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CONDITIONS FOR GLOBAL EXISTENCE OF SOLUTIONS OF ORDINARY DIFFERENTIAL, STOCHASTIC DIFFERENTIAL, AND PARABOLIC EQUATIONS

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First, we prove a necessary and sufficient condition for global in time existence of all solutions of an ordinary differential equation (ODE). It is a condition of one-sided estimate type that is formulated in terms of so-called proper functions on extended phase space. A generalization of this idea to stochastic differential equations (SDE) and parabolic equations (PE) allows us to prove similar necessary and sufficient conditions for global in time existence of solutions of special sorts: L^1-complete solutions of SDE (this means that they belong to a certain functional space of L^1 type) and the so-called complete Feller evolution families giving solutions of PE. The general case of equations on noncompact smooth manifolds is under consideration.

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1. Introduction. In this note, we consider some questions connected with global in time existence of solutions of various differential equations (ordinary, stochastic, and parabolic). The main goal is to obtain necessary and sufficient conditions.

At the moment, plenty of sufficient conditions for global existence of solutions can be found in the literature. We would like to point out the so-called conditions with one-sided estimates for ordinary differential equations (ODE), similar conditions with Lyapunov functions for parabolic equations (PE), and a certain very general condition of the same nature from [1] for stochastic differential equations (SDE). Recall that this sort of conditions deals with estimates on the derivative of a certain function $v(x)$ with respect to the right-hand side of the equation or with respect to the corresponding generator, where (say, in Euclidean space) $v(x) \to \infty$ as $x \to \infty$ (a particular case of proper function, see below).

It is shown in this note that after some modification and transition to extended phase space, conditions of this sort become necessary and sufficient or close to them. We deal with the general case of equations on finite-dimensional manifolds.

Notice that a necessary and sufficient condition for global existence of solutions of ODE on manifolds of two-sided sort (dealing with estimates on the norm of the right-hand side), based on a similar idea of passing to extended phase space, was obtained in [3] (see also [4, Section 1]).

In Section 2 of this note, we consider the case of ODE. In Theorem 2.4, we show that all solutions of an ODE are well posed globally if and only if there exists a proper function on the extended phase space whose derivative with respect to the natural (space-time) extension of the right-hand side is uniformly bounded in absolute value.

Analogous approach to SDE allows us to obtain a necessary and sufficient condition for global existence of solutions of a special type. Namely, those solutions are global and there exists a certain proper function such that having substituted the solution into the function, we obtain an integrable random variable (i.e., its expectation is finite). Thus the solution belongs to a functional space of L^1 type. In previous publications [5, 6], we call this property L^1-completeness of the stochastic flow (see the exact definitions below). Notice that a proper function is not specified a priori. A solution may not be L^1-complete with respect to the norm in the Euclidean space or with respect to the distance in a complete Riemannian manifold (particular cases of proper functions), but it may be L^1-complete with respect to some other proper function.

This sort of solutions is useful since the completeness and integrability properties are important for applications.

Recall that for a broad class of PE, the standard construction of a corresponding SDE is well known so that generalized solutions of a Cauchy problem for the PE are obtained via Feller evolution families (semigroups), generated by the SDE, if its solutions are global. If a solution of the SDE is L^1-complete, we call the Feller evolution family a complete one.

In Theorem 3.6, we obtain a necessary and sufficient condition for an SDE to have L^1-complete flow and so for the existence of a complete Feller evolution family giving special generalized solutions of the Cauchy problem for corresponding PE. This condition is formulated in terms of the existence of a certain proper function u on the extended phase space with properties analogous to those from Theorem 2.4 for the ODE (see above). In particular, the values of the space-time generator of the SDE on u are uniformly bounded.

In the proof of sufficiency, we start from some u on the extended phase space satisfying the conditions of Theorem 3.6, and construct a certain proper function v on the phase space giving the L^1-completeness of the flow. In the proof of necessity, we start from some v on the phase space giving the L^1-completeness, and construct a certain u on the extended phase space satisfying the conditions of Theorem 3.6.

2. The case of ordinary differential equations. Let M be a smooth manifold with dimension $n < \infty$.

Consider a certain smooth jointly in $t \in \mathbb{R}$, $m \in M$ vector field $X = X(t,m)$ on M. Its coordinate representation in a chart with respect to local coordinates $(q^1,\dots,q^n)$ takes the form $X = X^1(\partial/\partial q^1) + \cdots + X^n(\partial/\partial q^n)$. The vector field X can also be considered as the first-order differential operator on C^1-functions on M. For a function f, the value of the above operator is given as $Xf = X^1(\partial f/\partial q^1) + \cdots + X^n(\partial f/\partial q^n)$. The function Xf is also called the derivative of f along vector field X.

