral operator in a complex Banach space X and $F(\cdot)$ a complex-valued Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$). Then $f \in D(F(A))$ if and only if*

(i) *for any $g^* \in X^*$,*

$$\int_{\sigma(A)} |F(\lambda)|\, dv(f,g^*,\lambda) < \infty, \tag{2.19}$$

(ii)

$$\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |F(\lambda)|>n\}} |F(\lambda)|\, dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.20}$$

Observe that, for $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$ (on $\sigma(A)$), for any $f \in D(F(A))$, $g^* \in X^*$, and arbitrary Borel sets $\delta \subseteq \sigma$,

$$\begin{aligned}
&\int_{\sigma} |F(\lambda)|\, dv(f,g^*,\lambda) \quad \text{(see [3])} \\
&\quad \le 4 \sup_{\delta \subseteq \sigma} \left| \int_{\delta} F(\lambda)\, d\langle E_A(\lambda) f, g^*\rangle \right| \\
&\quad = 4 \sup_{\delta \subseteq \sigma} \left| \int_{\sigma} \chi_\delta(\lambda) F(\lambda)\, d\langle E_A(\lambda) f, g^*\rangle \right| \qquad \text{(by the properties of the } o.c.\text{)} \\
&\quad = 4 \sup_{\delta \subseteq \sigma} \left| \left\langle \int_{\sigma} \chi_\delta(\lambda) F(\lambda)\, dE_A(\lambda) f, g^* \right\rangle \right| \qquad \text{(by the properties of the } o.c.\text{)} \\
&\quad = 4 \sup_{\delta \subseteq \sigma} \left| \langle E_A(\delta) E_A(\sigma) F(A) f, g^*\rangle \right| \\
&\quad \le 4 \sup_{\delta \subseteq \sigma} \|E_A(\delta) E_A(\sigma) F(A) f\| \|g^*\| \\
&\quad \le 4 \sup_{\delta \subseteq \sigma} \|E_A(\delta)\| \|E_A(\sigma) F(A) f\| \|g^*\| \qquad \text{(by (2.17))} \\
&\quad \le 4M \|E_A(\sigma) F(A) f\| \|g^*\| \le 4M \|E_A(\sigma)\| \|F(A) f\| \|g^*\|.
\end{aligned} \tag{2.21}$$

In particular,

$$\begin{aligned}
&\int_{\sigma(A)} |F(\lambda)|\, dv(f,g^*,\lambda) \quad \text{(by (2.21))} \\
&\quad \le 4M \|E_A(\sigma(A))\| \|F(A) f\| \|g^*\| \\
&\qquad\qquad \text{(since } E_A(\sigma(A)) = I \text{ (} I \text{ is the \textit{identity operator} in } X\text{))} \\
&\quad \le 4M \|F(A) f\| \|g^*\|.
\end{aligned} \tag{2.22}$$

3. The Carleman classes of a scalar type spectral operator

Theorem 3.1. *Let A be a scalar type spectral operator in a complex reflexive Banach space X. If a sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfies condition* (WGR)*, equalities (2.6) hold, the scalar type spectral operators $T(t|A|)$ $(0<t<\infty)$ defined in the sense of the operational calculus for a scalar type spectral operator and the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on $[0,\infty)$ such that*

$$c_1L(\gamma_1\lambda)\le T(\lambda)\le c_2L(\gamma_2\lambda),\quad \lambda>R, \tag{3.1}$$

with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

Proof. First, we prove the replaceability of $T(\cdot)$ in (2.6) by a *nonnegative*, *continuous*, and *increasing* function satisfying (3.1) with some positive γ_1, γ_2, c_1, c_2, and a nonnegative $R\ge 0$.

