IJMMS 2004:9, 459–468
PII. S0161171204205038
http://ijmms.hindawi.com

RATE OF CONVERGENCE OF BOUNDED VARIATION FUNCTIONS BY A BÉZIER-DURRMEYER VARIANT OF THE BASKAKOV OPERATORS

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Received 6 May 2002

We consider a Bézier-Durrmeyer integral variant of the Baskakov operators and study the rate of convergence for functions of bounded variation.

2000 Mathematics Subject Classification: 41A36, 41A25, 26A45.

1. Introduction. Let $W(0,\infty)$ be the class of functions f which are locally integrable on $(0,\infty)$ and are of polynomial growth as $t\to\infty$, that is, for some positive r, there holds $f(t)=O(t^r)$ as $t\to\infty$. The Durrmeyer variant $\tilde{V}_n$ of the Baskakov operators associates to each function $f\in W(0,\infty)$ the series

$$\tilde{V}_n(f;x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)dt,\quad x\in[0,\infty),\tag{1.1}$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}\tag{1.2}$$

is the Baskakov basis function. Note that (1.1) is well defined, for $n\geq r+2$, provided that $f(t)=O(t^r)$ as $t\to\infty$. The operators (1.1) were first introduced by Sahai and Prasad [9]. They termed these operators as modified Lupaş operators. In 1991, Sinha et al. [10] improved and corrected the results of [9] and denoted $\tilde{V}_n$ as modified Baskakov operators. The rate of convergence of the operators (1.1) on functions of bounded variation was studied in [8, 11].

We mention that Agrawal and Thamer [2] considered the variant

$$M_n(f;x)=(n-1)\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-1}(t)f(t)dt+(1+x)^{-n}f(0)\tag{1.3}$$

of the operators (1.1) and studied its properties in subsequent papers [3, 4, 5]. See also [1]. The rate of convergence of the operators discussed by Agrawal and Thamer was studied by the first author in [7].

For each function $f \in W(0,\infty)$ and $\alpha \geq 1$, we consider the Bézier–type Baskakov-Durrmeyer operators $\widetilde{V}_{n,\alpha}$ as

$$\widetilde{V}_{n,\alpha}(f;x) = (n-1)\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \tag{1.4}$$

where

$$\begin{aligned} Q_{n,k}^{(\alpha)}(x) &= J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x), \\ J_{n,k}(x) &= \sum_{j=k}^{\infty} p_{n,j}(x). \end{aligned} \tag{1.5}$$

It is obvious that $\widetilde{V}_{n,\alpha}$ are positive linear operators and $\widetilde{V}_{n,\alpha}(1;x) = 1$. In the special case $\alpha = 1$, the operators $\widetilde{V}_{n,\alpha}$ reduce to the operators $\widetilde{V}_n \equiv \widetilde{V}_{n,1}$. Some basic properties of $J_{n,k}$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x)$ $(k = 0,1,2,\ldots)$;
(ii) $J'_{n,k}(x) = np_{n+1,k-1}(x)$ $(k = 1,2,3,\ldots)$;
(iii) $J_{n,k}(x) = n\int_0^x p_{n+1,k-1}(t)dt$ $(k = 1,2,3,\ldots)$;
(iv) $0 < \cdots < J_{n,k+1}(x) < J_{n,k}(x) < \cdots < J_{n,1}(x) < J_{n,0}(x) \equiv 1$ $(x > 0)$;
(v) $J_{n,k}$ is strictly increasing on $[0,\infty)$.

In this paper, we study the rate of convergence for the new sequence of operators (1.4), for functions f of bounded variation. Our result essentially generalizes and improves the results of [8, 11]. Furthermore, we find the limit of the sequence $\widetilde{V}_{n,\alpha}(f;x)$ for bounded locally integrable functions f having a discontinuity of the first kind at $x \in (0,\infty)$.