Definition 2.1. A curve $m(t)$ on M is called an integral curve of X if at any t, the vector $X_{m(t)}$ is equal to the derivative $\dot{m}(t)$.

Thus, the integral curves of X are defined by the ODE

$$\dot{m}(t) = X\big(t, m(t)\big). \tag{2.1}$$

Let $\gamma(t)$ be an integral curve of X such that $\gamma(0) = m$. It is well known that Xf is represented in terms of $\gamma(t)$ as follows: $Xf(m) = (d/dt)f(\gamma(t))|_{t=0}$.

Definition 2.2. A vector field X is called complete if all its integral curves are well posed for $t \in (-\infty, +\infty)$.

Denote by $m(s) : M \to M$, $s \in \mathbb{R}$, the flow of X. For any point $m \in M$ and time instant t, the orbit $m(s)(t,m) = m_{t,m}(s)$ of the flow is the solution of the equation

$$\dot{m}_{t,m}(s) = X(s, m_{t,m}(s)) \tag{2.2}$$

with the initial condition

$$m_{t,m}(t) = m. \tag{2.3}$$

Consider the extended phase space $M^+ = \mathbb{R} \times M$ with the natural projection $\pi^+ : M^+ \to M$, $\pi^+(t,m) = m$. Introduce the vector field X^+ on M^+ given at the point $(t,m) \in M^+$ as $X^+_{(t,m)} = (1, X(t,m))$. It is clear that its coordinate representation is given in the form $X^+ = \partial/\partial t + X^1(\partial/\partial q^1) + \cdots + X^n(\partial/\partial q^n)$. Hence, the corresponding differential operator on the space of C^1-smooth functions on M^+ takes the form $\partial/\partial t + X$.

Definition 2.3. A function f on a topological space $\mathbf{X}$ is called proper if the preimage of any relatively compact set in $\mathbb{R}$ is a relatively compact set in $\mathbf{X}$.

Recall that in any finite-dimensional space (in particular, in $\mathbb{R}$), a set is relatively compact if and only if it is bounded.

Examples of a proper function v^T are the norm in an Euclidean space (if $M = \mathbb{R}^n$) or the distance function on a complete Riemannian manifold (if it is smooth).

In what follows, we will mainly deal with proper functions on smooth manifolds.

Theorem 2.4. *A smooth vector field X on a finite-dimensional manifold M is complete if and only if there exists a smooth proper function $\varphi : M^+ \to \mathbb{R}$ such that the absolute value of the derivative $|X^+\varphi|$ of φ along X^+ is uniformly bounded, that is, $|X^+\varphi| = (|\partial/\partial t + X)\varphi| \le C$ at any $(t,m) \in M^+$ for a certain constant $C > 0$ that does not depend on (t,m).*

Proof

Sufficiency. Consider the flow $m^+(s) : M^+ \to M^+$, $s \in \mathbb{R}$, with the orbits $m^+(s)(t, m) = m^+_{(t,m)}(s)$ being the solutions of the equation

$$\dot{m}^+_{(t,m)}(s) = X^+\left(m^+_{(t,m)}(s)\right) \tag{2.4}$$

with initial conditions

$$m^+_{(t,m)}(t) = (t,m). \tag{2.5}$$

Consider the derivative $X^+\varphi$ of φ along X^+. At $(t,m) \in M^+$, we get the equality

$$X^+\varphi(t,m) = \frac{d}{ds}\varphi\left(m^+_{(t,m)}(s)\right)\Big|_{s=t} \tag{2.6}$$

(see above) and under the hypothesis of our theorem,

$$\left| \frac{d}{ds}\varphi\left(m^{+}_{(t,m)}(s)\right)\Big|_{s=t} \right| \le C. \tag{2.7}$$

Represent the values of φ along the orbit $m^{+}_{(t,m)}(s)$ as follows:

$$\varphi\left(m^{+}_{(t,x)}(s)\right) - \varphi(t,m) = \int_0^s \frac{d}{d\tau}\varphi\left(m^{+}_{(t,m)}(\tau)\right)d\tau. \tag{2.8}$$

From the last equality and from inequality (2.7), we evidently obtain that if s belongs to a finite interval, the values $\varphi(m^{+}_{(t,x)}(s))$ are bounded in $\mathbb{R}$. Then, since φ is proper, this means that the set $m^{+}_{(t,m)}(s)$ is relatively compact in M^{+}.

