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THE 3D HAPPEL MODEL FOR COMPLETE ISOTROPIC STOKES FLOW

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The creeping flow through a swarm of spherical particles that move with constant velocity in an arbitrary direction and rotate with an arbitrary constant angular velocity in a quiescent Newtonian fluid is analyzed with a 3D sphere-in-cell model. The mathematical treatment is based on the two-concentric-spheres model. The inner sphere comprises one of the particles in the swarm and the outer sphere consists of a fluid envelope. The appropriate boundary conditions of this non-axisymmetric formulation are similar to those of the 2D sphere-in-cell Happel model, namely, nonslip flow condition on the surface of the solid sphere and nil normal velocity component and shear stress on the external spherical surface. The boundary value problem is solved with the aim of the complete Papkovich-Neuber differential representation of the solutions for Stokes flow, which is valid in non-axisymmetric geometries and provides us with the velocity and total pressure fields in terms of harmonic spherical eigenfunctions. The solution of this 3D model, which is self-sufficient in mechanical energy, is obtained in closed form and analytical expressions for the velocity, the total pressure, the angular velocity, and the stress tensor fields are provided.

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1. Introduction. Stokes flow [8] characterizes the steady and non-axisymmetric flow of an incompressible, viscous fluid at low Reynolds number and is described by a pair of partial differential equations connecting the biharmonic velocity with the harmonic total pressure field. Fluid flow relative to assemblages of particles that conform to Stokes law represents an area of interest in many fields of science and technology. Thus, particle-fluid systems are encountered in many important applications. Because of the small size of the particles, spherical coordinates [12] approach efficiently the geometry of those suspensions for many interior and exterior flow problems. Then the flow caused by motion is considered to be axisymmetric. Nevertheless, more realistic and general models assume rotation beyond the translation in the assemblage where the rotational symmetry disappears. Eventually one has to deal with a full three dimensional (3D) Stokes flow in spherical coordinates. The introduction of the representation theory [16] serves to unify the method of attack on all 3D incompressible fluid motions since they provide us the flow fields for creeping flow in terms of harmonic and biharmonic potentials. The most famous differential solution for Stokes flow has been proposed by Papkovich (1932) and Neuber [14] and provides the flow fields in terms of harmonic functions [14, 16]. This representation is followed by the work at hand.

One of the largest physical areas with practical importance in flow hydrodynamics concerns the construction of particle-in-cell models for swarms of particles. The technique of cell models is based on the idea that a large enough concentration of particles within a fluid can be represented by many separate unit cells, where every cell contains one particle. Thus, the consideration of a full-dimensional porous media is being referred to as that of a single particle and its fluid cover. This way, the mathematical formulation of any physical problem is significantly simplified. Many efficient methods have been developed in order to solve this kind of problems in spherical and spheroidal coordinates, considering axial symmetry inherited by the geometry, such as numerical computation [1, 6, 10] and stream-function techniques [2, 3, 11, 13] or other analytic-function methods [4, 5, 7, 15]. Nevertheless, 3D flows have not been extensively faced. It is to this end that 3D particle-in-cell flow models serve as platforms, which capture the essential features of the transport process under consideration in an analytical formula.

In the present work, the solution of the non-axisymmetric (3D) Stokes flow problem in an assemblage of spherical particles, which translate and rotate, considering a sphere-in-cell model of Happel type [7], is obtained using the Papkovich-Neuber differential representation. The loss of symmetry is caused by the imposed rotation of the particles. The incentive for this is that the Happel-type boundary conditions (BCs) are more compatible with the physics of flow in a swarm since they ensure that each unit cell is energetically self-sufficient. On the other hand, it has the disadvantage that this formulation does not provide space filling, a difficulty that must be dealt with, when one tries to pass from the single unit cell to an assemblage of particles. In accordance with the concept introduced by Happel [7], two concentric spheres are considered. Under the assumption of very small Reynolds number and pseudosteady state, we investigate the creeping flow within the fluid cell contained between the two concentric spherical surfaces. The internal sphere is solid, moves with a constant uniform velocity, and rotates arbitrarily with a constant angular velocity in an otherwise quiescent spherical layer, which is confined by the external sphere that contains the spherical particle and the amount of fluid required to match the fluid's volume fraction of the swarm. This formulation is escorted by the appropriate BC on the two spherical surfaces; that is, nonslip flow on the inner sphere, and no normal flow and nil tangential stresses on the outer spherical envelope.

