

## ON CHUNG-TEICHER TYPE STRONG LAW FOR ARRAYS OF VECTOR-VALUED RANDOM VARIABLES

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We study the equivalence between the weak and strong laws of large numbers for arrays of row-wise independent random elements with values in a Banach space  $\mathcal{B}$ . The conditions under which this equivalence holds are of the Chung or Chung-Teicher types. These conditions are expressed in terms of convergence of specific series and  $o(1)$  requirements on specific weighted row-wise sums. Moreover, there are not any conditions assumed on the geometry of the underlying Banach space.

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Let  $(\Omega, \mathcal{F}, P)$  be a probability space and let  $\mathcal{B}$  be a real separable Banach space with norm  $\|\cdot\|$ . A strongly measurable transformation from  $\Omega$  to  $\mathcal{B}$  is said to be a  $\mathcal{B}$ -valued random variable or a random element. If  $E\|X\| < \infty$ , then the expected value is defined by the Bochner integral.

Let  $\{X_n, n \geq 1\}$  be a sequence of  $\mathcal{B}$ -valued random variables. Then  $\{X_n, n \geq 1\}$  is said to obey the strong law of large numbers (SLLN) if there exist sequences of real numbers  $\{a_n, n \geq 1\}$  and  $\{b_n, n \geq 1\}$  such that

$$\sum_{j=1}^n a_j (X_j - b_j) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \quad (1)$$

Sufficient conditions for SLLN use very often the geometry of a Banach space, that is, they assume that  $\mathcal{B}$  is a special-type space, for instance  $\mathcal{B}$  is of Rademacher type  $p$ ,  $1 < p \leq 2$ .

The space  $\mathcal{B}$  is of Rademacher type  $p$  if there exists a positive constant  $C$  such that

$$E \left\| \sum_{n=1}^{\infty} \varepsilon_n x_n \right\|^p \leq C \sum_{n=1}^{\infty} \|x_n\|^p \quad (2)$$

for each  $(x_1, x_2, \dots) \in C(B)$ , where  $\{\varepsilon_n, n \geq 1\}$  is a Bernoulli sequence, that is,  $\varepsilon_n, n \geq 1$ , are i.i.d. random variables and  $P[\varepsilon_n = 1] = P[\varepsilon_n = -1] = 1/2$ ,  $C(B) = \{(x_1, x_2, \dots) \in B^\infty : \sum_{n=1}^{\infty} \varepsilon_n x_n \text{ converges in probability}\}$ ,  $B^\infty = B \times B \times B \times \dots$ .

The sufficient conditions for SLLN for random elements taking value in a space of Rademacher type  $p$  were presented by Woyczyński [15], Hoffmann-Jørgensen and Pisier [6], Kuczmaszewska and Szynal [8], and Adler et al. [1].

The type of Marcinkiewicz-Zygmunt SLLN provides that for  $1 \leq \alpha < 2$  and a sequence  $\{X_n, n \geq 1\}$  of i.i.d.  $\mathcal{B}$ -valued random variables,

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n (X_i - EX_i) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty, \tag{3}$$

if and only if  $E\|X_1\| < \infty$  and the Banach space  $\mathcal{B}$  is of a Rademacher type  $p$  for  $\alpha < p \leq 2$  (cf. [15]).

The classical result of Hoffmann-Jørgensen and Pisier [6] proved that the assumption that a Banach space  $\mathcal{B}$  is the space of Rademacher type  $p$ ,  $1 \leq p \leq 2$ , is equivalent to the fact that the condition

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|^p}{n^p} < \infty \tag{4}$$

implies SLLN for a sequence of  $\mathcal{B}$ -valued independent random variables  $\{X_n, n \geq 1\}$  with  $EX_n = 0, n \geq 1$ .

In view of many statistical applications, it is important to consider the array-type SLLN.

Let  $\{k_n, n \geq 1\}$  be a strictly increasing sequence of positive integers. An array of  $\mathcal{B}$ -valued random variables  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  obeys the general array type of SLLN if

$$\sum_{i=1}^{k_n} a_{ni}(X_{ni} - c_{ni}) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty, \tag{5}$$

where  $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  and  $\{c_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  are suitable arrays of constants (weights) and  $\mathcal{B}$ -valued elements, respectively, and 0 denotes the zero-element in  $\mathcal{B}$ .

Hu and Taylor [7] considered SLLN for arrays of row-wise independent random variables  $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ .

Row-wise independence means that the random elements within each row are independent but no independence is assumed between rows.

In [3] Bozorgnia et al. obtained the Chung-type SLLN for arrays of row-wise independent random elements in a separable Banach space of Rademacher type  $p$ ,  $1 < p \leq 2$ . They proved the following result.

**THEOREM 1.** *Let  $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$  be an array of row-wise independent random elements in a separable Banach space of Rademacher type  $p$ ,  $1 < p \leq 2$ . Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a positive, even, and continuous function such that*

$$\frac{\varphi(|x|)}{|x|^r} \nearrow, \quad \frac{\varphi(|x|)}{|x|^{r+p-1}} \searrow \quad \text{as } |x| \nearrow, \tag{6}$$

for some integer  $r \geq 2$ .

Then the conditions

$$\begin{aligned}
 & EX_{ni} = 0, \quad 1 \leq i \leq n, n \geq 1, \\
 & \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\varphi(\|X_{ni}\|)}{\varphi(a_n)} < \infty, \quad \sum_{n=1}^{\infty} \left[ \sum_{i=1}^n E \left( \left\| \frac{X_{ni}}{a_n} \right\|^p \right) \right]^{pk} < \infty,
 \end{aligned} \tag{7}$$

for some positive integer  $k$ , imply

$$\frac{1}{a_n} \sum_{i=1}^n X_{ni} \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty, \tag{8}$$

where  $\{a_n, n \geq 1\}$  is a sequence of positive increasing real numbers such that

$$\lim_{n \rightarrow \infty} a_n = \infty. \tag{9}$$

This theorem generalizes Hu and Taylor’s result (cf. [7]) on the case of  $\mathcal{B}$ -valued random variables  $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$  taking value in a Banach space of Rademacher type  $p$ . Moreover, the assumptions of the function  $\varphi$  have some relationships with the geometric condition Rademacher type  $p$  of the Banach space.