Recall that by the classical solution existence theorem, the domain of any solution of ODE is an open interval in $\mathbb{R}$. In particular, for $s > 0$, the solution of the above Cauchy problem is well posed for $s \in [t,\varepsilon)$. If $\varepsilon > 0$ is finite, then the corresponding values of the solution belong to a relatively compact set in M, and so the solution is well posed for $s \in [t,\varepsilon]$. The same arguments are valid also for $s < 0$. Thus, the domain is both open and closed, and so it coincides with the entire real line $(-\infty,\infty)$.

Taking into account the construction of the vector field X^{+}, we can represent the integral curves $m^{+}_{(t,m)}(s)$ in the form $m^{+}_{(t,m)}(s) = (s, m_{t,m}(s))$. Hence from the global existence of integral curves of X^{+}, we obviously obtain the global existence of integral curves of X. So, the vector field X is complete.

Necessity. Let the vector field X be complete. Thus, all orbits $m_{t,m}(s)$ of the flow $m(s)$ are well posed on the entire real line. Specify a certain countable locally finite cover $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ of M, where all V_i are open and relatively compact. Such a cover does exist because any manifold is paracompact by definition and the finite-dimensional manifold M is locally compact. Introduce the functions $\psi_i : M \to \mathbb{R}$ by the formula

$$\psi_i(m) = \begin{cases} i & \text{if } m \in V_i, \\ 0 & \text{if } m \notin V_i. \end{cases} \tag{2.9}$$

Denote by $\{\phi_i\}_{i=1}^{\infty}$ the smooth partition of unity subordinated to the above cover and define the function ψ on M of the form $\psi(m) = \sum_{i=1}^{\infty}\psi_i(m)\phi_i(m)$. It is clear that $\psi(m)$ is smooth and proper by the construction. The construction of the function $\psi(m)$ is taken from [7].

Introduce the function $\Phi : M^{+} \to \mathbb{R}$ as follows. For any point $(t,m) \in M^{+}$, set $\Phi(t,m) = \psi(m_{t,m}(0))$. By the construction, the function Φ is constant along any orbit of the flow $m^{+}(s)$. Indeed, for $m^{+}(s)(t,m) = (s, m_{t,m}(s))$, the equality $m_{s,m_{t,m}(s)}(0) = m_{t,m}(0)$ holds.

Consider the function $\varphi : M^{+} \to \mathbb{R}$, $\varphi(t,m) = \Phi(t,m) + t$. Obviously, φ is smooth and proper. Consider $X^{+}\varphi$. By the construction of the vector field X^{+} and of the function φ, we get

$$X^{+}\varphi = X^{+}(\Phi(t,m)) + X^{+}t = 0 + 1 = 1. \tag{2.10}$$

Thus, we have proven the necessity part of our theorem for $C = 1$. This completes the proof. □

3. The case of stochastic differential and parabolic equations. In this section, we introduce the notion of L^1-completeness of a stochastic flow and the corresponding notion of a complete Feller evolution family and prove necessary and sufficient conditions for L^1-completeness of the flow and so for the existence of complete Feller evolution families. For this, we combine the ideas of a necessary and sufficient condition for completeness of a vector field from Section 2 and Elworthy's sufficient condition for completeness of a stochastic flow from [1, item IX.6A].

Let M be a finite-dimensional manifold. Consider a stochastic dynamical system (SDS) on M (see [1]) with the generator $\mathcal{A}(t,x)$ that is an elliptic (but not necessarily strongly elliptic) operator on the space of smooth enough functions on M. In local coordinates, the SDS is described in terms of a SDE with smooth coefficients in Itô or in Stratonovich form. Since the coefficients are smooth, we can pass from Stratonovich to Itô equation and vice versa.

Denote by $\xi(s) : M \to M$ the stochastic flow of the above-mentioned SDS. For any point $x \in M$ and time instant $t \geq 0$, the orbit $\xi_{t,x}(s)$ of this flow is the unique solution of the above equation with initial conditions $\xi_{t,x}(t) = x$. As the coefficients of the equation are smooth, this is a strong solution and so a Markov diffusion process given on a certain random time interval. Below, we denote the probability space, where the flow is defined, by $(\Omega, \mathcal{F}, \mathrm{P})$ and suppose that it is complete. We also deal with separable realizations of all processes.

Specify $T \in (0,\infty)$.

Definition 3.1. The flow $\xi(s)$ is complete on $[0,T]$ if $\xi_{t,x}(s)$ exists for any couple (t,x) and for all $s \in [t,T]$.