2. The main results. As a main result, we derive the following estimate on the rate of convergence.

Theorem 2.1. *Assume that $f \in W(0,\infty)$ is a function of bounded variation on every finite subinterval of $(0,\infty)$. Furthermore, let $\alpha \geq 1$, $\lambda > 2$, and $x \in (0,\infty)$ be given. Then, for each $r \in \mathbb{N}$, there exists a constant $M(f,\alpha,r,x)$ such that for sufficiently large n, the Bézier–type Baskakov-Durrmeyer operators $\widetilde{V}_{n,\alpha}$ satisfy the estimate*

$$\begin{aligned} &\left|\widetilde{V}_{n,\alpha}(f;x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| \\ &\quad \leq \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}\left|f(x+)-f(x-)\right| \\ &\qquad + \frac{2\alpha\lambda(1+x)+x}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{M(f,\alpha,r,x)}{n^r}, \end{aligned} \tag{2.1}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-) & (0 \leq t < x), \\ 0 & (t = x), \\ f(t) - f(x+) & (x < t < \infty), \end{cases} \tag{2.2}$$

and $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Remark 2.2. The exponent r in the O-term of (2.1) can be chosen arbitrary large.

As an immediate consequence of Theorem 2.1, we obtain in the special case $\alpha = 1$ the following estimate which improves the results of [8, 11].

Corollary 2.3. *Under the assumptions of Theorem 2.1, there holds, for sufficiently large n,*

$$\begin{aligned} &\left|\tilde{V}_n(f;x)-\frac{1}{2}[f(x+)+f(x-)]\right| \\ &\quad\le \frac{(10+11x)}{2\sqrt{nx(1+x)}}|f(x+)-f(x-)| \\ &\qquad+\frac{2\lambda(1+x)+x}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+\frac{M(f,1,r,x)}{n^r}, \end{aligned} \tag{2.3}$$

where g_x is defined as in Theorem 2.1.

Theorem 2.4. *Let $x \in (0,\infty)$. If $f \in L(0,\infty)$ has a discontinuity of the first kind at x, then*

$$\lim_{n\to\infty}\tilde{V}_{n,\alpha}(f;x)=\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-). \tag{2.4}$$

3. Auxiliary results. In order to prove our main result, we will need the following lemmas. Throughout the paper, for each real x, let $\psi_x(t) = t - x$.

Lemma 3.1 (see [6]). *Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables with finite variance such that the expectation $E(\xi_i) = a_1 \in \mathbb{R} \equiv (-\infty,\infty)$, and the variance $V(\xi_i) = b_1^2 > 0$. Assume that $E|\xi_i - a_1|^3 < \infty$. Then there exists a constant c with $1/\sqrt{2\pi} < c < 0.82$ such that, for all $n = 1,2,3,\ldots$ and all $t \in \mathbb{R}$,*

$$\left|P\left(\frac{1}{b_1\sqrt{n}}\sum_{k=1}^{n}(\xi_i-a_1)\le t\right)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-u^2/2}du\right|\le c\frac{E|\xi_i-a_1|^3}{\sqrt{n}b_1^3}. \tag{3.1}$$

Lemma 3.2 (see [10]). *For each fixed $x \in [0,\infty)$ and $m \in \mathbb{N}_0$, the central moments $\tilde{V}_n(\psi_x^m;x)$ of the Baskakov-Durrmeyer operators (1.1) satisfy*

$$\tilde{V}_n(\psi_x^m;x)=O\left(n^{-\lfloor (m+1)/2\rfloor}\right)\quad (n\longrightarrow\infty). \tag{3.2}$$

In particular,

$$\tilde{V}_n(1;x)=1,\qquad \tilde{V}_n(\psi_x^2;x)=\frac{2(n-1)x(1+x)}{(n-2)(n-3)}+\frac{2(1+2x)^2}{(n-2)(n-3)}. \tag{3.3}$$