The Papkovich-Neuber representation is employed in order to solve the above boundary value problem. In order to achieve that, we calculate the Papkovich-Neuber eigensolutions, generated by the appropriate spherical eigenfunctions [9]. That way, we determine the flow fields as a full series expansion via the Papkovich-Neuber representation, which represents the velocity and the total pressure fields in terms of harmonic functions. After the imposition of the required BCs, the solution is obtained in a closed 3D form. Once the velocity and the total pressure fields are calculated, the angular velocity and the stress tensor fields are also obtained.

Section 2 provides the mathematical statement of the Stokes flow problem where the Papkovich-Neuber differential representation is presented and the 3D Happel-type BCs are given for the corresponding sphere-in-cell model. Section 3 discusses the eigenfunctions for the Papkovich-Neuber harmonic potentials in spherical coordinates. The Stokes flow fields are also provided as full series expansions. The aforementioned Happel-type problem is solved explicitly in Section 4, where the results are presented in 3D closed form. Section 5 is dedicated to a discussion of the results drawn in this work. The necessary material, which makes this work self-dependent such as identities, useful recurrence relations associated with the Legendre functions and related functions, is collected in the appendix.

2. The 3D Happel sphere model. In terms of the transformation $\zeta = \cos\theta$, $-1 \le \zeta \le 1$, the following expressions for the relation between the Cartesian coordinates and the spherical coordinate system [12] are obtained:

$$x_1 = r\zeta, \qquad x_2 = r\sqrt{1-\zeta^2}\cos\varphi, \qquad x_3 = r\sqrt{1-\zeta^2}\sin\varphi, \tag{2.1}$$

where $0 \le r < +\infty$, $0 \le \theta \le \pi$, and $0 \le \varphi < 2\pi$. We define the sphere B_r for $r > 0$ as the set

$$B_r = \{\mathbf{r} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \le r^2\}. \tag{2.2}$$

Then, the outward unit normal vector on the surface of the sphere $r = r_0 > 0$ is furnished by the formula

$$\hat{\mathbf{n}}(r_0) = \zeta\hat{\mathbf{x}}_1 + \sqrt{1-\zeta^2}\cos\varphi\hat{\mathbf{x}}_2 + \sqrt{1-\zeta^2}\sin\varphi\hat{\mathbf{x}}_3 = \frac{\mathbf{r}(r_0)}{r_0} \equiv \hat{\mathbf{r}}. \tag{2.3}$$

In order to construct tractable mathematical models for the flow systems involving particles, it is necessary to resort to a number of simplifications. A dimensionless criterion, which determines the relative importance of inertial and viscous effects, is the Reynolds number Re. Stokes equations for the pseudosteady, non-axisymmetric, creeping flow (Re $\ll 1$) of incompressible (density $\rho =$ const.), viscous (dynamic viscosity $\mu =$ const.) fluids connect the vector velocity field $\mathbf{v}(\mathbf{r})$ with the scalar total pressure field $\mathrm{P}(\mathbf{r})$ [8]. Considering Stokes flow around particles embedded within smooth, bounded domains $\Omega(\mathbb{R}^3)$, these equations appear as

$$\mu\Delta\mathbf{v}(\mathbf{r}) = \nabla\mathrm{P}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.4}$$

$$\nabla\cdot\mathbf{v}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2.5}$$

The total pressure is harmonic, while the velocity is biharmonic and divergence-free. Equation (2.4) states that, for creeping flow, the pressure compensates the viscous forces, while (2.5) preserves the incompressibility of the fluid. The harmonic vorticity field $\boldsymbol{\omega}(\mathbf{r})$ is obtained via

$$\boldsymbol{\omega}(\mathbf{r}) = \frac{1}{2}\nabla\times\mathbf{v}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2.6}$$