Definition 3.2. The flow $\xi(s)$ is complete if it is complete on any interval $[0,T] \subset \mathbb{R}$.

Consider the space of bounded measurable functions on M with the norm $\|f\| = \sup_{x\in M} |f(x)|$. If the flow $\xi(t)$ is complete, it is possible to construct on this space the evolution family $S(t,s)$ (the semigroup, if $\mathcal{A}$ is autonomous) defined by the formula

$$[S(t,s)f](x) = Ef(\xi_{t,x}(s)), \tag{3.1}$$

where E is the mathematical expectation.

Definition 3.3. An evolution family is called a Feller one if for any $t \geq 0$, $s \geq t$, the operators $S(t,s)$ are contracting and send any continuous bounded positive function into a continuous bounded positive one.

It is a well-known fact that if the flow $\xi(s)$ is complete, (3.1) is a Feller evolution family. Notice, in addition, that in this case, evidently $S(t,s)1 = 1$ for all $0 \leq t \leq s < \infty$.

Consider the following Cauchy problem for the parabolic PDE on M:

$$\frac{\partial u}{\partial s} = \mathcal{A}u, \tag{3.2}$$

$$u(0,x) = u_0(x). \tag{3.3}$$

If the Feller evolution family (3.1) exists (i.e., the flow $\xi(s)$ is complete), the function

$$u(s,x) = [S(0,s)u_0](x) = Eu_0(\xi_{0,x}(s)) \tag{3.4}$$

is a generalized solution of (3.3) (see, e.g., [1, 8]). If it is smooth enough, it is a classical solution of (3.3).

On the other hand, starting from the Cauchy problem (3.3) with $\mathscr{A}$ from a broad class of operators, one can construct an SDS whose stochastic flow, if it is complete, determines generalized solutions (3.4) of (3.3). We refer the reader, for example, to [8] for details.

Thus, if we find conditions for the existence of the Feller evolution family, this will give us conditions for global existence of solutions of a SDE, describing the trajectories $\xi_{t,x}(s)$, and of generalized solutions of the Cauchy problem (3.3).

Definition 3.4. If the flow $\xi(s)$ is complete so that formula (3.1) is well posed, we say that the operator $\mathscr{A}$ generates the Feller evolution family $S(t,s)$.

Below, we will find necessary and sufficient conditions for $\mathscr{A}$ to generate a Feller evolution family of some special sort, called complete Feller evolution family. This corresponds to a special type of completeness of the flow $\xi(s)$, called L^1-completeness.

Definition 3.5. The flow $\xi(s)$ is called L^1-complete and, respectively, the evolution family $S(t,s)_{s\geq t\geq 0}$ is called a complete Feller one if

(i) the flow $\xi(s)$ is complete and so the operators $S(t,s)$ from (3.1) form a Feller evolution family on the space of bounded continuous functions on the manifold M;

and for any $0 < T < \infty$,

(ii) there exists a smooth proper positive function $v^T : M \to \mathbb{R}_+$ such that $S(t,T)v^T$ is well posed, that is, $Ev^T(\xi_{t,x}(T)) = [S(t,T)v^T](x) < \infty$ for all $x \in M$, $t \in [0,T]$;

(iii) for any $K > 0$, there exists a compact set $C_{K,T} \subset M$, depending on K and T, such that from the inequality $Ev^T(\xi_{t,x}(T)) = [S(t,T)v^T](x) < K$, it follows that $x \in C_{K,T}$;

(iv) the map $(t,x) \mapsto Ev^T(\xi_{t,x}(T)) = [S(t,T)v^T](x)$ is C^1-smooth in t and C^2-smooth in x.

In the cases of the norm in a Euclidean space and the distance function on a complete Riemannian manifold (as it is mentioned in Section 2, they are proper functions), the variable $\xi_{t,x}(T)$ satisfying Definition 3.5(ii) belongs to the ordinary functional space L^1. Notice that a flow may not be L^1-complete with respect to the norm and the distance, but may be L^1-complete with respect to some other proper function. We emphasize that a flow is L^1-complete if there exists at least one proper function satisfying Definition 3.5. See a more detailed discussion in [5, 6].