Remark 3.3. Note that, given any $\lambda > 2$ and any $x > 0$, for all n sufficiently large, we have the estimate

$$\tilde{V}_n(\psi_x^2;x)<\frac{\lambda x(1+x)}{n}. \tag{3.4}$$

Lemma 3.4 (see [13]). *For all $x > 0$ and $n, k \in \mathbb{N}$, there holds*

$$J^{\alpha}_{n,k}(x)p_{n,k}(x) \le Q^{(\alpha)}_{n,k}(x) \le \alpha p_{n,k}(x) < \frac{\alpha\sqrt{1+x}}{\sqrt{2enx}}. \tag{3.5}$$

Throughout, let

$$K_{n,\alpha}(x,t) = (n-1)\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x)p_{n,k}(t), \tag{3.6}$$

$$\lambda_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)\,dt. \tag{3.7}$$

With this definition, for each function $f \in W(0,\infty)$, there holds, for all sufficiently large n,

$$\tilde{V}_{n,\alpha}(f;x) = \int_0^{\infty} K_{n,\alpha}(x,t)f(t)\,dt. \tag{3.8}$$

Note that, in particular,

$$\lambda_{n,\alpha}(x,\infty) = \int_0^{\infty} K_{n,\alpha}(x,u)\,du = 1. \tag{3.9}$$

Lemma 3.5. *For each $\lambda > 2$ and, for all sufficiently large n, there exist, for all $x \in (0,\infty)$,*

$$\lambda_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)\,dt \le \frac{\lambda\alpha x(1+x)}{n(x-y)^2} \quad (0 \le y < x), \tag{3.10}$$

$$1-\lambda_{n,\alpha}(x,z) = \int_z^{\infty} K_{n,\alpha}(x,t)\,dt \le \frac{\lambda\alpha x(1+x)}{n(z-x)^2} \quad (x < z < \infty). \tag{3.11}$$

Proof. First we prove (3.10). There holds

$$\begin{aligned}\int_0^y K_{n,\alpha}(x,t)\,dt &\le \int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}\,dt \\ &\le (x-y)^{-2}\tilde{V}_{n,\alpha}(\psi_x^2;x) \\ &\le \alpha(x-y)^{-2}\tilde{V}_{n,1}(\psi_x^2;x),\end{aligned} \tag{3.12}$$

where we applied Lemma 3.4. Now (3.10) is a consequence of Remark 3.3. The proof of (3.11) is similar. □

Lemma 3.6 (see [13]). *Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution*

$$P(\xi_1 = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} \quad (k \in \mathbb{N}), \tag{3.13}$$

where $x > 0$ is a parameter. Then,

$$E(\xi_1) = x, \qquad E(\xi_1 - E\xi_1)^2 = x(1+x), \qquad E\left|\xi_1 - E\xi_1\right|^3 \le 3x(1+x)^2. \tag{3.14}$$

Lemma 3.7. *For all $x \in (0,\infty)$ and $k = 0,1,2,\dots,$ there hold*

$$\left|J^{\alpha}_{n,k}(x) - J^{\alpha}_{n-1,k+1}(x)\right| \le \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}, \tag{3.15}$$

$$\left|J^{\alpha}_{n,k}(x) - J^{\alpha}_{n-1,k}(x)\right| \le \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}. \tag{3.16}$$

Proof. First we prove (3.15). Proceeding along the lines of [8, Lemma 2.8] and [12], it is easily checked that

$$\begin{aligned}\left|J_{n,k}(x) - J_{n-1,k+1}(x)\right| &\le \frac{2(0.82)E\left|\xi_1 - E\xi_1\right|^3}{\sqrt{n}(x(1+x))^{3/2}} + \frac{x}{\sqrt{2\pi nx(1+x)}} \\ &\le \frac{2(0.82)\cdot 3x(1+x)^2}{\sqrt{n}(x(1+x))^{3/2}} + \frac{x}{2\sqrt{nx(1+x)}} \\ &\le \frac{10+11x}{2\sqrt{nx(1+x)}},\end{aligned} \tag{3.17}$$

where we used Lemmas 3.1 and 3.6. Application of the inequality $|a^{\alpha} - b^{\alpha}| \le \alpha|a-b|$, for $0 \le a$, $b \le 1$, and $\alpha \ge 1$ yields (3.15). The proof of (3.16) is similar. □