Some results which consider the problem of equivalence between weak law of large numbers (WLLN) and SLLN for a sequence  $\{X_n, n \geq 1\}$  of independent  $\mathcal{B}$ -valued random variables can be found in Kuelbs and Zinn [10], de Acosta [4], Etemadi [5], Mikosch and Norvaiša [11, 12], Wang et al. [14], and Kuczmaszewska and Szynal [9].

Now, we recall some definitions and a lemma which will be used in the paper.

**DEFINITION 2.** A double array  $\{a_{ni}, i \geq 1, n \geq 1\}$  of real numbers is said to be a Toeplitz array if  $\lim_{n \rightarrow \infty} a_{ni} = 0$  for each  $i \geq 1$  and  $\sum_{i=1}^{\infty} |a_{ni}| \leq C$  for all  $n \geq 1$ , where  $C > 0$ .

In further consideration, we need an extension of the concept of stochastic domination by a random variable to an array of  $\mathcal{B}$ -valued random variables.

An array  $\{X_{ni}, i \geq 1, n \geq 1\}$  of  $\mathcal{B}$ -valued random variables is stochastically dominated by the random element  $X$  if there exists a constant  $D > 0$  such that

$$P[\|X_{ni}\| > x] \leq DP[D\|X\| > x] \tag{10}$$

for all  $x \geq 0, i \geq 1$ , and  $n \geq 1$ .

We also need some inequalities which will be very important in our consideration. The following lemma presents one of them.

**LEMMA 3** (cf. Yurinskii [16]). *Let  $X_1, X_2, \dots, X_n$  be independent  $\mathcal{B}$ -valued random variables with  $E\|X_i\| < \infty, i = 1, 2, \dots, n$ . Let  $\mathcal{F}$  be a  $\sigma$ -field generated by  $(X_1, X_2, \dots, X_k), k = 1, 2, \dots, n$ , and let  $\mathcal{F}_0 = \{\Omega, \emptyset\}$ . Then for  $1 \leq k \leq n$  and  $S_n = \sum_{i=1}^n X_i$ ,*

$$|E(\|S_n\| | \mathcal{F}_k) - E(\|S_n\| | \mathcal{F}_{k-1})| \leq \|X_k\| + E\|X_k\|. \tag{11}$$

**THEOREM 4.** *Let  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for all  $1 \leq i \leq k_n, n \geq 1$ , and for some increasing*

sequence  $\{k_n, n \geq 1\}$  of positive integers. Let  $\varphi_{ni} : \mathbb{R} \rightarrow \mathbb{R}_+$  and  $\psi_{ni} : \mathbb{R} \rightarrow \mathbb{R}_+$  be positive, even, and continuous functions, which for constants  $\alpha_{ni} \geq 1, 0 < \beta_{ni} \leq 2, K_{ni} > 0,$  and  $M_{ni} > 0, 1 \leq i \leq k_n, n \geq 1,$  satisfy the following conditions:

$$|x_1| \leq |x_2| \Rightarrow \frac{\varphi_{ni}(|x_1|)}{|x_1|^{\alpha_{ni}}} \leq K_{ni} \frac{\varphi_{ni}(|x_2|)}{|x_2|^{\alpha_{ni}}}, \tag{12}$$

$$|x_1| \leq |x_2| \Rightarrow \frac{|x_1|^{\beta_{ni}}}{\psi_{ni}(|x_1|)} \leq M_{ni} \frac{|x_2|^{\beta_{ni}}}{\psi_{ni}(|x_2|)}. \tag{13}$$

Suppose that for some array  $\{a_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  of nonzero reals and  $k \geq 1/2,$

$$\sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} M_{ni} \frac{\psi_{ni}(\|X_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})} \right)^k < \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[\|X_{ni}\| \geq c_{ni}] < \infty, \tag{15}$$

for some array  $\{c_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  of positive numbers such that

$$\sum_{n=1}^{\infty} \sum_{i=1}^n K_{ni}^2 \cdot \varphi_{ni}(c_{ni}) \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty. \tag{16}$$

Then

$$\sum_{i=1}^{k_n} a_{ni} X_{ni} \xrightarrow{P} 0, \quad n \rightarrow \infty, \tag{17}$$

if and only if

$$\sum_{i=1}^{k_n} a_{ni} X_{ni} \rightarrow 0 \quad a.s., \quad n \rightarrow \infty. \tag{18}$$

**PROOF.** Let  $X'_{ni} = X_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]$  and  $X^*_{ni} = X'_{ni} - EX'_{ni}.$

Now we introduce the following notation:

$$S_n = \sum_{i=1}^{k_n} a_{ni} X_{ni}, \quad S'_n = \sum_{i=1}^{k_n} a_{ni} X'_{ni}. \tag{19}$$