Consider the direct products $\tilde{M} = [0,\infty)\times M$ and $M^T = [0,T]\times M$. Let $\pi^T : M^T \to M$ be the natural projection, $\pi^T(t,x) = x$. On the manifold $\tilde{M}$, consider diffusion processes $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$ satisfying the conditions $\eta_{(t,x)}(t) = (t,x)$. These processes have the same infinitesimal operator that on the space of smooth functions on M^T coincides

with $\mathcal{A}^T$ defined by the formula

$$\mathcal{A}^T = \frac{\partial}{\partial t} + \mathcal{A}. \tag{3.5}$$

It is obvious that if $\xi_{t,x}(s)$ exists for all initial data t, x and for all $s \in [t,\infty)$, $\eta_{(t,x)}(s)$ also exists for $s \in [t,\infty)$ and for all initial points $(t,x) \in M^T$. Then the Feller evolution family

$$[\tilde{S}(t,s)g](t,x) = Eg(\eta_{(t,x)}(s)), \quad s \geq t \geq 0, \tag{3.6}$$

on the space of continuous bounded functions on $\tilde{M}$ is well posed. Notice that $\tilde{S}(t,s)$ for $t \leq s \leq T$ is well posed for the functions $g : M^T \to \mathbb{R}$.

Theorem 3.6. *The flow $\xi(s)$ is L^1-complete and so the operator $\mathcal{A}$ generates a complete Feller evolution family $S(t,s)_{s\geq t\geq 0}$ on M if and only if for any $0 \leq T < \infty$, there exists a smooth proper positive function $u^T : M^T \to \mathbb{R}_+$ such that at any $(t,x) \in M^T$, the following conditions are satisfied:*

(1) *$\mathcal{A}^T u^T \leq C^T$, where C^T is a certain positive constant depending on T;*

(2) *$[\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) < \infty$ and*

$$\left|[\tilde{S}(t,T)u^T](t,x) - u^T(t,x)\right| < C_1, \tag{3.7}$$

where $C_1 > 0$ is a certain constant depending on T;

(3) *the function $[\tilde{S}(t,T)u^T](t,x)$ is C^1-smooth in t and C^2-smooth in x.*

Proof

Sufficiency. Assume that there exists a smooth proper positive function $u^T(t,x)$ on M^T such that $\mathcal{A}^T_{(t,x)}u^T \leq C$ for all points of M^T. Then, from the theorem from [1, item IX.6A], it follows that for any $0 \leq T < \infty$, the process $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$ exists for all initial points $(t,x) \in M^T$ and all $s \in [t,T]$. Since it is valid for any $0 \leq T < \infty$, this means that the flow $\xi(s)$ is complete. Then there exists the Feller evolution family $S(t,s)_{s\geq t\geq 0}$, acting on the space of continuous bounded functions on M by the formula $[S(t,s)f](x) = Ef(\xi_{t,x}(s))$.

Consider the function

$$v^T(x) = u^T(T,x). \tag{3.8}$$

By the construction, it is obviously smooth and positive. We show that it is proper. Consider an arbitrary compact $D \subset \mathbb{R}_+$. One can easily see that $(v^T)^{-1}(D) \subset \pi^T((u^T)^{-1})(D)$. Then, from the properness of u^T and from the continuity of the map π^T, it follows that the set $\pi^T((u^T)^{-1})(D)$ is compact.

Lemma 3.7. *The relation $[S(t,T)v^T](x) = [\tilde{S}(t,T)u^T](t,x)$ holds for any $t \in [0,T]$ and $x \in M$.*

Proof of Lemma 3.7. Consider $[S(t,T)v^T](x)$. Taking into account the construction of $S(t,T)$ and the equality $v^T = u^T(T,x)$, we get

$$[S(t,T)v^T](x) = Ev^T(\xi_{t,x}(T)) = Eu^T(T,\xi_{t,x}(T)). \tag{3.9}$$

On the other hand, from the construction of the diffusion process $\eta_{(t,x)}(s) = (s,\xi_{t,x}(s))$, it follows that

$$Eu^T(T,\xi_{t,x}(T)) = Eu^T(\eta_{(t,x)}(T)), \tag{3.10}$$

and by the definition of $\tilde{S}(t,T)$, we get

$$Eu^T(\eta_{(t,x)}(T)) = [\tilde{S}(t,T)u^T](t,x). \tag{3.11}$$

□

From Lemma 3.7 and from condition (3) of Theorem 3.6, we immediately obtain that the map $(t,x) \mapsto [S(t,T)v^T](x)$ is C^1 in t and C^2 in x. Hence, condition (iv) of Definition 3.5 is fulfilled.

We show that $[S(t,T)v^T](x)$ is bounded. From condition (2) of Theorem 3.6 and from Lemma 3.7, we get $|[S(t,T)v^T](x) - u^T(t,x)| < C_1$. This means that

$$-C_1 + u^T(t,x) < [S(t,T)v^T](x) < C_1 + u^T(t,x). \tag{3.12}$$

Hence, $[S(t,T)v^T](x) < \infty$.