Equation (2.4) is the time-independent, simplified Navier-Stokes equation. Therefore, by virtue of the Papkovich-Neuber (3D) differential representation of the solution for Stokes flow [14, 16], there exist harmonic functions $\boldsymbol{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$, the vector and scalar Papkovich-Neuber potentials, respectively, such that

$$\begin{aligned} \mathbf{v}(\mathbf{r}) &= \mathbf{\Phi}(\mathbf{r}) - \frac{1}{2}\nabla\left(\mathbf{r}\cdot\mathbf{\Phi}(\mathbf{r}) + \Phi_0(\mathbf{r})\right), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \\ \mathrm{P}(\mathbf{r}) &= \mathrm{P}_0 - \mu\nabla\cdot\mathbf{\Phi}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \end{aligned} \tag{2.7}$$

whereas P_0 is a constant pressure of reference usually assigned at a convenient point, while $\mathbf{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$ satisfy

$$\Delta\mathbf{\Phi}(\mathbf{r}) = \mathbf{0}, \quad \Delta\Phi_0(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2.8}$$

The total pressure is produced by the summation of the thermodynamic pressure $p(\mathbf{r})$ and the gravitational pressure force $\rho g h$ (g is the acceleration of the gravity):

$$\mathrm{P}(\mathbf{r}) = p(\mathbf{r}) + \rho g h, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.9}$$

where h specifies an arbitrarily chosen height of reference.

The stress tensor $\tilde{\mathbf{\Pi}}(\mathbf{r})$ is taken to be

$$\tilde{\mathbf{\Pi}}(\mathbf{r}) = -p(\mathbf{r})\tilde{\mathbf{I}} + \mu\left[\nabla\otimes\mathbf{v}(\mathbf{r}) + \left(\nabla\otimes\mathbf{v}(\mathbf{r})\right)^{\top}\right], \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.10}$$

where $\tilde{\mathbf{I}}$ stands for the unit dyadic and the symbol "$\top$" denotes transposition.

The gradient ∇ and the Laplacian Δ assume the expressions

$$\begin{aligned} \nabla &= \hat{\mathbf{r}}\frac{\partial}{\partial r} - \frac{\sqrt{1-\zeta^2}}{r}\hat{\boldsymbol{\zeta}}\frac{\partial}{\partial \zeta} + \frac{1}{r\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\varphi}, \\ \Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial}{\partial\zeta}\right] + \frac{1}{r^2(1-\zeta^2)}\frac{\partial^2}{\partial\varphi^2}, \end{aligned} \tag{2.11}$$

while $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\zeta}}$, $\hat{\boldsymbol{\varphi}}$ are the coordinate vectors of the spherical system for $r > 0$, $|\zeta| \le 1$, and $\varphi \in [0, 2\pi)$.

2.1. The Happel-type BCs for a 3D sphere-in-cell model. The general 3D solution of Papkovich-Neuber (equations (2.7)-(2.8)) is employed here. According to the idea of particle-in-cell model described in the introduction, we are interested solving the creeping flow within a fluid cell limited between two concentric spherical surfaces. Thus, we examine the flow of one particle in the assemblage, neglecting the interaction with other particles or with the bounded walls of a container. This way, we avoid technical complications and additional terminology that will lead us to cumbersome in use results.