Note that using this notation, condition (12) on the Borel functions  $\varphi_{ni},$  and assumptions (15) and (16), we have

$$\begin{aligned} \sum_{n=1}^{\infty} P[S_n \neq S'_n] &= \sum_{n=1}^{\infty} P \left[ \left\| \sum_{i=1}^{k_n} a_{ni} X_{ni} I[\|X_{ni}\| > |a_{ni}^{-1}|] \right\| > \varepsilon \right] \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[\|X_{ni}\| I[\|X_{ni}\| > |a_{ni}^{-1}|] \neq 0] \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[\|X_{ni}\| > |a_{ni}^{-1}|] \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} EI[|X_{ni}| > |a_{ni}^{-1}|] \cdot I[|X_{ni}| \geq c_{ni}] \\
 &\quad + \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} EI[|X_{ni}| > |a_{ni}^{-1}|] \cdot I[|X_{ni}| < c_{ni}] \\
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[|X_{ni}| \geq c_{ni}] \\
 &\quad + \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E \left\{ \left( \frac{|X_{ni}|^{\alpha_{ni}}}{(|a_{ni}^{-1}|)^{\alpha_{ni}}} \right)^2 I[|X_{ni}| > |a_{ni}^{-1}|] \cdot I[|X_{ni}| < c_{ni}] \right\} \\
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[|X_{ni}| \geq c_{ni}] + \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} K_{ni}^2 \cdot \varphi_{ni}(c_{ni}) \frac{E\varphi_{ni}(|X_{ni}|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty.
 \end{aligned} \tag{20}$$

Thus the two sequences  $\{S_n, n \geq 1\}$  and  $\{S'_n, n \geq 1\}$  are equivalent.

Now we must prove that

$$E\|S'_n\| \rightarrow 0, \quad n \rightarrow \infty. \tag{21}$$

First we will show that

$$\|S'_n\| - E\|S'_n\| \xrightarrow{P} 0, \quad n \rightarrow \infty. \tag{22}$$

Using the Markov inequality, the Marcinkiewicz-Zygmunt inequality in its Banach space version (cf. de Acosta [4] or Berger [2]), and assumptions (12) and (14), for any  $\varepsilon > 0$ , we get

$$\begin{aligned}
 P[|\|S'_n\| - E\|S'_n\|| > \varepsilon] &\leq \varepsilon^{-2k} E \|\|S'_n\| - E\|S'_n\|\|^2 \\
 &\leq \varepsilon^{-2k} A_k E \left( \sum_{i=1}^{k_n} \|a_{ni} X'_{ni}\|^2 \right)^k = \varepsilon^{-2k} A_k E \left( \sum_{i=1}^{k_n} \frac{\|X'_{ni}\|^2}{(a_{ni}^{-1})^2} \right)^k \\
 &= \varepsilon^{-2k} A_k E \left( \sum_{i=1}^{k_n} \frac{\|X'_{ni}\|^{\beta_{ni}}}{(|a_{ni}^{-1}|)^{\beta_{ni}}} \cdot \frac{\|X'_{ni}\|^{2-\beta_{ni}}}{(|a_{ni}^{-1}|)^{2-\beta_{ni}}} \right)^k \\
 &\leq \varepsilon^{-2k} A_k E \left( \sum_{i=1}^{k_n} M_{ni} \frac{\psi_{ni}(\|X'_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})} \right)^k \\
 &\quad \times \varepsilon^{-2k} A_k E \left( \sum_{i=1}^{k_n} M_{ni} \frac{\psi_{ni}(\|X_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})} \right)^k = o(1).
 \end{aligned} \tag{23}$$

Thus we conclude that (22) holds and, together with (17) and the equivalence between  $\{S_n, n \geq 1\}$  and  $\{S'_n, n \geq 1\}$ , gives (21).

Now, we will show that  $\|S'_n\| \rightarrow 0$  a.s., as  $n \rightarrow \infty$ . By (21) it is enough to prove that

$$\|S'_n\| - E\|S'_n\| \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \tag{24}$$

As before, using the Markov inequality, the Marcinkiewicz-Zygmunt inequality, condition (13), and assumption (14), we have

$$\begin{aligned} \sum_{n=1}^{\infty} P[|\|S'_n\| - E\|S'_n\|| > \varepsilon] &\leq \varepsilon^{-2k} \sum_{n=1}^{\infty} E \|\|S'_n\| - E\|S'_n\|\|^{2k} \\ &\leq \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} \frac{\|X'_{ni}\|^2}{(a_{ni}^{-1})^2} \right)^k \\ &= \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} \frac{\|X'_{ni}\|^{\beta_{ni}}}{(|a_{ni}^{-1}|)^{\beta_{ni}}} \cdot \frac{\|X'_{ni}\|^{2-\beta_{ni}}}{(|a_{ni}^{-1}|)^{2-\beta_{ni}}} \right)^k \\ &\leq \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} M_{ni} \frac{\psi_{ni}(\|X_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})} \right)^k < \infty. \end{aligned} \tag{25}$$

Hence, by the Borel-Cantelli lemma, we obtain (24), which, by the equivalence between  $\{S_n, n \geq 1\}$  and  $\{S'_n, n \geq 1\}$ , completes the proof.  $\square$

Note that if we put in Theorem 4  $\varphi_{ni} = \varphi$  and  $\psi_{ni} = \psi$ , where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ ,  $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$  are positive, even, and continuous functions such that

$$\frac{\varphi(|x|)}{|x|^\alpha} \nearrow, \quad \frac{\psi(|x|)}{|x|^\beta} \searrow \quad \text{as } |x| \nearrow, \tag{26}$$

for some  $\alpha \geq 1$  and  $0 < \beta \leq 2$ , we get the following result.