Suppose that $[S(t,T)v^T](x) < K$. Then, from Lemma 3.7, we get $Eu^T(\eta_{(t,x)}(T)) < K$. Taking into account condition (2), we see that

$$|Eu^T(\eta_{(t,x)}(T)) - u^T(t,x)| < C_1, \tag{3.13}$$

that is,

$$-C_1 + Eu^T(\eta_{(t,x)}(T)) < u^T(t,x) < C_1 + Eu^T(\eta_{(t,x)}(T)). \tag{3.14}$$

Recall that the function u^T is positive, hence

$$0 < u^T(t,x) < C_1 + K. \tag{3.15}$$

Thus, the values $u^T(t,x)$ belong to the compact $[0,C_1+K] \subset \mathbb{R}_+$, and if $[S(t,T)v^T](x) < K$, $x \in \pi^T((u^T)^{-1}([0,C_1+K]))$, while the last set is compact since u^T is proper and the map π^T is continuous.

So, conditions (i), (ii), (iii), and (iv) of Definition 3.5 are satisfied, that is, $S(t,s)_{s\geq t\geq 0}$ is a complete Feller evolution family and $\xi(s)$ is L^1-complete.

Necessity. Let $\xi(s)$ be L^1-complete and so $S(t,s)_{s\geq t\geq 0}$ a complete Feller evolution family. For any $0 \leq T < \infty$, denote by $v^T : M \to \mathbb{R}_+$ the smooth proper positive function from Definition 3.5. Construct the function $u^T : M^T \to \mathbb{R}$ by the formula

$$u^T(t,x) = [S(t,T)v^T](x) = Ev^T(\xi_{t,x}(T)). \tag{3.16}$$

This function is C^1-smooth in t and C^2-smooth in x by condition (iv) of Definition 3.5. It is also obvious that the function $u^T(t,x)$ is positive.

We show that $\mathscr{A}^T u^T = 0$. To prove this, we modify some technical approaches of [2, Chapter VIII].

Consider the sets $\tilde{W}_n = (v^T)^{-1}([0,n])$, $n \in \mathbb{N}$. Since the function v^T is proper, the sets $\tilde{W}_n$ are compact. Moreover, it is easy to see that the family of compacts $\tilde{W}_n$ forms a cover of the manifold M such that $\tilde{W}_n \subset \tilde{W}_{n+1}$ for any n. For $x \in \tilde{W}_n$, denote by $\tilde{\tau}_n$ the first exit time of $\xi_{t,x}(s)$ from $\tilde{W}_n$.

Consider

$$Eu^T((t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)) = Eu^T(\eta_{(t,x)}((t+\Delta t)\wedge\tilde{\tau}_n)), \tag{3.17}$$

where, according to usual notations of the probability theory, $(a\wedge b)_\omega = \min(a_\omega, b_\omega)$, $\omega \in \Omega$. From the construction of $\eta_{(t,x)}(s)$, it follows that if $\tilde{\tau}_n$ is the first exit time of $\xi_{t,x}(s)$ from the compact $\tilde{W}_n$, $\tilde{\tau}_n$ is also the first exit time of $\eta_{(t,x)}(s)$ from the compact $[0,T]\times\tilde{W}_n$ on the manifold M^T.

Since the processes are considered up to the first exit times from compacts, we may use the Itô formula and the fact that in this case, the expectation of Itô integral on the interval $[t,(t+\Delta t)\wedge\tilde{\tau}_n]$ equals zero. Thus, we obtain

$$\begin{aligned} &Eu^T((t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)) \\ &\quad = u^T(t,x) + E\int_t^{(t+\Delta t)\wedge\tilde{\tau}_n} \mathscr{A}^T u^T(\eta_{(t,x)}(s))\,ds. \end{aligned} \tag{3.18}$$

Notice that $Eu^T((t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)) = u^T(t,x)$. Indeed, by the construction of the function u^T,

$$\begin{aligned} &Eu^T((t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)) \\ &\quad = E(Ev^T(\xi_{(t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)}(T))) \\ &\quad = E(Ev^T(\xi_{t,x}(T))) = Ev^T(\xi_{t,x}(T)) = u^T(t,x). \end{aligned} \tag{3.19}$$

Then, from (3.18), we get

$$\begin{aligned} 0 &= Eu^T((t+\Delta t)\wedge\tilde{\tau}_n, \xi_{t,x}((t+\Delta t)\wedge\tilde{\tau}_n)) - u^T(t,x) \\ &= E\int_t^{(t+\Delta t)\wedge\tilde{\tau}_n} \mathscr{A}^T u^T(\eta_{(t,x)}(s))\,ds. \end{aligned} \tag{3.20}$$