Let

$$f\in\bigcup_{t>0}T(t|A|)\quad\left(\bigcap_{t>0}T(t|A|)\right). \tag{3.2}$$

Then, for some (any) $0<t<\infty$, $f\in D(T(t|A|))$, which, according to Proposition 2.1, implies, in particular, that, for any $g^*\in X^*$,

$$\int_{\sigma(A)}T(t|\lambda|)\,dv(f,g^*,\lambda)<\infty. \tag{3.3}$$

For any $g^*\in X^*$,

$$\int_{\sigma(A)}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)<\infty. \tag{3.4}$$

Indeed,

$$\begin{aligned}
&\int_{\sigma(A)}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)\\
&\quad=\int_{\{\lambda\in\sigma(A)|t|\lambda|\le R\}}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)+\int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)\\
&\quad\le L(\gamma_1R)v(f,g^*,\sigma(A))+\int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)\quad\text{(by (2.18))}\\
&\quad\le L(\gamma_1R)4M\|f\|\|g^*\|+\int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}}L(\gamma_1t|\lambda|)\,dv(f,g^*,\lambda)\quad\text{(by (3.1))}\\
&\quad\le L(\gamma_1R)4M\|f\|\|g^*\|+\frac{1}{c_1}\int_{\{\lambda\in\sigma(A)|t|\lambda|>R\}}F(t|\lambda|)\,dv(f,g^*,\lambda)\\
&\quad\le L(\gamma_1R)4M\|f\|\|g^*\|+\frac{1}{c_1}\int_{\sigma(A)}F(t|\lambda|)\,dv(f,g^*,\lambda)\quad\text{(by (3.3))}\\
&\quad<\infty.
\end{aligned} \tag{3.5}$$

Further,

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid t|\lambda|\le R,\ L(\gamma_1 t|\lambda|)>n\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) = 0 \tag{3.6}$$

for all sufficiently large natural n's since, when $t|\lambda| \le R$, $L(\gamma_1 t|\lambda|) \le L(\gamma_1 R)$.

On the other hand,

$$\begin{aligned}
&\int_{\{\lambda\in\sigma(A) \mid t|\lambda|>R,\ L(\gamma_1 t|\lambda|)>n\}} L(\gamma_1 t|\lambda|)\, dv(f,g^*,\lambda) \quad \text{(by (3.1))}\\
&\quad\le \frac{1}{c_1}\int_{\{\lambda\in\sigma(A) \mid t|\lambda|>R,\ T(t|\lambda|)>c_1 n\}} T(t|\lambda|)\, dv(f,g^*,\lambda) \quad \text{(by (2.21))}\\
&\quad\le \frac{1}{c_1}\big\|E_A(\{\lambda\in\sigma(A) \mid T(t|\lambda|)>c_1 n\})T(t|A|)f\big\|\|g^*\|\\
&\qquad\qquad\qquad \text{(by the continuity of the } s.m.)\\
&\quad\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{3.7}$$

Therefore, by Proposition 2.1, $f \in D(L(\gamma_1 t|A|))$.

Thus, we have proved the inclusions

$$\begin{aligned}
\bigcup_{t>0} D(T(t|A|)) &\subseteq \bigcup_{t>0} D(L(t|A|)),\\
\bigcap_{t>0} D(T(t|A|)) &\subseteq \bigcap_{t>0} D(L(t|A|)).
\end{aligned} \tag{3.8}$$

Similarly, one can derive from (3.1) the inverse inclusions:

$$\begin{aligned}
\bigcup_{t>0} D(T(t|A|)) &\supseteq \bigcup_{t>0} D(L(t|A|)),\\
\bigcap_{t>0} D(T(t|A|)) &\supseteq \bigcap_{t>0} D(L(t|A|)).
\end{aligned} \tag{3.9}$$

Thus,

$$\begin{aligned}
\bigcup_{t>0} D(T(t|A|)) &= \bigcup_{t>0} D(L(t|A|)),\\
\bigcap_{t>0} D(T(t|A|)) &= \bigcap_{t>0} D(L(t|A|)).
\end{aligned} \tag{3.10}$$

Let $f \in C_{\{m_n\}}(A)$ $(C_{(m_n)}(A))$. Then $f \in C^\infty(A)$ and, for a certain (an arbitrary) $\alpha > 0$, there is a $c > 0$ such that