**COROLLARY 5.** *Let  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for all  $1 \leq i \leq k_n, n \geq 1$ , and for any increasing sequence  $\{k_n, n \geq 1\}$  of positive integers. Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$  and  $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$  be positive, even, and continuous functions satisfying (26) for some  $\alpha \geq 1$  and  $0 < \beta \leq 2$ .*

*Suppose that for some array  $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  of nonzero reals and  $k \geq 1/2$ ,*

$$\sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} \frac{\psi(\|X_{ni}\|)}{\psi(a_{ni}^{-1})} \right)^k < \infty, \quad \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[\|X_{ni}\| \geq c_{ni}] < \infty, \tag{27}$$

*for some array  $\{c_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  of positive numbers such that*

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \varphi(c_{ni}) \frac{E\varphi(\|X_{ni}\|)}{\varphi^2(a_{ni}^{-1})} < \infty. \tag{28}$$

*Then (17) is equivalent to (18).*

*Putting  $\psi(x) = |x|^p, 1 < p \leq \beta \leq 2$ , and  $c_{ni} = |a_{ni}^{-1}|$ , we obtain the following result for a separable Banach space of Rademacher type  $p$ .*

**COROLLARY 6.** Let  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  be an array of row-wise independent  $\mathfrak{B}$ -valued random variables in a separable Banach space of Rademacher type  $p, 1 < p \leq \beta \leq 2$ , with  $EX_{ni} = 0$  for all  $1 \leq i \leq k_n, n \geq 1$ , and for some increasing sequence  $\{k_n, n \geq 1\}$  of positive integers. Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a positive, even, and continuous function such that

$$\frac{\varphi(|x|)}{|x|^\alpha} \nearrow \text{ as } |x| \nearrow, \tag{29}$$

for some  $\alpha \geq 1$ .

Then, for some array  $\{a_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  of nonzero reals and some integer  $k \geq 1$ , the conditions

$$\sum_{n=1}^{\infty} \left[ \sum_{i=1}^{k_n} \frac{E(\|X_{ni}\|^p)}{|a_{ni}|^{-p}} \right]^k < \infty, \tag{30}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\varphi(\|X_{ni}\|)}{\varphi(a_{ni}^{-1})} < \infty \tag{31}$$

imply (18).

**PROOF.** Putting  $M_{ni} = K_{ni} = 1$  and using (20), we see that it is enough to show that Theorem 4 holds for  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  and  $k \geq 2$ .

Indeed, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} \frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p} \right)^k \\ & \leq \sum_{n=1}^{\infty} \sum^* \binom{k}{s_1, \dots, s_{k_n}} E \left( \frac{\|X'_{n1}\|^p}{|a_{n1}^{-1}|^p} \right)^{s_1} E \left( \frac{\|X'_{n2}\|^p}{|a_{n2}^{-1}|^p} \right)^{s_2} \cdots E \left( \frac{\|X'_{nk_n}\|^p}{|a_{nk_n}^{-1}|^p} \right)^{s_{k_n}}, \end{aligned} \tag{32}$$

where the sum  $\sum^* \binom{k}{s_1, \dots, s_{k_n}}$  is over all choices of  $\{s_1, s_2, \dots, s_{k_n}\}, s_i \in \{0, 1, 2, \dots, k\}$ , such that  $\sum_{i=1}^{k_n} s_i = k$ . Choose  $n$  sufficiently large so that  $k_n > k$ . Let  $m = m(s_1, s_2, \dots, s_{k_n})$  be a number of  $s_i \neq 0$ . We see that  $m$  takes all the values from the set  $\{1, 2, \dots, k\}$ . Changing the order in our sum, we can express the right-hand side of (32) in the following form:

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^k \sum_{\substack{1 \leq i_j \leq k_n, \\ j=1,2,\dots,m \\ i_j \neq i_k \forall k \neq j}} \binom{k}{s_{i_1}, \dots, s_{i_m}} E \left( \frac{\|X'_{ni_1}\|^p}{|a_{ni_1}^{-1}|^p} \right)^{s_{i_1}} E \left( \frac{\|X'_{ni_2}\|^p}{|a_{ni_2}^{-1}|^p} \right)^{s_{i_2}} \cdots E \left( \frac{\|X'_{ni_m}\|^p}{|a_{ni_m}^{-1}|^p} \right)^{s_{i_m}} \\ & \leq \sum_{n=1}^{\infty} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq k_n} E \left( \frac{\|X'_{ni_1}\|^p}{|a_{ni_1}^{-1}|^p} \right) E \left( \frac{\|X'_{ni_2}\|^p}{|a_{ni_2}^{-1}|^p} \right) \cdots E \left( \frac{\|X'_{ni_k}\|^p}{|a_{ni_k}^{-1}|^p} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \left( \sum_{m=1}^{k-1} \sum_{\substack{1 \leq i_j \leq k_n, \\ j=1,2,\dots,m \\ i_j \neq i_k \forall k \neq j}} * \binom{k}{s_1, \dots, s_{i_m}} \left( \prod_{h=1}^L E \left( \frac{\|X'_{ni_{j_h}}\|^p}{|a_{ni_{j_h}}^{-1}|^p} \right)^{s_{i_{j_h}}} \right) \right) \\
 & \cdot \left( \prod_{j=1}^N E \frac{\|X'_{ni_{h_j}}\|^p}{|a_{ni_{h_j}}^{-1}|^p} \right) \Bigg\}, \tag{33}
 \end{aligned}$$

where  $L =$  number of  $s_i \geq 2, N =$  number of  $s_i = 1,$  and  $\{s_{i_1}, \dots, s_{i_m}\} = \{s_{i_{j_h}}, h = 1, \dots, L\} \cup \{s_{i_{h_j}}, s_{i_{h_j}} = 1, h = 1, \dots, N\}, \{s_{i_{j_h}}, h = 1, \dots, L\} \cap \{s_{i_{h_j}}, s_{i_{h_j}} = 1, h = 1, \dots, N\} = \emptyset.$

But

$$\left( \frac{\|X'_{ni_j}\|^p}{|a_{ni_j}^{-1}|^p} \right)^{s_{i_j}} \leq \frac{\|X'_{ni_j}\|^p}{|a_{ni_j}^{-1}|^p}, \quad E \frac{\|X'_{ni_j}\|^p}{|a_{ni_j}^{-1}|^p} \leq \sum_{i=1}^{k_n} \frac{E \|X'_{ni_j}\|^p}{|a_{ni_j}^{-1}|^p} \quad \text{for } 1 \leq i_j \leq k_n, \tag{34}$$

so the right-hand side of (33) can be estimated as follows:

$$\begin{aligned}
 & C \sum_{n=1}^{\infty} \left\{ \left( \sum_{i=1}^{k_n} E \frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p} \right)^k + \left( \sum_{i=1}^{k_n} E \frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p} \right)^L \cdot \left( \sum_{i=1}^{k_n} E \frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p} \right)^M \right\} \\
 & \leq C \sum_{n=1}^{\infty} \left( \sum_{i=1}^{k_n} E \frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p} \right)^k < \infty. \tag{35}
 \end{aligned}$$