Two concentric spheres of radii a and b, $a < b$, are considered. The inner one, indicated by S_a, at $r = a$, is solid and is moving with a constant translational velocity $\boldsymbol{U}$ in the main directions of a sphere. Furthermore, it is rotating, also arbitrarily, with a constant angular velocity $\mathbf{\Omega}$. The difference between the velocity $\boldsymbol{U}$ and the mean interstitial velocity through a swarm of spherical particles must be taken into account when we refer to the assemblage, since the specific model is not space filling. Consequently, by definition, the uniform velocity and the constant rotation are dictated by

$$\boldsymbol{U} = U\left(\zeta\hat{\mathbf{r}} - \sqrt{1-\zeta^2}\hat{\boldsymbol{\zeta}}\right) = U\hat{\mathbf{x}}_1, \qquad \boldsymbol{\Omega} = \sum_{i=1}^{3}\Omega_i\hat{\mathbf{x}}_i, \tag{2.12}$$

respectively. The outer sphere at $r = b$, indicated by S_b, represents the fictitious boundary of the unit cell (sphere-in-cell) that is used to model flow through the swarm of spherical particles. The volume of the fluid cell is chosen so that the solid volume fraction in the cell equals the solid volume fraction of the swarm. The imposed rotation (along with the translation) in (2.12) generates the 3D flow in the fluid layer between the two spheres.

The BCs, which are applied, are analogous to those of the Happel sphere-in-cell model for axisymmetric flows [7]. Indeed, assuming pseudosteady state, the 3D Happel-type BCs can be expressed as follows:

(i) BC (1):

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{U} + \boldsymbol{\Omega}\times\mathbf{r} \quad \text{for } \mathbf{r}\in S_a(r=a), \tag{2.13}$$

(ii) BC (2):

$$\hat{\mathbf{r}}\cdot\mathbf{v}(\mathbf{r}) = 0 \quad \text{for } \mathbf{r}\in S_b(r=b), \tag{2.14}$$

(iii) BC (3):

$$\hat{\mathbf{r}}\cdot\tilde{\mathbf{\Pi}}(\mathbf{r})\cdot(\tilde{\mathbf{I}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}) = \mathbf{0} \quad \text{for } \mathbf{r}\in S_b(r=b). \tag{2.15}$$

Equation (2.13) expresses the nonslip flow condition on the solid particle of the swarm, whereas (2.14) implies that there is no flow across the boundary of the fluid envelope S_b. Furthermore, the shear stress is assumed to nil on the external sphere, as shown by (2.15), a condition that secures the nonexchange of mechanical energy with the environment. This completes the statement of a well-posed Happel-type boundary value problem within 3D domains, $\mathbf{r}\in\Omega(\mathbb{R}^3)$.

Our purpose is to solve the aforementioned non-axisymmetric Happel flow problem in spherical domains with the aim of the Papkovich-Neuber differential representation and obtain the basic flow fields.

3. Papkovich-Neuber flow fields: the sphere (3D). We introduce the set of the $2n+1$ linearly independent surface spherical harmonic eigenfunctions $Y_n^{ms}(\hat{\mathbf{r}})$ of degree n ($n = 0,1,2,\ldots$) and of order m ($m = 0,1,2,\ldots,n$) in terms of the associated Legendre functions $P_n^m(\zeta)$ of the first kind [9] via the formulae

$$Y_n^{ms}(\hat{\mathbf{r}}) = P_n^m(\zeta)\begin{cases}\cos m\varphi, & s=e,\\ \sin m\varphi, & s=o,\end{cases} \quad m = 0,1,2,\ldots,n,\ |\zeta|\le 1,\ \varphi\in[0,2\pi), \tag{3.1}$$

for $n = 0,1,2,\ldots$, which satisfy the orthonormalization relations

$$\oint_{S^2} Y_n^{ms}(\hat{\mathbf{r}})Y_{n'}^{m's'}(\hat{\mathbf{r}})\,dS(\hat{\mathbf{r}}) = \frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'}\delta_{mm'}\delta_{ss'}\frac{1}{\varepsilon_m}. \tag{3.2}$$

Here, δ_{ij} denotes the Kronecker delta, ε_m stands for the Neumann factor ($\varepsilon_m = 1$, $m = 0$ and $\varepsilon_m = 2$, $m \geq 1$), s comprises the even (e) or the odd (o) character of the spherical surface harmonics, and S^2 is the unit sphere in $\mathbb{R}^3$. For the same values of n and m, the associated Legendre functions of the first kind [9] are defined by the following derivatives:

$$P_n^m(\zeta) = \frac{(1-\zeta^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\zeta^{n+m}}(\zeta^2-1)^n, \quad |\zeta| < 1. \tag{3.3}$$

The index n denotes the degree and m denotes the order.