$$\|A^n f\| \le c\alpha^n m_n, \quad n = 0,1,2,\dots. \tag{3.11}$$

For any $g^* \in X^*$,

$$\begin{aligned}\int_{\sigma(A)} T\left(\frac{1}{2\alpha}|\lambda|\right) dv(f,g^*,\lambda) &= \int_{\sigma(A)} \sum_{n=0}^{\infty} \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f,g^*,\lambda) \\ &\qquad \text{(by the \textit{monotone convergence theorem})} \\ &= \sum_{n=0}^{\infty} \int_{\sigma(A)} \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f,g^*,\lambda) \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n\alpha^n m_n} \int_{\sigma(A)} |\lambda|^n dv(f,g^*,\lambda) \quad \text{(by (2.22))} \\ &\le \sum_{n=0}^{\infty} \frac{1}{2^n\alpha^n m_n} 4M\|A^n f\|\|g^*\| \quad \text{(by (3.11))} \\ &\le 4Mc \sum_{n=0}^{\infty} \frac{1}{2^n}\|g^*\| = 8Mc\|g^*\| < \infty.\end{aligned} \tag{3.12}$$

Let

$$\Delta_n := \{\lambda \in \sigma(A) \mid |\lambda| \le n\}, \quad n = 0,1,2,\dots. \tag{3.13}$$

By the properties of the *o.c.*, $T((1/2\alpha)|A|)E_A(\Delta_n)$, $n = 0,1,2,\dots$, is a bounded operator on X and

$$\begin{aligned}\left\|T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_n)\right\| &\le 4M \sum_{k=0}^{\infty} \frac{n^k}{2^k\alpha^k m_k} \\ &\left(\text{by condition (WGR), there is a } C = C(\alpha,n) > 0 : \frac{n^k}{\alpha^k m_k} \le C,\ k = 0,1,\dots\right) \\ &\le 4MC \sum_{k=0}^{\infty} \frac{1}{2^k} = 8MC.\end{aligned} \tag{3.14}$$

For any $1 \le m < n$,

$$\begin{aligned}
&\left|\left\langle T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_n)f - T\left(\frac{1}{2\alpha}|A|\right)E_A(\Delta_m)f, g^*\right\rangle\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the properties of the } o.c.\text{)} \\
&\left|\left\langle \int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\le n\}} T\left(\frac{1}{2\alpha}|\lambda|\right)dE_A(\lambda)f, g^*\right\rangle\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the properties of the } o.c.\text{)} \\
&\quad = \left|\int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\le n\}} T\left(\frac{1}{2\alpha}|\lambda|\right)d\langle E_A(\lambda)f, g^*\rangle\right| \\
&\quad \le \int_{\{\lambda\in\sigma(A)\,|\,m<|\lambda|\}} T\left(\frac{1}{2\alpha}|\lambda|\right)dv(f,g^*,\lambda) \quad \text{(by (3.12))} \\
&\quad \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{3.15}$$

Since a *reflexive* Banach space is *weakly complete* (see, e.g., [3]), we infer that the sequence $\{T((1/2\alpha)|A|)E_A(\Delta_n)f\}_{n=1}^{\infty}$ *weakly converges* in X. This, considering the fact that, by the continuity of the $s.m.$,

$$E_A(\Delta_n)f \longrightarrow f \quad \text{as } n \longrightarrow \infty \tag{3.16}$$

and the *closedness* of the operator $T((1/2\alpha)|A|)$, implies

$$f \in D\left(T\left(\frac{1}{2\alpha}|A|\right)\right). \tag{3.17}$$

Therefore,

$$f \in \bigcup_{t>0} D(T(t|A|)) \quad \left(\bigcap_{t>0} D(T(t|A|)),\ \text{resp.}\right), \tag{3.18}$$

which proves the inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\subseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\subseteq \bigcap_{t>0} D(T(t|A|)).
\end{aligned} \tag{3.19}$$

Now, we are to prove the inverse inclusions.

Let

$$f \in \bigcup_{t>0} D(T(t|A|)) \quad \left(\bigcap_{t>0} D(T(t|A|))\right). \tag{3.20}$$

Then, for a certain (any) $t > 0$, $f \in D(T(t|A|))$.

We infer from the latter that $f \in C^{\infty}(A)$.