Therefore, assumption (14) of Theorem 4 holds. Moreover, by (31), we get (15) and (16) of Theorem 4. We also note that if  $\mathcal{B}$  is a Banach space of Rademacher type  $p, 1 < p \leq 2,$  we have the following estimation:

$$P[\|S_n\| \geq \varepsilon] \leq \varepsilon^{-p} E \|S_n\|^p \leq \varepsilon \sum_{i=1}^{k_n} E \|a_{ni} X_{ni}\|^p \leq \varepsilon^{-p} \sum_{i=1}^{k_n} \frac{E(\|X_{ni}\|^p)}{|a_{ni}|^{-p}} = o(1). \tag{36}$$

This fact, together with (20), completes the proof. □

Now we present the result which gives the sufficient conditions for the equivalence of (17) and (18) in the Chung-Teicher terms (cf. Teicher [13]).

**THEOREM 7.** *Let  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for all  $1 \leq i \leq k_n, n \geq 1,$  and for some increasing sequence  $\{k_n, n \geq 1\}$  of positive integers. Let  $\varphi_{ni} : \mathbb{R} \rightarrow \mathbb{R}_+$  be positive, even, and*



continuous functions which, for constants  $\alpha_{ni} \geq 1$ ,  $0 < \beta_{ni} \leq 2$ ,  $K_{ni} > 0$ , and  $M_{ni} > 0$ ,  $n \geq 1$ ,  $i \geq 1$ , satisfy (12) and

$$|x_1| \leq |x_2| \Rightarrow \frac{|x_1|^{\beta_{ni}}}{\varphi_{ni}(|x_1|)} \leq M_{ni} \frac{|x_2|^{\beta_{ni}}}{\varphi_{ni}(|x_2|)}. \tag{37}$$

Suppose that for some array  $\{a_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  of nonzero reals,

$$\sum_{n=1}^{\infty} \sum_{i=2}^{k_n} M_{ni} \frac{E\varphi_{ni}(\|X_{ni}\|)}{P\varphi_{ni}(a_{ni}^{-1})} \sum_{j=1}^{i-1} M_{nj} \frac{E\varphi_{nj}(\|X_{nj}\|)}{\varphi_{nj}(a_{nj}^{-1})} < \infty, \tag{38}$$

$$\sum_{i=1}^{k_n} M_{ni} \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1), \tag{39}$$

$$\sum_{i=1}^{k_n} K_{ni} \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1), \tag{40}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[\|X_{ni}\| \geq c_{ni}] < \infty, \tag{41}$$

for some array  $\{c_{ni}, i \geq 1, n \geq 1\}$  of positive numbers such that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} M_{ni}^2 \cdot \varphi_{ni}(c_{ni}) \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty, \tag{42}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} K_{ni}^2 \cdot \varphi_{ni}(c_{ni}) \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty. \tag{43}$$

Then (17) is equivalent to (18).

**PROOF.** Let  $X'_{ni} = X_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]$ ,  $X^*_{ni} = X'_{ni} - EX'_{ni}$ ,  $S'_n = \sum_{i=1}^{k_n} a_{ni}X'_{ni}$ , and  $S_n^* = \sum_{i=1}^{k_n} a_{ni}X^*_{ni}$ . By (20), we state that  $\{S_n, n \geq 1\}$  and  $\{S_n^*, n \geq 1\}$  are equivalent.

Moreover, we have by (40)

$$\begin{aligned} \left\| \sum_{i=1}^{k_n} a_{ni}EX_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|] \right\| &\leq \sum_{i=1}^{k_n} |a_{ni}| \cdot \|EX_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]\| \\ &= \sum_{i=1}^{k_n} |a_{ni}| \cdot \|EX_{ni}I[\|X_{ni}\| > |a_{ni}^{-1}|]\| \\ &\leq \sum_{i=1}^{k_n} \frac{E\|X_{ni}\|I[\|X_{ni}\| > |a_{ni}^{-1}|]}{|a_{ni}^{-1}|} \\ &\leq \sum_{i=1}^{k_n} \frac{E\|X_{ni}\|^{\alpha_{ni}}I[\|X_{ni}\| > |a_{ni}^{-1}|]}{|a_{ni}^{-1}|^{\alpha_{ni}}} \\ &\leq \sum_{i=1}^{k_n} K_{ni} \cdot \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1). \end{aligned} \tag{44}$$

Now we define

$$Y_{ni} = E(\|S_n^*\| | \mathcal{F}_{ni}) - E(\|S_n^*\| | \mathcal{F}_{ni-1}), \tag{45}$$

where  $\mathcal{F}_{ni} = \sigma(X_{n1}^*, X_{n2}^*, \dots, X_{ni}^*)$  and  $\mathcal{F}_{n0} = \{\emptyset, \Omega\}$ . Then we have

$$\|S_n^*\| - E\|S_n^*\| = \sum_{i=1}^{k_n} Y_{ni} \tag{46}$$

and we note that  $\{Y_{ni}, 1 \leq i \leq k_n\}$  is a sequence of martingale differences for a fixed  $n$ .