Multiply both sides of (3.20) by $1/\Delta t$ and find the limit as $\Delta t \to 0$. We obtain

$$0 = \lim_{\Delta t\to 0}\frac{1}{\Delta t}E\int_t^{(t+\Delta t)\wedge\tilde{\tau}_n} \mathscr{A}^T u^T(\eta_{(t,x)}(s))\,ds. \tag{3.21}$$

Taking into account (3.5), one can easily transform the last equality to the form

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\int_t^{(t+\Delta t)\wedge\tilde{\tau}_n}\left[\frac{\partial u^T(s,\xi_{t,x}(s))}{\partial s} + \mathscr{A}u^T(s,\xi_{t,x}(s))\right]ds = 0. \tag{3.22}$$

The function u^T and its derivatives are considered here on the compact set $[0,T]\times \tilde{W}_n$, and so they are bounded. Hence, we can apply Lebesgue's theorem to get to the limit under the mathematical expectation and also to obtain that there exists a value $s' \in [t,(t+\Delta t)\wedge \tilde{\tau}_n]$ such that

$$\begin{aligned}&\int_t^{(t+\Delta t)\wedge\tilde{\tau}_n}\left[\frac{\partial u^T(s,\xi_{t,x}(s))}{\partial s}+\mathcal{A}u^T(s,\xi_{t,x}(s))\right]ds\\&\quad=\left[\frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s}+\mathcal{A}u^T(s',\xi_{t,x}(s'))\right]((t+\Delta t)\wedge\tilde{\tau}_n-t).\end{aligned}\tag{3.23}$$

One can easily see that

$$(t+\Delta t)\wedge\tilde{\tau}_n-t=((t+\Delta t)-t)\wedge(\tilde{\tau}_n-t)=\Delta t\wedge(\tilde{\tau}_n-t).\tag{3.24}$$

As a result, we obtain

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\left(\frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s}+\mathcal{A}u^T(s',\xi_{t,x}(s'))(\Delta t\wedge(\tilde{\tau}_n-t))\right)\\&\quad=E\lim_{\Delta t\to 0}\left(\frac{\partial u^T(s',\xi_{t,x}(s'))}{\partial s}+\mathcal{A}u^T(s',\xi_{t,x}(s'))\frac{\Delta t\wedge(\tilde{\tau}_n-t)}{\Delta t}\right)=0.\end{aligned}\tag{3.25}$$

Notice that here $\tilde{\tau}_n-t>0$ a.s. by the definition of the first exit time. Also, $\tilde{\tau}_n-t$ is bounded and does not depend on Δt. So,

$$\lim_{\Delta t\to 0}\frac{\tilde{\tau}_n-t}{\Delta t}=\infty.\tag{3.26}$$

From the last equality, it obviously follows that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}(\Delta t\wedge(\tilde{\tau}_n-t))=1\wedge\lim_{\Delta t\to\infty}\frac{\tilde{\tau}_n-t}{\Delta t}=1.\tag{3.27}$$

Since $s'\in[t,(t+\Delta t)\wedge\tilde{\tau}_n]$ and since we can apply the Lebesgue's theorem, $s'\to t$ when $\Delta t\to 0$.

Thus, equality (3.25) takes the form

$$\frac{\partial u^T(t,\xi_{t,x}(t))}{\partial t}+\mathcal{A}u^T(t,\xi_{t,x}(t))=0.\tag{3.28}$$

This means that

$$\mathcal{A}^T u^T(t,x)=0.\tag{3.29}$$

Lemma 3.8. *The function u^T on M^T is proper.*

Proof of Lemma 3.8. Suppose that u^T is not proper. Then there exists a sequence $(t_k,x_k)\in M^T$ such that $0<u^T(t_k,x_k)<K$ for all k, where $0<K<\infty$ is a certain real number, and $v^T(x_k)\to\infty$ as $k\to\infty$. Since v^T is proper, this means that x_k leaves any specified compact in M. But, if $0<u^T(t_k,x_k)<K$, by the construction of the function u^T, we get $[S(t_k,T)v^T](x_k)<K$, and so by condition (iii) of Definition 3.5, x_k must belong to a certain compact $C_{K,T}$. □

Lemma 3.9. *For any $t \in [0,T]$, $x \in M$, the equality $Eu^T(\eta_{(t,x)}(T)) = Ev^T(\xi_{t,x}(T))$ takes place.*