4. Proofs of the main results

Proof of Theorem 2.1. Our starting point is the identity

$$\begin{aligned} f(t) = {}& \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) + \left(\operatorname{sign}(t-x) + \frac{\alpha-1}{\alpha+1}\right)\frac{f(x+)-f(x-)}{2} \\ &+ g_x(t) + \delta_x(t)\left(f(x) - \frac{f(x+)+f(x-)}{2}\right),\end{aligned} \tag{4.1}$$

where $\delta_x(t) = 1\ (t = x)$ and $\delta_x(t) = 0\ (t \ne x)$ (see [12, Equation (28)]). Since $\tilde{V}_{n,\alpha}(\delta_x;x) = 0$, we conclude that

$$\begin{aligned}&\left|\tilde{V}_{n,\alpha}(f;x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-)\right]\right| \\ &\quad \le \frac{1}{2}\left|\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) + \frac{\alpha-1}{\alpha+1}\right| \left|f(x+) - f(x-)\right| + \left|\tilde{V}_{n,\alpha}(g_x;x)\right|.\end{aligned} \tag{4.2}$$

First, we obtain

$$\begin{aligned}\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) &= (n-1)\sum_{j=0}^{\infty} Q^{(\alpha)}_{n,j}(x)\left(\int_x^{\infty} p_{n,j}(t)\,dt - \int_0^x p_{n,j}(t)\,dt\right) \\ &= (n-1)\sum_{j=0}^{\infty} Q^{(\alpha)}_{n,j}(x)\left(\int_0^{\infty} p_{n,j}(t)\,dt - 2\int_0^x p_{n,j}(t)\,dt\right) \\ &= 1 - 2(n-1)\sum_{j=0}^{\infty} Q^{(\alpha)}_{n,j}(x)\int_0^x p_{n,j}(t)\,dt.\end{aligned} \tag{4.3}$$

Using

$$\sum_{j=0}^{k} p_{n-1,j}(x) = (n-1)\int_x^\infty p_{n,k}(t)\,dt, \tag{4.4}$$

we conclude that

$$\begin{aligned}\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) &= 1-2\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x)\left(1-\sum_{k=0}^{j} p_{n-1,k}(x)\right)\\ &= -1+2\sum_{k=0}^{\infty} p_{n-1,k}(x)\sum_{j=k}^{\infty} Q_{n,j}^{(\alpha)}(x)\\ &= -1+2\sum_{k=0}^{\infty} p_{n-1,k}(x)J_{n,k}^{\alpha}(x)\end{aligned} \tag{4.5}$$

since $\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x) = 1$. Therefore, we obtain

$$\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1} = 2\sum_{k=0}^{\infty} p_{n-1,k}(x)J_{n,k}^{\alpha}(x)-\frac{2}{\alpha+1}\sum_{k=0}^{\infty} Q_{n-1,k}^{(\alpha+1)}(x) \tag{4.6}$$

since $\sum_{k=0}^{\infty} Q_{n-1,k}^{(\alpha+1)}(x) = 1$. By the mean value theorem, it follows that

$$Q_{n-1,k}^{(\alpha+1)}(x) = J_{n-1,k}^{\alpha+1}(x)-J_{n-1,k+1}^{\alpha+1}(x) = (\alpha+1)p_{n-1,k}(x)\gamma_{n,k}^{\alpha}(x), \tag{4.7}$$

where $J_{n-1,k+1}(x) < \gamma_{n,k}(x) < J_{n-1,k}(x)$. Hence,

$$\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1} = 2\sum_{k=0}^{\infty} p_{n-1,k}(x)\left(J_{n,k}^{\alpha}(x)-\gamma_{n,k}^{\alpha}(x)\right), \tag{4.8}$$

where

$$J_{n,k}^{\alpha}(x)-J_{n-1,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x)-\gamma_{n,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x)-J_{n-1,k+1}^{\alpha}(x). \tag{4.9}$$