Now we are going to prove that

$$E\|S_n^*\| \rightarrow 0, \quad n \rightarrow \infty. \tag{47}$$

First we will show that

$$\|S_n^*\| - E\|S_n^*\| \xrightarrow{P} 0, \quad n \rightarrow \infty. \tag{48}$$

Using Chebyshev's inequality, [Lemma 3](#), and assumption [\(39\)](#), we get, for any  $\varepsilon > 0$ ,

$$\begin{aligned} &P[|\|S_n^*\| - E\|S_n^*\|| > \varepsilon] \\ &\leq \varepsilon^{-2} E(\|S_n^*\| - E\|S_n^*\|)^2 = \varepsilon^{-2} E\left(\sum_{i=1}^{k_n} Y_{ni}\right)^2 = \varepsilon^{-2} \sum_{i=1}^{k_n} E(Y_{ni}^2) \\ &\leq \varepsilon^{-2} \sum_{i=1}^{k_n} E(\|a_{ni}X_{ni}^*\| + E\|a_{ni}X_{ni}^*\|)^2 \leq 8\varepsilon^{-2} \sum_{i=1}^{k_n} a_{ni}^2 E\|X'_{ni}\|^2 \\ &= 8\varepsilon^{-2} \sum_{i=1}^{k_n} E \frac{\|X'_{ni}\|^{\beta_{ni}}}{|a_{ni}^{-1}|^{\beta_{ni}}} \cdot \frac{\|X'_{ni}\|^{2-\beta_{ni}}}{|a_{ni}^{-1}|^{2-\beta_{ni}}} \leq 8\varepsilon^{-2} \sum_{i=1}^{k_n} M_{ni} \frac{E\varphi_{ni}(\|X'_{ni}\|)}{\varphi_{ni}(a_{ni})} = o(1). \end{aligned} \tag{49}$$

Thus, we conclude that [\(48\)](#) holds and, together with [\(17\)](#), [\(20\)](#), and [\(44\)](#), gives [\(47\)](#).

Now we want to show that  $\|S_n^*\| \rightarrow 0$  a.s., as  $n \rightarrow \infty$ .

By [\(47\)](#) it is enough to prove that

$$\|S_n^*\| - E\|S_n^*\| \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \tag{50}$$

Taking into account the identity

$$(\|S_n^*\| - E\|S_n^*\|)^2 = \sum_{i=1}^{k_n} Y_{ni}^2 + 2 \sum_{i=2}^{k_n} Y_{ni} \sum_{j=1}^{i-1} Y_{nj} \tag{51}$$

and using the notation

$$Z_{ni} = Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] - E(Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] | \mathcal{F}_{ni-1}), \quad 1 \leq i \leq k_n, \tag{52}$$

we have, by Chebyshev's inequality, [Lemma 3](#), and assumption (42),

$$\begin{aligned}
 \sum_{n=1}^{\infty} P \left[ \left| \sum_{i=1}^{k_n} Z_{ni} \right| > \varepsilon \right] &\leq \varepsilon^{-2} \sum_{n=1}^{\infty} E \left( \sum_{i=1}^{k_n} Z_{ni} \right)^2 = \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E(Z_{ni}^2) \\
 &\leq C \cdot \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E(Y_{ni}^4 I[\|X_{ni}\| < c_{ni}]) \\
 &\leq C \cdot \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E\|a_{ni} X'_{ni}\|^4 I[\|X_{ni}\| < c_{ni}] \tag{53} \\
 &\leq C \cdot \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E \left( \frac{\|X'_{ni}\|^{2\beta_{ni}}}{|a_{ni}^{-1}|^{2\beta_{ni}}} \cdot \frac{\|X'_{ni}\|^{4-2\beta_{ni}}}{|a_{ni}^{-1}|^{4-2\beta_{ni}}} I[\|X_{ni}\| < c_{ni}] \right) \\
 &\leq C \cdot \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} M_{ni} \cdot \varphi_{ni}(c_{ni}) \cdot \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty.
 \end{aligned}$$

Hence, by the Borel-Cantelli Lemma, we obtain

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] - \sum_{i=1}^{k_n} E(Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] | \mathcal{F}_{ni-1}) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \tag{54}$$

Using [Lemma 3](#), we can note by assumption (39) that

$$\begin{aligned}
 &\sum_{i=1}^{k_n} E(Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] | \mathcal{F}_{ni-1}) \\
 &\leq 8 \sum_{i=1}^{k_n} E(\|a_{ni} X'_{ni}\|^2) \leq \sum_{i=1}^{k_n} E \left( \frac{\|X'_{ni}\|^{\beta_{ni}}}{|a_{ni}^{-1}|^{\beta_{ni}}} \cdot \frac{\|X'_{ni}\|^{2-\beta_{ni}}}{|a_{ni}^{-1}|^{2-\beta_{ni}}} \right) \tag{55} \\
 &\leq \sum_{i=1}^{k_n} M_{ni} \cdot \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1),
 \end{aligned}$$

which, together with (54), allows us to state that

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \tag{56}$$

To prove that

$$\sum_{i=1}^{k_n} Y_{ni}^2 \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty, \tag{57}$$

we only need to show that

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[\|X_{ni}\| \geq c_{ni}] \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \tag{58}$$

Indeed, by (41) and Lemma 3, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} P \left[ \left| \sum_{i=1}^{k_n} Y_{ni}^2 I[|X_{ni}| \geq c_{ni}] \right| \geq \varepsilon \right] \\ & \leq \varepsilon^{-1} \sum_{n=1}^{\infty} E \left| \sum_{i=1}^{k_n} Y_{ni}^2 I[|X_{ni}| \geq c_{ni}] \right| \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} E \|a_{ni} X'_{ni}\|^2 I[|X_{ni}| \geq c_{ni}] \quad (59) \\ & \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} EI[|X_{ni}| \geq c_{ni}] \leq \varepsilon^{-1} \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[|X_{ni}| \geq c_{ni}] < \infty, \end{aligned}$$

and, by the Borel-Cantelli Lemma, we get (57). To close this proof, we must show that

$$\sum_{i=2}^{k_n} Y_{ni} \sum_{j=1}^{i-1} Y_{nj} \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \quad (60)$$