Due to physical requirements concerning Stokes flows, the fields must be regular for $\zeta = \pm 1$. Therefore, the terms involving the associated Legendre functions of the second kind are excluded and the corresponding eigenfunctions should be eliminated. Consequently, every harmonic function belongs to the kernel of the Laplace operator Δ and in spherical coordinates, this linear space can be expressed as a complete set of the internal (i) and the external (e) solid spherical harmonics in the absence of singularities for $\zeta = \pm 1$, that is,

$$u_n^{(i)ms}(\mathbf{r}) = r^n Y_n^{ms}(\hat{\mathbf{r}}), \quad u_n^{(e)ms}(\mathbf{r}) = r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}}), \quad n \geq 0,\ m = 0,1,\ldots,n,\ s = e,o, \tag{3.4}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$.

Eventually, the complete representation of the Papkovich-Neuber potentials that appear in (2.8) assume the form

$$\boldsymbol{\Phi}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[\mathbf{e}_n^{(i)ms} u_n^{(i)ms}(\mathbf{r}) + \mathbf{e}_n^{(e)ms} u_n^{(e)ms}(\mathbf{r})\right], \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{3.5}$$

$$\Phi_0(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[d_n^{(i)ms} u_n^{(i)ms}(\mathbf{r}) + d_n^{(e)ms} u_n^{(e)ms}(\mathbf{r})\right], \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{3.6}$$

Note that

$$\mathbf{e}_n^{(i/e)ms} = a_n^{(i/e)ms}\hat{\mathbf{x}}_1 + b_n^{(i/e)ms}\hat{\mathbf{x}}_2 + c_n^{(i/e)ms}\hat{\mathbf{x}}_3 \tag{3.7}$$

and $d_n^{(i/e)ms}$ denote the vector and scalar constant coefficients of the harmonic potentials $\boldsymbol{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$, respectively, whereas $n \geq 0$, $m = 0,1,\ldots,n$, and $s = e,o$.

Substituting the potentials $\boldsymbol{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$, $\mathbf{r} \in \Omega(\mathbb{R}^3)$, given by (3.5) and (3.6), respectively, to the Papkovich-Neuber representation (2.7), we derive the following relation for the velocity field as full series expansion of the aforementioned eigenfunctions, that is,

$$\begin{aligned}\mathbf{v}(\mathbf{r}) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\Big\{&\mathbf{e}_n^{(i)ms} u_n^{(i)ms}(\mathbf{r}) - \left[\left(\mathbf{e}_n^{(i)ms}\cdot\mathbf{r}\right) + d_n^{(i)ms}\right]\nabla u_n^{(i)ms}(\mathbf{r}) \\ &+ \mathbf{e}_n^{(e)ms} u_n^{(e)ms}(\mathbf{r}) - \left[\left(\mathbf{e}_n^{(e)ms}\cdot\mathbf{r}\right) + d_n^{(e)ms}\right]\nabla u_n^{(e)ms}(\mathbf{r})\Big\}\end{aligned} \tag{3.8}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$, while for the total pressure field, we obtain

$$\mathrm{P}(\mathbf{r}) = \mathrm{P}_0 - \mu \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=e,o} \{\mathbf{e}_n^{(i)ms} \cdot \nabla u_n^{(i)ms}(\mathbf{r}) + \mathbf{e}_n^{(e)ms} \cdot \nabla u_n^{(e)ms}(\mathbf{r})\}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{3.9}$$

Of course, all kinds of singularities have been excluded.