Lemma 3.7 implies that

$$\left|\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1}\right| \le \frac{\alpha(10+11x)}{\sqrt{nx(1+x)}} \quad \text{for } x\in(0,\infty). \tag{4.10}$$

In order to complete the proof of the theorem, we need an estimate of $\tilde{V}_{n,\alpha}(g_x;x)$. We use the integral representation (3.8) and decompose $[0,\infty)$ into three parts as follows:

$$\begin{aligned}\tilde{V}_{n,\alpha}(g_x;x) &= \left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}+\int_{x+x/\sqrt{n}}^{\infty}\right)K_{n,\alpha}(x,t)g_x(t)\,dt\\ &= I_1+I_2+I_3, \quad \text{say.}\end{aligned} \tag{4.11}$$

We start with I_2. For $t\in[x-x/\sqrt{n},x+x/\sqrt{n}]$, we have

$$|g_x(t)| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x), \tag{4.12}$$

and therefore

$$|I_2| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x) \le \frac{1}{n} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x). \tag{4.13}$$

Next we estimate I_1. Let $y = x - x/\sqrt{n}$. Using integration by parts with (3.7), we have

$$I_1 = \int_0^y g_x(t)\, d_t \lambda_{n,\alpha}(x,t) = g_x(y)\lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t)\, d_t g_x(t). \tag{4.14}$$

Since $|g_x(y)| = |g_x(y) - g_x(x)| \le \bigvee_y^x (g_x)$, we conclude that

$$|I_1| \le \bigvee_y^x (g_x)\lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t)\, d_t\left(-\bigvee_t^x (g_x)\right). \tag{4.15}$$

Since $y = x - x/\sqrt{n} \le x$, (3.10) implies that

$$|I_1| \le \frac{\lambda\alpha x(1+x)}{n(x-y)^2} \bigvee_y^x (g_x) + \frac{\lambda\alpha x(1+x)}{n} \int_0^y \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x (g_x)\right). \tag{4.16}$$

Integrating the last term by parts, we get

$$|I_1| \le \frac{\lambda\alpha x(1+x)}{n}\left(x^{-2}\bigvee_0^x (g_x) + 2\int_0^y \frac{\bigvee_t^x (g_x)}{(x-t)^3} dt\right). \tag{4.17}$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we obtain

$$\begin{aligned}\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)(x-t)^{-3} dt &= \sum_{k=1}^{n-1} \int_{x/\sqrt{k+1}}^{x/\sqrt{k}} \bigvee_{x-t}^x (g_x) t^{-3} dt \\ &\le \frac{1}{2x^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x).\end{aligned} \tag{4.18}$$

Hence,

$$|I_1| \le \frac{2\lambda\alpha(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x). \tag{4.19}$$

Finally, we estimate I_3. We let

$$\tilde{g}_x(t) = \begin{cases} g_x(t) & (0 \le t \le 2x), \\ g_x(2x) & (2x < t < \infty) \end{cases} \tag{4.20}$$

and divide $I_3 = I_{31} + I_{32}$, where

$$I_{31} = \int_{x+x/\sqrt{n}}^{\infty} K_{n,\alpha}(x,t)\tilde{g}_x(t)\,dt, \qquad I_{32} = \int_{2x}^{\infty} K_{n,\alpha}(x,t)\left[g_x(t)-g_x(2x)\right]dt. \tag{4.21}$$

With $y = x + x/\sqrt{n}$, the first integral can be written in the form

$$I_{31} = \lim_{R\to+\infty}\Big\{g_x(y)\left[1-\lambda_{n,\alpha}(x,y)\right]+\tilde{g}_x(R)\left[\lambda_{n,\alpha}(x,R)-1\right] + \int_y^R \left[1-\lambda_{n,\alpha}(x,t)\right]d_t\tilde{g}_x(t)\Big\}. \tag{4.22}$$