Indeed, for an arbitrary $N = 0, 1, 2, \ldots$ and any $g^* \in X^*$,

$$\begin{aligned}\int_{\sigma(A)} \frac{t^N}{m_N} |\lambda|^N \, dv(f, g^*, \lambda) &\le \int_{\sigma(A)} \sum_{k=0}^{\infty} \frac{[t|\lambda|]^k}{m_k} \, dv(f, g^*, \lambda) \\ &= \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) \\ &\qquad\qquad \text{(by Proposition 2.1),} \\ &< \infty.\end{aligned} \tag{3.21}$$

Further, for any $N = 0, 1, 2, \ldots,$

$$\begin{aligned}&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid (t^N/m_N)|\lambda|^N > n\}} \frac{t^N}{m_N} |\lambda|^N \, dv(f, g^*, \lambda) \\ &\quad \le \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid T(t|\lambda|) > n\}} T(t|\lambda|) \, dv(f, g^*, \lambda) \quad \text{(by Proposition 2.1),} \\ &\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.22}$$

By Proposition 2.1, (3.21) and (3.22) imply that

$$f \in C^{\infty}(A). \tag{3.23}$$

Further, by (2.22),

$$\begin{aligned}&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) \quad \text{(by (2.22))} \\ &\quad \le 4M \| T(t|A|) f \| < \infty.\end{aligned} \tag{3.24}$$

By (2.22),

$$\begin{aligned}0 < c :&= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} T(t|\lambda|) \, dv(f, g^*, \lambda) + 1 \\ &\le 4M \| T(t|A|) f \| < \infty.\end{aligned} \tag{3.25}$$

Whence, for any $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
c &\geq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\sigma(A)} \frac{t^n}{m_n} |\lambda|^n \, dv(f, g^*, \lambda) \\
&\geq \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \int_{\sigma(A)} \lambda^n \, d\langle E_A(\lambda) f, g^* \rangle \right| \\
&\qquad \text{(by the properties of the } o.c.\text{)} \\
&\geq \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\sigma(A)} \lambda^n \, dE_A(\lambda) f, g^* \right\rangle \right| \qquad (3.26) \\
&\qquad \text{(by the properties of the } o.c.\text{)} \\
&= \frac{t^n}{m_n} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} |\langle A^n f, g^* \rangle| \\
&\qquad \text{(as follows from the \textit{Hahn-Banach theorem})} \\
&= \frac{t^n}{m_n} \|A^n f\|.
\end{aligned}$$

Thus, for some (any) $t > 0$,

$$\|A^n f\| \leq c \left(\frac{1}{t} \right)^n m_n, \quad n = 0, 1, 2, \ldots. \tag{3.27}$$

Hence,

$$f \in C_{\{m_n\}}(A) \quad (C_{(m_n)}(A), \text{ resp.}), \tag{3.28}$$

which proves the inverse inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)).
\end{aligned} \tag{3.29}$$

From (3.19) and (3.29), we infer equalities (2.6). □

Remark 3.2. Observe that the assumption of the *reflexivity* of the space X was utilized for proving the inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\subseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\subseteq \bigcap_{t>0} D(T(t|A|))
\end{aligned} \tag{3.30}$$

only.

The inverse inclusions

$$\begin{aligned}
C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\
C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|))
\end{aligned} \tag{3.31}$$

hold regardless whether X is reflexive or not.

4. The Gevrey classes of a scalar type spectral operator. Let $0 < \beta < \infty$. As is easily seen, the sequence $m_n = [n!]^\beta$, $n = 0,1,2,\ldots$, satisfies condition (WGR) and, thus, the function

$$T(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n}{[n!]^\beta}, \quad 0 \le \lambda < \infty, \tag{4.1}$$

is well defined.

According to *Stirling's formula*,

$$n^{\beta n} \sim (2\pi n)^{-\beta/2} e^{\beta n} [n!]^\beta \quad \text{as } n \longrightarrow \infty. \tag{4.2}$$

Hence, there is such a $C = C(\beta) \ge 1$ such that

$$[n!]^\beta \le n^{\beta n} \le C(2\pi n)^{-\beta/2} e^{\beta n}[n!]^\beta \le C e^{\beta n}[n!]^\beta, \quad n = 0,1,2,\ldots. \tag{4.3}$$