Using the fact that  $\{Y_{ni} \sum_{j=1}^{i-1} Y_{nj}, 2 \leq i \leq k_n\}$  is a sequence of martingale differences for each  $n$ , we have, by Chebyshev's inequality, Lemma 3, and assumption (38),

$$\begin{aligned} & \sum_{n=1}^{\infty} P \left[ \left| \sum_{i=2}^{k_n} Y_{ni} \sum_{j=1}^{i-1} Y_{nj} \right| > \varepsilon \right] \\ & \leq \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=2}^{k_n} E \left[ (\|a_{ni} X_{ni}^*\| + E \|a_{ni} X_{ni}^*\|)^2 \cdot \left( \sum_{j=1}^{i-1} Y_{nj} \right)^2 \right] \\ & \leq \varepsilon^{-2} \sum_{n=1}^{\infty} \sum_{i=2}^{k_n} E (\|a_{ni} X_{ni}^*\| + E \|a_{ni} X_{ni}^*\|)^2 \cdot \sum_{j=1}^{i-1} E(Y_{nj}^2) \quad (61) \\ & \leq C \sum_{n=1}^{\infty} \sum_{i=2}^{k_n} E \|a_{ni} X'_{ni}\|^2 \sum_{j=1}^{i-1} E \|a_{nj} X'_{nj}\|^2 C \sum_{n=1}^{\infty} \sum_{i=2}^{k_n} M_{ni} \frac{E \varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} \\ & \quad \times \sum_{j=1}^{i-1} M_{nj} \frac{E \varphi_{nj}(\|X_{nj}\|)}{\varphi_{nj}(a_{nj}^{-1})} < \infty, \end{aligned}$$

which, together with the Borel-Cantelli Lemma, implies (60).

Thus, we have proved that

$$\sum_{i=1}^{k_n} a_{ni} (X'_{ni} - EX'_{ni}) \rightarrow 0 \quad \text{a.s., } n \rightarrow \infty. \quad (62)$$

But  $\{\sum_{i=1}^{k_n} a_{ni} X'_{ni}, n \geq 1\}$  and  $\{\sum_{i=1}^{k_n} a_{ni} X_{ni}, n \geq 1\}$  are equivalent and (44) holds, so we get (18). □

Now we consider an array  $\{X_{ni}, i \geq 1, n \geq 1\}$  of independent random elements which are stochastically dominated by a random element  $X$  in the sense of (10).

**COROLLARY 8.** *Let  $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for all  $1 \leq i \leq k_n, n \geq 1$ , and some increasing*

sequence  $\{k_n, n \geq 1\}$  of positive integers stochastically dominated by a random element  $X$  in the sense of (10). Let  $E\|X\|^p < \infty, 1 < p \leq 2$ . Assume that  $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$  is an array of nonzero reals.

Suppose that for some increasing sequence  $\{k_n, n \geq 1\}$  of positive integers,

$$\sum_{n=1}^{\infty} \sum_{i=2}^{k_n} |a_{ni}|^p \sum_{j=1}^{i-1} |a_{nj}|^p < \infty, \tag{63}$$

$$\sum_{i=1}^{k_n} |a_{ni}|^p = o(1), \tag{64}$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} P[|X| \geq c_{ni}] < \infty, \tag{65}$$

for some array  $\{c_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  of positive numbers such that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} c_{ni}^p \cdot |a_{ni}|^{2p} < \infty. \tag{66}$$

Then (17) is equivalent to (18).

**PROOF.** Put  $\varphi_{ni}(x) = |x|^p, 1 < p \leq 2$ , for all  $1 \leq i \leq k_n, n \geq 1$ . Then there exist  $\alpha$  and  $\beta$  such that  $1 \leq \alpha < p < \beta \leq 2$ , for which (12) and (37) hold with  $\alpha_{ni} = \alpha, \beta_{ni} = \beta$ , and  $M_{ni} = K_{ni} = 1, i \geq 1, n \geq 1$ . Moreover, (10) and  $E\|X\|^p < \infty$  imply  $E\|X_{ni}\|^p < \infty$  for all  $1 \leq i \leq k_n$  and  $n \geq 1$ . Therefore, we see that assumptions (38)-(43) of Theorem 7 are fulfilled ((38) is fulfilled by (63), (39) and (40) by (64), (41) by (65), and (42)-(43) by (66)). □

**COROLLARY 9.** Let  $\{X_{ni}, (1 \leq i \leq k_n, n \geq 1)\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for  $1 \leq i \leq k_n, n \geq 1$ , stochastically dominated by a random element  $X$  in the sense of (10). Assume that  $\{a_{ni}, i \geq 1, n \geq 1\}$  is a Toeplitz array such that  $a_{ni} \neq 0, i \geq 1, n \geq 1$ , and for some  $\gamma$  such that  $\gamma p > 2, 1 < p \leq 2$ ,

$$\sup_{i \geq 1} |a_{ni}| = O(n^{-\gamma}). \tag{67}$$

If  $E\|X\|^p < \infty$ , then (17) is equivalent to (18).

**PROOF.** To prove this result, it is enough to show that under assumption (67), conditions (14), (15), and (16) of Theorem 4 are fulfilled.

Indeed, we see that for  $\psi_{ni}(x) = \varphi_{ni}(x) = |x|^p$  and  $M_{ni} = K_{ni} = 1, i \geq 1, n \geq 1, k = 1$ , and  $k_n = n, n \geq 1$ , we have

$$\sum_{n=1}^{\infty} E \left( \sum_{i=1}^n |a_{ni}|^p \|X_{ni}\|^p \right) \leq C \sum_{n=1}^{\infty} \left( \sup_{i \geq 1} |a_{ni}| \right)^p \sum_{i=1}^n E\|X\|^p \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\gamma p - 1}} < \infty. \tag{68}$$

So, condition (14) is fulfilled.