Once the velocity is calculated, the vorticity field, dictated by (2.6), is easily confirmed to be expressed as

$$\boldsymbol{\omega}(\mathbf{r}) = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=e,o} \{\nabla u_n^{(i)ms}(\mathbf{r}) \times \mathbf{e}_n^{(i)ms} + \nabla u_n^{(e)ms}(\mathbf{r}) \times \mathbf{e}_n^{(e)ms}\}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{3.10}$$

while in view of the velocity field (3.7), equation (2.10) implies

$$\begin{aligned} \tilde{\mathbf{\Pi}}(\mathbf{r}) = -p(\mathbf{r})\tilde{\mathbf{I}} - \mu \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=e,o} \{&[(\mathbf{e}_n^{(i)ms} \cdot \mathbf{r}) + d_n^{(i)ms}] \nabla \otimes \nabla u_n^{(i)ms}(\mathbf{r}) \\ &+ [(\mathbf{e}_n^{(e)ms} \cdot \mathbf{r}) + d_n^{(e)ms}] \nabla \otimes \nabla u_n^{(e)ms}(\mathbf{r})\} \end{aligned} \tag{3.11}$$

for $\mathbf{r} \in \Omega(\mathbb{R}^3)$. The unit dyadic in both the Cartesian and the spherical coordinates is furnished by

$$\begin{aligned} \tilde{\mathbf{I}} &= \hat{\boldsymbol{x}}_1 \otimes \hat{\boldsymbol{x}}_1 + \hat{\boldsymbol{x}}_2 \otimes \hat{\boldsymbol{x}}_2 + \hat{\boldsymbol{x}}_3 \otimes \hat{\boldsymbol{x}}_3 \\ &= \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \hat{\boldsymbol{\zeta}} \otimes \hat{\boldsymbol{\zeta}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}} \end{aligned} \tag{3.12}$$

and the thermodynamic pressure which appears in the form of the stress tensor (3.11) is calculated from (2.9).

The basic identities that were used to obtain the formulae (3.8)–(3.11), as well as the connection formulae between the coordinate vectors of the Cartesian and the spherical system, are summarized in the appendix.

4. Solution with 3D Happel-type sphere-in-cell model. The point of this section is to solve the 3D Stokes sphere-in-cell model with the Happel-type BCs (2.13)–(2.15), in view of relations (3.8) and (3.11). Since the vector character of the vector harmonic eigenfunctions is reflected upon the constant coefficients, which are written in Cartesian coordinates, we are obliged to work in the Cartesian system. This is attainable and requires the expression of the flow fields in terms of constants and surface spherical harmonics. In order to do that, it is necessary to express the gradient of the internal and external solid spherical harmonics (3.4) as a function of surface spherical harmonics. This is possible since the $\nabla u_n^{(i)ms}(\mathbf{r})$ and the $\nabla u_n^{(e)ms}(\mathbf{r})$ for every $n \geq 0$, $m = 0, 1, \ldots, n$, $s = e, o$, and $\mathbf{r} \in \Omega(\mathbb{R}^3)$ belong to the subspace produced by the surface spherical harmonics provided by (3.1). After long and tedious calculations, taking advantage of certain recurrence relations for the associated Legendre functions of the first kind and of special identities (see, also, the appendix), we arrive at very useful expressions for the internal solid spherical harmonic eigenfunctions provided by (3.4). This program furnishes