Taking this into account, we infer

$$\begin{aligned} \sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} &\le \sum_{n=0}^{\infty} \frac{\lambda^n}{n^{\beta n}} \le T(\lambda) \le C \sum_{n=0}^{\infty} \frac{(e^\beta \lambda)^n}{n^{\beta n}} = C \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{(2e^\beta\lambda)^n}{n^{\beta n}} \\ &\le C \sup_{n\ge 0} \frac{(2e^\beta\lambda)^n}{n^{\beta n}} \sum_{n=0}^{\infty} \frac{1}{2^n} = 2C \sup_{n\ge 0} \frac{(2e^\beta\lambda)^n}{n^{\beta n}}, \quad 0 \le \lambda < \infty. \end{aligned} \tag{4.4}$$

Now, we consider the family of functions

$$\rho_\lambda(x) := \frac{\lambda^x}{x^{\beta x}}, \quad 0 \le x < \infty,\ 1 \le \lambda < \infty\ (0^0 := 1). \tag{4.5}$$

It is easy to make sure that the function $\rho_\lambda(\cdot)$ attains its maximum value on $[0,\infty)$ at the point $x_\lambda = e^{-1}\lambda^{1/\beta}$.

Therefore,

$$\sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} \le \sup_{x\ge 0} \frac{\lambda^x}{x^{\beta x}} = \rho_\lambda(x_\lambda) = e^{\beta e^{-1}\lambda^{1/\beta}}. \tag{4.6}$$

For $\lambda \ge e^\beta$, let N be the *integer part* of $x_\lambda = e^{-1}\lambda^{1/\beta}$.

Hence, $N \ge 1$ and

$$\begin{aligned} \sup_{n\ge 0} \frac{\lambda^n}{n^{\beta n}} &\ge \frac{\lambda^N}{N^{\beta N}} = \exp\left(N\ln\lambda - \beta N \ln N\right) \\ &\ge \exp\left((x_\lambda - 1)\ln\lambda - \beta x_\lambda \ln x_\lambda\right) = \frac{1}{\lambda} e^{\beta e^{-1}\lambda^{1/\beta}}, \quad \lambda \ge e^\beta. \end{aligned} \tag{4.7}$$

Obviously, for all sufficiently large positive λ's,

$$e^{-(\beta e^{-1}/2)\lambda^{1/\beta}} \le \frac{1}{\lambda}. \tag{4.8}$$

Based on (4.4), (4.6), (4.7), and (4.8), for all sufficiently large positive λ's,

$$\begin{aligned} e^{(\beta^\beta(e^{-\beta}/2^\beta)\lambda)^{1/\beta}} &\le T(\lambda) \le 2C \sup_{n\ge 0} \frac{(2e^\beta\lambda)^n}{n^{\beta n}} \le 2C \sup_{x\ge 0} \rho_{2e^\beta\lambda}(x) \\ &= 2Ce^{\beta e^{-1}(2e^\beta\lambda)^{1/\beta}} \le e^{(4\beta^\beta\lambda)^{1/\beta}}. \end{aligned} \tag{4.9}$$

Thus, by Theorem 3.1, in the considered case, the function $T(\lambda)$ can be replaced by $e^{\lambda^{1/\beta}}$ $(0 \le \lambda < \infty)$ and we arrive at the following.

Corollary 4.1. *Let A be a scalar type spectral operator in a complex reflexive Banach space and $0 < \beta < \infty$. Then*

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &= \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &= \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \end{aligned} \tag{4.10}$$

In particular, for $\beta = 1$, Corollary 4.1 gives the description of the *analytic* and *entire* vectors of the scalar type spectral operator A.

Corollary 4.1 generalizes the corresponding result of [8] (see also [9, 10]) for a *normal operator* in a complex Hilbert space.

Observe that the inclusions

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &\supseteq \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &\supseteq \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \end{aligned} \tag{4.11}$$

are valid without the assumption of the *reflexivity* of X (see Remark 3.2).

5. A theorem of the Paley-Wiener type.

Consider the self-adjoint differential operator $A = i(d/dx)$ (i is the *imaginary unit*) in the complex Hilbert space $L^2(-\infty,\infty)$. With the unitary equivalence of this operator and the operator of multiplication by the independent variable x in view, by Theorem 3.1 as well as by [9, 10], we arrive at the following theorem of the Paley-Wiener type [18, 22].