For  $c_{ni} = D/a_{ni}$ ,  $i \geq 1$ ,  $n \geq 1$ , we get (15) and (16):

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{i=1}^n P \left[ \|X_{ni}\| \geq \frac{D}{a_{ni}} \right] \\ & \leq \sum_{n=1}^{\infty} \sum_{i=1}^n P \left[ D\|X\| \geq \frac{D}{a_{ni}} \right] \leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^p E\|X\|^p \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\gamma p - 1}} < \infty, \quad (69) \\ & \sum_{n=1}^{\infty} \sum_{i=1}^n \left| \frac{D}{a_{ni}} \right|^p |a_{ni}|^{2p} E\|X_{ni}\|^p \leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^p E\|X\|^p \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\gamma p - 1}} < \infty. \end{aligned}$$

□

Similarly, we can prove the next corollary.

**COROLLARY 10.** Let  $\{X_{ni}, (1 \leq i \leq n, n \geq 1)\}$  be an array of row-wise independent  $\mathcal{B}$ -valued random variables with  $EX_{ni} = 0$  for  $1 \leq i \leq k_n, n \geq 1$ , and some increasing sequence  $\{k_n, n \geq 1\}$  of positive integers stochastically dominated by a random element  $X$  in the sense of (10). Assume that  $\{a_{ni}, i \geq 1, n \geq 1\}$  is a Toeplitz array such that  $a_{ni} \neq 0, i \geq 1, n \geq 1$ , and for some  $\gamma$  such that  $\gamma q > 2, 0 < q \leq 2$ ,

$$\sum_{i=1}^{k_n} |a_{ni}|^q = O(n^{-\gamma}). \quad (70)$$

If  $E\|X\|^q < \infty$ , then (17) is equivalent to (18).

Using the fact that  $\mathcal{B}$  is a Banach space of Rademacher type  $p$ , we see that (67) implies (17) under  $E\|X\|^p < \infty$ . Indeed, we see that

$$P \left[ \left\| \sum_{i=1}^n a_{ni} X_{ni} \right\| \geq \varepsilon \right] \leq \varepsilon^{-p} \sum_{i=1}^n E\|a_{ni} X_{ni}\|^p \leq C \sum_{i=1}^n |a_{ni}|^p E\|X\|^p \leq \frac{C}{n^{\gamma p - 1}} = o(1). \quad (71)$$

So, we can formulate the next result.

**COROLLARY 11.** Let  $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$  be an array of row-wise independent random elements taking values in a Banach space of Rademacher type  $p, 1 < p \leq 2$ , with  $EX_{ni} = 0$  for  $i \geq 1, n \geq 1$ , stochastically dominated by a random element  $X$  in the sense of (10) and  $E\|X\|^p < \infty$ . Assume that  $\{a_{ni}, i \geq 1, n \geq 1\}$  is a Toeplitz array. If for some  $\gamma$  satisfying  $\gamma p > 2$  and  $a_{ni} \neq 0, i \geq 1, n \geq 1$ , (67) holds, then (18) is fulfilled.

**EXAMPLE 12.** Let  $\{X_n, n \geq 1\}$  be a sequence of independent random variables which are stochastically dominated by a random variable  $X$  such that  $EX = 0$  and  $E|X|^\gamma < \infty$  for some  $\gamma, 1 \leq \alpha < \gamma < \beta \leq 2$ . Set  $X_{ni} = X_i^i/n^{(1+\varepsilon)i/\gamma}$  and  $a_{ni} = 1/n^{i/\gamma}$ , for  $1 \leq i \leq n, n \geq 1$ .

We will verify that conditions (14), (15), and (16) of Theorem 4 hold with  $k_n = n, n \geq 1, c_{ni} = n^{i/\gamma}$ , and  $\varphi_{ni}(x) = \psi_{ni}(x) = |x|^{\gamma/i}, 1 \leq i \leq n, n \geq 1, k = 1$ .

We note that the functions  $\varphi_{ni}$  and  $\psi_{ni}, 1 \leq i \leq n, n \geq 1$ , satisfy conditions (12) and (13) with  $\alpha_{ni} = \alpha/i, \beta_{ni} = \beta/i$ , and  $M_{ni} = K_{ni} = 1$ .

To verify (14), note that

$$\sum_{n=1}^{\infty} E \left( \sum_{i=1}^n M_{ni} \frac{\psi_{ni}(|X_{ni}|)}{\psi_{ni}(a_{ni}^{-1})} \right) \leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_{ni}|^y}{n} \leq C \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_i|^y}{n^{1+\epsilon} \cdot n} \leq C \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty, \tag{72}$$

and to verify (15), note that

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{i=1}^n P[|X_{ni}| > n^{i/y}] \\ & \leq \sum_{n=1}^{\infty} \sum_{i=1}^n P \left[ \frac{|X_i|^i}{n^{(1+\epsilon)i/y}} > n^{i/y} \right] \leq \sum_{n=1}^{\infty} \sum_{i=1}^n P[|X_i|^i > n^{(2+\epsilon)i/y}] \leq C \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty. \end{aligned} \tag{73}$$

Moreover, we see that

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{i=1}^n K_{ni}^2 \cdot \varphi_{ni}(c_{ni}) \frac{E\varphi_{ni}(|X_{ni}|)}{\varphi_{ni}^2(a_{ni}^{-1})} \\ & \leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{nE|X_{ni}|^{y/i}}{n^2} \leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E|X_i|^y}{n^{2+\epsilon}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty. \end{aligned} \tag{74}$$

To complete the proof, we note that the real number space is of Rademacher type  $p$  with  $p = 2$ , so by the estimations (36), (72), and (73), we have (17). Thus, by Theorem 4, we have

$$\sum_{i=1}^{\infty} a_{ni} X_{ni} = \sum_{i=1}^n n^{-(2+\epsilon)i/y} X_i^i \rightarrow 0, \quad \text{a.s., } n \rightarrow \infty. \tag{75}$$

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