$$
\begin{aligned}
\nabla u_n^{(i)me}(\mathbf{r}) &= \frac{1}{2}\left[(n+m)(n+m-1)Y_{n-1}^{(m-1)e}(\hat{\boldsymbol{r}}) - Y_{n-1}^{(m+1)e}(\hat{\boldsymbol{r}})\right]r^{n-1}\hat{\mathbf{x}}_2 \\
&\quad -\frac{1}{2}\left[(n+m)(n+m-1)Y_{n-1}^{(m-1)o}(\hat{\boldsymbol{r}}) + Y_{n-1}^{(m+1)o}(\hat{\mathbf{r}})\right]r^{n-1}\hat{\mathbf{x}}_3 \\
&\quad +(n+m)Y_{n-1}^{me}(\hat{\mathbf{r}})r^{n-1}\hat{\mathbf{x}}_1, \quad n \ge 0,\ m = 1,\dots,n,\ \mathbf{r}\in\Omega(\mathbb{R}^3), \\
\nabla u_n^{(i)mo}(\mathbf{r}) &= \frac{1}{2}\left[(n+m)(n+m-1)Y_{n-1}^{(m-1)o}(\hat{\mathbf{r}}) - Y_{n-1}^{(m+1)o}(\hat{\mathbf{r}})\right]r^{n-1}\hat{\mathbf{x}}_2 \\
&\quad +\frac{1}{2}\left[(n+m)(n+m-1)Y_{n-1}^{(m-1)e}(\hat{\mathbf{r}}) + Y_{n-1}^{(m+1)e}(\hat{\mathbf{r}})\right]r^{n-1}\hat{\mathbf{x}}_3 \\
&\quad +(n+m)Y_{n-1}^{mo}(\hat{\mathbf{r}})r^{n-1}\hat{\mathbf{x}}_1, \quad n \ge 0,\ m = 1,\dots,n,\ \mathbf{r}\in\Omega(\mathbb{R}^3)
\end{aligned}
\tag{4.1}
$$

and for the case $m = 0$,

$$
\nabla u_n^{(i)0e}(\mathbf{r}) = \left[-Y_{n-1}^{1e}(\hat{\mathbf{r}})\hat{\mathbf{x}}_2 - Y_{n-1}^{1o}(\hat{\mathbf{r}})\hat{\mathbf{x}}_3 + nY_{n-1}^{0e}(\hat{\mathbf{r}})\hat{\mathbf{x}}_1\right]r^{n-1}, \quad n \ge 0,\ \mathbf{r}\in\Omega(\mathbb{R}^3). \tag{4.2}
$$

Similarly, for the external solid spherical harmonic eigenfunctions, the following relations hold true:

$$
\begin{aligned}
\nabla u_n^{(e)me}(\mathbf{r}) &= \frac{1}{2}\left[(n-m+1)(n-m+2)Y_{n+1}^{(m-1)e}(\hat{\mathbf{r}}) - Y_{n+1}^{(m+1)e}(\hat{\mathbf{r}})\right]r^{-(n+2)}\hat{\mathbf{x}}_2 \\
&\quad -\frac{1}{2}\left[(n-m+1)(n-m+2)Y_{n+1}^{(m-1)o}(\hat{\mathbf{r}}) + Y_{n+1}^{(m+1)o}(\hat{\mathbf{r}})\right]r^{-(n+2)}\hat{\mathbf{x}}_3 \\
&\quad -(n-m+1)Y_{n+1}^{me}(\hat{\mathbf{r}})r^{-(n+2)}\hat{\mathbf{x}}_1, \quad n \ge 0,\ m = 1,\dots,n,\ \mathbf{r}\in\Omega(\mathbb{R}^3), \\
\nabla u_n^{(e)mo}(\mathbf{r}) &= \frac{1}{2}\left[(n-m+1)(n-m+2)Y_{n+1}^{(m-1)o}(\hat{\mathbf{r}}) - Y_{n+1}^{(m+1)o}(\hat{\mathbf{r}})\right]r^{-(n+2)}\hat{\mathbf{x}}_2 \\
&\quad +\frac{1}{2}\left[(n-m+1)(n-m+2)Y_{n+1}^{(m-1)e}(\hat{\mathbf{r}}) + Y_{n+1}^{(m+1)e}(\hat{\mathbf{r}})\right]r^{-(n+2)}\hat{\mathbf{x}}_3 \\
&\quad -(n-m+1)Y_{n+1}^{mo}(\hat{\mathbf{r}})r^{-(n+2)}\hat{\mathbf{x}}_1, \quad n \ge 0,\ m = 1,\dots,n,\ \mathbf{r}\in\Omega(\mathbb{R}^3)
\end{aligned}
\tag{4.3}
$$

and in the same way for the case $m = 0$ and for $\mathbf{r}\in\Omega(\mathbb{R}^3)$,

$$
\nabla u_n^{(e)0e}(\mathbf{r}) = \left[-Y_{n+1}^{1e}(\hat{\mathbf{r}})\hat{\mathbf{x}}_2 - Y_{n+1}^{1o}(\hat{\mathbf{r}})\hat{\mathbf{x}}_3 - (n+1)Y_{n+1}^{0e}(\hat{\mathbf{r}})\hat{\mathbf{x}}_1\right]r^{-(n+2)}, \quad n \ge 0. \tag{4.4}
$$