Theorem 5.1. *Let $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers satisfying condition* (WGR)*, then*

$$f \in C_{\{m_n\}}(A)\ (C_{(m_n)}(A)) \Longleftrightarrow \int_{-\infty}^{\infty} |\hat{f}(x)|^2 T^2(t|x|)\,dx < \infty \tag{5.1}$$

($\hat{f}$ is the Fourier transform of f) for some (any) $0 < t < \infty$, the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on $[0,\infty)$ and satisfying (3.1) with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

The only natural question to be answered now is how the abstract smoothness relative to the differential operator A in $L^2(-\infty,\infty)$ reveals itself as the smoothness in the ordinary sense.

For any $f \in W_2^n(I)$, where I is an interval of the real axis and $W_2^n(I) = H^n(I)$ is the nth-order *Sobolev space* [20], let $f(\cdot)$ be the representative of the equivalence class f continuously differentiable $n-1$ times and such that $f^{(n-1)}(\cdot)$ is *absolutely continuous* on I.

For

$$f \in W_2^\infty(-\infty,\infty) := \bigcap_{n=0}^{\infty} W_2^n(-\infty,\infty), \tag{5.2}$$

let $f(\cdot)$ be the infinite-differentiable representative of the equivalence class f such that

$$\int_{-\infty}^{\infty} |f^{(n)}(t)|^2\,dt < \infty, \quad n = 0,1,2,\dots. \tag{5.3}$$

Let

$$\begin{aligned}
\hat{C}_{\{m_n\}}(-\infty,\infty) &\overset{\text{def}}{=} \Big\{ f \in W_2^\infty(-\infty,\infty) \mid \forall [a,b] \subseteq (-\infty,\infty)\ \exists \alpha > 0, \\
&\qquad \exists c > 0 : \max_{a \le t \le b} \|f^{(n)}(t)\| \le c\alpha^n m_n,\ n = 0,1,2,\dots \Big\}, \\
\hat{C}_{(m_n)}(-\infty,\infty) &\overset{\text{def}}{=} \Big\{ f \in W_2^\infty(-\infty,\infty) \mid \forall [a,b] \subseteq (-\infty,\infty),\ \forall \alpha > 0 \\
&\qquad \exists c > 0 : \max_{a \le t \le b} \|f^{(n)}(t)\| \le c\alpha^n m_n,\ n = 0,1,2,\dots \Big\}.
\end{aligned} \tag{5.4}$$

We will impose upon the sequence $\{m_n\}_{n=0}^\infty$ an additional condition.

(DI) There are an $L > 0$ and a $\gamma > 1$ such that

$$m_{n+1} \le L\gamma^n m_n, \quad n = 0,1,2,\dots.$$

Note that the name (DI) originates from the words *"differentiation invariant"* since, as is easily verifiable, under this condition, the Carleman classes $C_{\{m_n\}}(-\infty,\infty)$ and $C_{(m_n)}(-\infty,\infty)$ along with a function $f(\cdot)$ contain its first derivative, $f'(\cdot)$.

Observe that, for $0 \le \beta < \infty$, the Gevrey sequence $m_n = [n!]^\beta$, $n = 0,1,2,\dots$, meets condition (DI) with any $\gamma > 1$. Indeed, in this case, $m_{n+1}/m_n = (n+1)^\beta$, $n = 0,1,2,\dots$.

Lemma 5.2. *Let a sequence of positive numbers* $\{m_n\}_{n=0}^\infty$ *satisfy condition* (DI). *Then*

$$\begin{aligned}
C_{\{m_n\}}(A) &\subseteq \hat{C}_{\{m_n\}}(-\infty,\infty), \\
C_{(m_n)}(A) &\subseteq \hat{C}_{(m_n)}(-\infty,\infty).
\end{aligned} \tag{5.5}$$

Proof. Let $f \in C_{\{m_n\}}(A)$ $(C_{(m_n)}(A))$, Then

$$f \in W_2^\infty(-\infty,\infty), \tag{5.6}$$

and for some (any) $\alpha > 0$, there is a $c > 0$ such that

$$\|f\|_{L^2(-\infty,\infty)} = \left[\int_{-\infty}^{\infty} |f^{(n)}(x)|^2\,dx\right]^{1/2} \le c\alpha^n m_n, \quad n = 0,1,2,\dots. \tag{5.7}$$