Proof of Lemma 3.9. Recall that $\eta_{(t,x)}(s) = (s, \xi_{t,x}(s))$, and so

$$Eu^T(\eta_{(t,x)}(T)) = Eu^T(T, \xi_{t,x}(T)). \tag{3.30}$$

By the construction of u^T,

$$Eu^T(T, \xi_{t,x}(T)) = E(Ev^T(\xi_{T,\xi_{t,x}(T)}(T))). \tag{3.31}$$

Taking into account the properties of the mathematical expectation and the evolution property of $\xi_{t,x}(s)$, we obtain

$$E(Ev^T(\xi_{T,\xi_{t,x}(T)}(T))) = Ev^T(\xi_{t,x}(T)). \tag{3.32}$$

□

From the construction of $S(t,s)$ and $\tilde{S}(t,s)$, it follows that

$$[\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) = Ev^T(\xi_{t,x}(T)) = [S(t,T)v^T](x) = u^T(t,x). \tag{3.33}$$

Then, from (iv) of Definition 3.5, we derive that $[\tilde{S}(t,T)u^T](t,x)$ is C^1-smooth in t and C^2-smooth in x. Condition (3) is fulfilled.

Notice, in addition, that $|[\tilde{S}(t,T)u^T](t,x) - u^T(t,x)| = 0$, that is, it is less than any positive constant. This means that Condition (2) is fulfilled.

This completes the proof of necessity and of Theorem 3.6 □

Remark 3.10. The similarity between the assertions of Theorems 2.4 and 3.6 becomes more clear if one passes from the Cauchy problem (3.3) to the corresponding abstract Cauchy problem, that is, to the first-order ODE in the Banach space. Then the assertion of Theorem 3.6 is very close to the reformulation of Theorem 2.4 for solutions of the abstract Cauchy problem (i.e., for generalized solutions of (3.3)).

Corollary 3.11. *The flow $\xi(s)$ is L^1-complete, and so the operator $\mathcal{A}$ generates the complete Feller evolution family $S(t,s)_{s\geq t\geq 0}$ on M if and only if for any $0 \leq T < \infty$, there exists a smooth positive proper function $u^T : M^T \to \mathbb{R}_+$ such that at any point $(t,x) \in M^T$, the following conditions are satisfied:*

(1) *$\mathcal{A}^T u^T \leq C^T$, where $C^T \geq 0$ is a certain constant depending on T;*

(2) *$[\tilde{S}(t,T)u^T](t,x) = Eu^T(\eta_{(t,x)}(T)) < \infty$ and*

$$[\tilde{S}(t,T)u^T](t,x) = u^T(t,x); \tag{3.34}$$

(3) *the function $[\tilde{S}(t,T)u^T](t,x)$ is C^1-smooth in t and C^2-smooth in x.*

Proof. Notice that in the proof of necessity in Theorem 3.6, we first proved the equality $[\tilde{S}(t,T)u^T](t,x) = u^T(t,x)$, that is, condition (2) of Corollary 3.11, from which we derived that condition (2) of Theorem 3.6 was satisfied. Thus, we only need to modify the proof of sufficiency under the assumption that condition (2) of Theorem 3.6 is replaced by that of Corollary 3.11.

The proof that the Feller evolution family $S(t,s)$ on the space of continuous bounded functions on M exists is absolutely the same as for the conditions of Theorem 3.6.

We construct $v^T(x) = u^T(T,x)$ and show that $[S(t,T)v^T](x)$ is bounded. From condition (2) of the corollary and from Lemma 3.7, we obtain $[\tilde{S}(t,T)u^T](t,x) - u^T(t,x) = [S(t,T)v^T](x) - u^T(t,x) = 0$. Hence

$$[S(t,T)v^T](x) = u^T(t,x). \tag{3.35}$$

Thus, $[S(t,T)v^T](x) < \infty$.

From equality (3.35), it also follows that the map $(t,x) \mapsto [S(t,T)v^T](x)$ is smooth.

Suppose that $[S(t,T)v^T](x) < K$. Then, from (3.35), since u^T is positive, we get $0 < u^T(t,x) < K$. Thus, the values $u^T(t,x)$ belong to the compact set $[0,K] \subset \mathbb{R}_+$. Hence, from $[S(t,T)v^T](x) < K$, it follows that $x \in \pi^T((u^T)^{-1}([0,K]))$, while the last set is compact since u^T is proper and π^T is continuous.

So, conditions (i), (ii), (iii), and (iv) of Definition 3.5 are satisfied. Hence, $\xi(s)$ is L^1-complete and $S(t,s)_{s\geq t\geq 0}$ is a complete Feller evolution family. □

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