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A COMMUTATIVITY-OR-FINITENESS CONDITION FOR RINGS

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We show that a ring with only finitely many noncentral subrings must be either commutative or finite.

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1. Preliminaries.

In [1] the following theorem was proved.

Theorem 1.1. *If R is a periodic ring having only finitely many noncentral subrings of zero divisors, then R is either finite or commutative.*

In view of this result, it was conjectured that any ring with only finitely many noncentral subrings is either finite or commutative. It is our principal goal to prove this conjecture; and in the process we provide a new proof of Theorem 1.1.

For any ring R, the symbols N, D, Z, and $\wp(R)$ will denote, respectively, the set of nilpotent elements, the set of zero divisors, the center, and the prime radical; and R will be called reduced if $N = \{0\}$.

We will require the following three lemmas.

Lemma 1.2 [5, Corollary 5]**.** *If R is any ring in which N is finite, then $R/\wp(R) = B \oplus C$, where B is reduced and C is a direct sum of finitely many total matrix rings over finite fields.*

Lemma 1.3. *If R is any ring in which $\wp(R) \subseteq Z$, then zero divisors in $R/\wp(R)$ may be lifted to zero divisors in R.*

Proof. We show that if $\bar{x} = x + \wp(R)$ is a zero divisor in $\bar{R} = R/\wp(R)$, then x is a zero divisor in R. Clearly, this is the case for $\bar{x} = \bar{0}$, so we assume that $\bar{x} \neq \bar{0}$ and that $\bar{y} \in \bar{R} \setminus \{\bar{0}\}$ is such that $\bar{x}\bar{y} = \bar{0}$. Then, $xy \in \wp(R)$, and there exists n such that $(xy)^n = 0$. Moreover, since $xyr \in \wp(R)$ for all $r \in R$, we have $xy \in Z$ and $xyr \in Z$ for all $r \in R$.

Since $y \notin \wp(R)$, y is not strongly nilpotent, therefore there exist $r_1, r_2, \ldots, r_{n-1} \in R$ such that $yr_1yr_2 \cdots yr_{n-1}y \neq 0$. Since xy and all xyr_i are central, we get $x^n yr_1yr_2 \cdots yr_{n-1}y = xyr_1x^{n-1}yr_2 \cdots y = xyr_1xyr_2x^{n-2} \cdots y = xyr_1xyr_2 \cdots xyr_{n-1}xy = (xy)^n r_1r_2 \cdots r_{n-1} = 0$; hence x^n is a zero divisor in R, and so is x. □

Lemma 1.4 [3]**.** *If R is a periodic ring with $N \subseteq Z$, then R is commutative.*

2. The main theorems.

We begin with results on rings having only finitely many noncentral subrings of zero divisors, which we will call D-nearly central rings (DNC-rings).

Lemma 2.1. *If R is any DNC-ring, then $N \subseteq Z$ or N is finite.*

Proof. Assume $N \setminus Z \neq \phi$ and let $a \in N \setminus Z$. By [1, Lemma 3], a has finite additive order and hence a generates a finite noncentral subring of zero divisors. Thus, $N \setminus Z$ is finite; and since $u + N \cap Z \subseteq N \setminus Z$ for all $u \in N \setminus Z$, we see that $N \cap Z$ is also finite. □

Lemma 2.2. *Let R be an infinite DNC-ring with N finite. Then every nil ideal of R is central.*

Proof. Suppose I is a noncentral nil ideal. Since I is finite, the two-sided annihilator $A(I)$ is of finite index in R, hence is infinite. Moreover, for any subring S of $A(I)$, $S + I$ is a noncentral subring of zero divisors of R; thus, there are only finitely many subrings $S + I$ with S a subring of $A(I)$. It follows at once that $A(I)$ has only finitely many finite subrings, since otherwise there exist infinitely many distinct finite subrings $S_1, S_2, \ldots$, such that $S_1 + I = S_2 + I = \cdots = S_n + I = \cdots$ is a finite ring with infinitely many subrings.

It is well known that every infinite ring has infinitely many subrings (cf. [6]); hence $A(I)$ has infinitely many infinite subrings, and so does every infinite subring of $A(I)$. Thus $A(I)$ has an infinite strictly decreasing sequence $S_1 \supset S_2 \supset \cdots$ of subrings; and for some j, $S_j + I = S_i + I$ for all $i \geq j$. For each $i \geq j$, choose $a_i \in S_i \setminus S_{i+1}$ and write $a_i = b_{i+1} + c_i$, where $b_{i+1} \in S_{i+1}$ and $c_i \in I$. Since I is finite, there exist h, k such that $j \leq h < k$ and $c_h = c_k$. It follows that $a_h = b_{h+1} + a_k - b_{k+1}$ so that $a_h \in S_{h+1}$, a contradiction. Therefore, I must be central. □

Lemma 2.3. *If R is any DNC-ring, then either R is finite or $N \subseteq Z$.*

Proof. Let R be any infinite DNC-ring. By Lemmas 2.1 and 2.2, we may assume that N is finite and $\wp(R) \subseteq Z$; and it follows by Lemma 1.3 that $\bar{R} = R/\wp(R)$ is a DNC-ring. By Lemma 1.2, $\bar{R} = B \oplus C$, where B is an infinite reduced ring and C is a direct sum of finitely many total matrix rings over finite fields. Suppose $C \neq \{0\}$ and contains the total matrix ring K_n over the finite field K. We may assume $n > 1$; otherwise K_n can be incorporated into B. Now B has infinitely many subrings; and for each subring S of B, $S + Ke_{1n}$ is a noncentral subring of zero divisors of $\bar{R}$. Since $\bar{R}$ is a DNC-ring, we must therefore have $C = \{0\}$ so that $\bar{R}$ is reduced. It follows that in R, $N = \wp(R) \subseteq Z$. □

Remark 2.4. Theorem 1.1 follows at once from Lemmas 1.4 and 2.3. Thus, we have produced a proof of Theorem 1.1 which is very different from the one in [1].

We now arrive at our main result.

Theorem 2.5. *If R is any ring with only finitely many noncentral subrings, then R is finite or commutative.*

Proof. Suppose R is a counterexample with the minimum possible number of noncentral subrings. Then R is infinite and every proper infinite subring of R is commutative. Since R is a DNC-ring, Lemmas 1.4 and 2.3 show that all finite subrings of R are commutative as well; hence R is a so-called one-step noncommutative ring—a noncommutative ring in which every proper subring is commutative.

Note that if H is any set of pairwise noncommuting elements of R, the centralizers of the elements of H are pairwise distinct noncentral subrings of R. Hence any set of pairwise noncommuting elements is finite, and by [2, Theorem 2.1], Z has finite index in R. But by a theorem of Ikeda [4], a one-step noncommutative ring with $[R:Z] < \infty$ must be finite. Thus, we have contradicted our assumption that R was a counterexample. $\square$

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