We fix a finite segment $[a,b]$ of the real axis. Then, according to the *Sobolev embedding theorems* [20] (see also [22, 23]), the space $W_2^1(a,b)$ is *continuously embedded* into $C[a,b]$, that is, for some $M>0$ and any $f\in W_2^1(a,b)$,

$$\max_{a\le t\le b}|f(x)|\le M\|f\|_{W_2^1(a,b)}\le M\Big[\|f\|_{L^2(a,b)}+\|f'\|_{L^2(a,b)}\Big]. \tag{5.8}$$

Since $f\in C_{\{m_n\}}(A)$ $(C_{(m_n)}(A))$. Then, obviously, $f^{(n)}\in W_2^1(a,b)$ for any $n=0,1,2,\ldots.$ Therefore, for an arbitrary $n=0,1,2,\ldots,$

$$\begin{aligned}
\max_{a\le t\le b}|f^{(n)}(x)| &\le M\|f\|_{W_2^1(a,b)}\le M\Big[\|f^{(n)}\|_{L^2(a,b)}+\|f^{(n+1)}\|_{L^2(a,b)}\Big]\\
&\le M\Big[\|f^{(n)}\|_{L^2(-\infty,\infty)}+\|f^{(n+1)}\|_{L^2(-\infty,\infty)}\Big]\\
&\le M\big[c\alpha^n m_n+c\alpha^{n+1}m_{n+1}\big]\quad \text{(by (DI))}\\
&\le M\big[c\alpha^n m_n+c\alpha^{n+1}L\gamma^n m_n\big]=Mc\big[1+L\alpha\gamma^n\big]\alpha^n m_n\\
&\text{(considering that } \gamma>1\text{, there is a } c_1>0 \text{ such that } \gamma>1,\ c_1>0)\\
&\le c_1(\gamma\alpha)^n m_n,\quad n=0,1,2,\ldots.
\end{aligned} \tag{5.9}$$

□

Based on this Lemma, we obtain the following proposition.

Proposition 5.3. *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying* (WGR) *and* (DI). *If $f\in L^2(-\infty,\infty)$ is such that, for some (any) $0<t<\infty$,*

$$\int_{-\infty}^{\infty}|\hat f(x)|^2T^2(t|x|)\,dx<\infty, \tag{5.10}$$

there is a representative $f(\cdot)$ of the equivalence class f such that $f(\cdot)\in C^\infty(-\infty,\infty)$,

$$\begin{gathered}
\int_{-\infty}^{\infty}|f^{(n)}(x)|^2\,dx<\infty,\quad n=0,1,2,\ldots,\\
f(\cdot)\in C_{\{m_n\}}(-\infty,\infty)\ \big(C_{(m_n)}(-\infty,\infty)\big),
\end{gathered} \tag{5.11}$$

the function $T(\cdot)$ being replaceable by any nonnegative, continuous, and increasing function $L(\cdot)$ defined on $[0,\infty)$ and satisfying (3.1) with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

Corollary 5.4. *Let $0<\beta<\infty$. If $f\in L^2(-\infty,\infty)$ is such that, for some (any) $0<t<\infty$,*

$$\int_{-\infty}^{\infty}|\hat f(x)|^2e^{2t|x|^{1/\beta}}\,dx<\infty, \tag{5.12}$$

there is a representative $f(\cdot)$ of the equivalence class f such that $f(\cdot)\in C^\infty(-\infty,\infty)$,

$$\begin{gathered}
\int_{\infty}^{\infty}|f^{(n)}(x)|^2\,dx<\infty,\quad n=0,1,2,\ldots,\\
f(\cdot)\in \mathscr{E}^{\{\beta\}}(-\infty,\infty)\ \big(\mathscr{E}^{(\beta)}(-\infty,\infty)\big).
\end{gathered} \tag{5.13}$$

In particular, for $\beta=1$, we obtain sufficient conditions for the *real analyticity* and *entireness.*

6. Remarks. It is to be noted that, in [10] (see also [8, 9]), not only were equalities (2.6) for a *normal operator* in a complex Hilbert space proved to hold in the set-theoretical sense but also in the topological sense, the sets $C_{\{m_n\}}(A)$ and $C_{(m_n)}(A)$ considered as the *inductive* and, respectively, *projective* limits of the Banach spaces

$$C_{\alpha[m_n]}(A) := \left\{ f \in C^\infty(A) \mid \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n = 0,1,\dots \right\}, \tag{6.1}$$