By (3.11), we conclude that

$$\begin{aligned}|I_{31}| &\le \frac{\lambda\alpha x(1+x)}{n}\lim_{R\to+\infty}\left\{\frac{\bigvee_x^y(g_x)}{(y-x)^2}+\frac{|\tilde{g}_x(R)|}{(R-x)^2}+\int_y^R\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(\tilde{g}_x)\right)\right\}\\ &= \frac{\lambda\alpha x(1+x)}{n}\left\{\frac{\bigvee_x^y(g_x)}{(y-x)^2}+\int_y^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(g_x)\right)\right\}.\end{aligned} \tag{4.23}$$

In a similar way as above we obtain

$$\int_y^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(g_x)\right) \le x^{-2}\bigvee_x^{2x}(g_x)-\frac{\bigvee_x^y(g_x)}{(y-x)^2}+x^{-2}\sum_{k=1}^{n-1}\bigvee_x^{x+x/\sqrt{k}}(g_x) \tag{4.24}$$

which implies the estimate

$$|I_{31}| \le \frac{2\lambda\alpha(1+x)}{nx}\sum_{k=1}^{n}\bigvee_x^{x+x/\sqrt{k}}(g_x). \tag{4.25}$$

We proceed with I_{32}. By assumption, there exists an integer r such that $f(t) = O(t^{2r})$ as $t\to\infty$. Thus, for a certain constant $M>0$, depending only on f, x, and r, we have

$$\begin{aligned}|I_{32}| &\le M(n-1)\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_{2x}^{\infty}p_{n,k}(t)t^{2r}dt\\ &\le \alpha M(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_{2x}^{\infty}p_{n,k}(t)t^{2r}dt,\end{aligned} \tag{4.26}$$

where we used Lemma 3.4. Obviously, $t \ge 2x$ implies $t \le 2(t-x)$ and it follows that

$$|I_{32}| \le 2^{2r}\alpha M(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)(t-x)^{2r}dt = 2^{2r}\alpha M\tilde{V}_n(\psi_x^{2r};x). \tag{4.27}$$

By Lemma 3.2, the central moments of the Baskakov-Durrmeyer operators (1.1) satisfy $\tilde{V}_n(\psi_x^{2r};x) = O(n^{-r})(n \to \infty)$, and we obtain

$$I_{32} = O(n^{-r}) \quad (n \longrightarrow \infty). \tag{4.28}$$

Collecting the estimates (4.13), (4.19), (4.25), and (4.28) yields with regard to (4.11)

$$|\tilde{V}_{n,\alpha}(g_x;x)| \le \frac{2\lambda\alpha(1+x)+x}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+O(n^{-r}) \quad (n \longrightarrow \infty). \tag{4.29}$$

Finally, combining (4.2), (4.10), and (4.29), we obtain (2.1). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. Since the function ψ_x^2 given by $\psi_x^2(t) = (t-x)^2$ is of bounded variation on every finite subinterval of $[0,\infty)$, we deduce from Theorem 2.1 that, for all $x \in (0,\infty)$,

$$\lim_{n\to\infty}\tilde{V}_{n,\alpha}(\psi_x^2;x) = 0. \tag{4.30}$$

If $f \in L_\infty(0,\infty)$, then g_x defined as in (2.2) is also bounded and is continuous at the point x. By the Korovkin theorem, we conclude that

$$\lim_{n\to\infty}\tilde{V}_{n,\alpha}(g_x;x) = g_x(x) = 0. \tag{4.31}$$

Therefore, the right-hand side of inequality (4.2) tends to zero as $n \to \infty$. This completes the proof of Theorem 2.4. □

Acknowledgment. The authors are thankful to the referee for the valuable comments that helped to improve this paper. Moreover, the referee provided further important references.

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