$0 < \alpha < \infty$, with the norms

$$\|f\|_{C_{\alpha[m_n]}(A)} := \sup_{n\ge 0} \frac{\|A^n f\|}{\alpha^n m_n} \tag{6.2}$$

and the sets $\bigcup_{t>0} D(T(t|A|))$ and $\bigcap_{t>0} D(T(t|A|))$ as the *inductive* and, respectively, *projective* limits of the Hilbert spaces

$$H_{t[T]}(A) := D(T(t|A|)), \quad 0 < t < \infty, \tag{6.3}$$

with inner products

$$(f,g)_{H_{t[T]}(A)} := (T(t|A|)f, T(t|A|)g), \quad 0 < t < \infty. \tag{6.4}$$

Observe also that, in [11] (see also [10]), similar results were obtained for the *generator of a bounded analytic semigroup* in a Banach space.

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References

[1] T. Carleman, *Édition Complète des Articles de Torsten Carleman*, Institut Mathématique Mittag-Leffler, Djursholm, Suède, 1960.

[2] N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64** (1958), 217–274.

[3] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York, 1958.

[4] ———, *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*, Interscience Publishers, New York, 1963, with the assistance of William G. Bade and Robert G. Bartle.

[5] ———, *Linear Operators. Part III: Spectral Operators*, Interscience Publishers, New York, 1971.

[6] M. Gevrey, *Sur la nature analytique des solutions des équations aux dérivées partielles*, Ann. Ec. Norm. Sup. Paris **35** (1918), 129–196.

[7] R. W. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.

[8] V. I. Gorbachuk, *Spaces of infinitely differentiable vectors of a nonnegative selfadjoint operator*, Ukrainian Math. J. **35** (1983), 531–535.

[9] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Mathematics and its Applications (Soviet Series), vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991.

[10] V. I. Gorbachuk and A. V. Knyazyuk, *Boundary values of solutions of operator-differential equations*, Russian Math. Surveys **44** (1989), 67–111.

[11] A. V. Knyazyuk, *Boundary values of infinitely differentiable semigroups*, Akad. Nauk Ukrain. SSR Inst. Mat. Preprint (1985), no. 69, 47 (Russian).

[12] H. Komatsu, *Ultradistributions. I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.

[13] ———, *Ultradistributions and hyperfunctions*, Hyperfunctions and Pseudo-Differential Equations (Proc. Conf. on the Theory of Hyperfunctions and Analytic Functionals and Applications, RIMS, Kyoto University, Kyoto, 1971; dedicated to the memory of André Martineau), Lecture Notes in Math., vol. 287, Springer, Berlin, 1973, pp. 164–179.

[14] ———, *Microlocal analysis in Gevrey classes and in complex domains*, Microlocal Analysis and Applications (Montecatini Terme, 1989), Lecture Notes in Math., vol. 1495, Springer, Berlin, 1991, pp. 161–236.

[15] S. Mandelbrojt, *Series de Fourier et Classes Quasi-Analytiques de Fonctions*, Gauthier-Villars, Paris, 1935 (French).

[16] M. V. Markin, *On an abstract evolution equation with a spectral operator of scalar type*, Int. J. Math. Math. Sci. **32** (2002), no. 9, 555–563.

[17] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.

[18] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, American Mathematical Society, New York, 1934.

[19] A. I. Plesner, *Spectral Theory of Linear Operators*, Izdat. “Nauka”, Moscow, 1965.

[20] S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Translations of Mathematical Monographs, vol. 7, American Mathematical Society, Rhode Island, 1963.

[21] J. Wermer, *Commuting spectral measures on Hilbert space*, Pacific J. Math. **4** (1954), 355–361.

[22] K. Yosida, *Functional Analysis*, Die Grundlehren der Mathematischen Wissenschaften, vol. 123, Academic Press, New York, 1965.

[23] R. J. Zimmer, *Essential Results of Functional Analysis*, Chicago Lectures in Mathematics, University of Chicago Press, Illinois, 1990.

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