Here, it is important to remark that by definition for $|\zeta| \le 1$ and $\varphi \in [0,2\pi)$,

$$
Y_{-n}^{ms}(\hat{\mathbf{r}}) \equiv 0, \quad n \ge 0,\ m = 0,1,\dots,n,\ s = e,o, \tag{4.5}
$$

while

$$
Y_{n}^{ms}(\hat{\mathbf{r}}) \equiv 0, \quad n \ge 0,\ m > n,\ s = e,o. \tag{4.6}
$$

Our intention is to write the velocity field (3.8) and the stress tensor (3.11) in an appropriate form so that the application of the BCs (2.13)–(2.15) can provide us with easy-to-handle relations and thus obtain the unknown constant coefficients. In order to apply the BCs (2.13)–(2.15), we use the expressions (3.8) and (3.11), the formulae (4.1)–(4.6), the outward unit normal vector in Cartesian coordinates (2.3), and the orthogonality relation (3.2) as well as certain recurrence relations (see the appendix) for the Legendre and the trigonometric functions. After some extensive algebra, one obtains a complicated system of linear algebraic equations involving the unknown constant coefficients.

Homogeneity of the system, which is constituted by the constant coefficients that correspond to a velocity field of degree greater than two and nonvanishment of the relevant determinant, reveals that

$$\begin{aligned} &\mathbf{e}_n^{(i)ms} = \mathbf{e}_n^{(e)ms} = \mathbf{0}, \quad n \geq 3,\ m = 0,1,\dots,n,\ s = e,o, \\ &d_n^{(i)ms} = d_n^{(e)ms} = 0, \quad n \geq 4,\ m = 0,1,\dots,n,\ s = e,o. \end{aligned} \tag{4.7}$$

Consequently, our results are reduced up to the second degree for the velocity field and instead of the series (3.8), we recover a closed form. Some easy algebra leads us also to the vanishing of many of the remaining constant coefficients.

Finally, further examination of the constant coefficients that survive, in view of definitions (2.12) and (3.7) which were noted earlier, implies that

$$\begin{aligned} &c_1^{(i)1e} - b_1^{(i)1o} = 2\Omega_1, \\ &a_1^{(i)1o} - c_1^{(i)0e} = 2\Omega_2, \\ &a_1^{(i)1e} - b_1^{(i)0e} = -2\Omega_3, \\ &-a^3 a_2^{(i)0e} + \left(2 - \frac{3a}{b}\right) a_0^{(e)0e} = 3aU, \\ &d_0^{(e)0e} = -3a_1^{(e)0e} = -3b_1^{(e)1e} = -3c_1^{(e)1o}, \\ &-\frac{b^5}{5} a_2^{(i)0e} + d_1^{(e)0e} = 0, \\ &a_0^{(i)0e} - d_1^{(i)0e} + \frac{2}{b} a_0^{(e)0e} = 0, \\ &a_2^{(i)0e} + 5d_3^{(i)0e} = 0, \\ &a^5 a_2^{(i)0e} + 3a^5 d_3^{(i)0e} + 3d_1^{(e)0e} + a^2 a_0^{(e)0e} = 0. \end{aligned} \tag{4.8}$$

By virtue of the relations (4.8), setting the rest of constant coefficients to nil, the flow fields (3.8)–(3.11) take their final form after the substitution of the calculated constant coefficients. Thus, inserting the solution of (4.8) into the relations for the flow fields, using formulae (4.1)–(4.6), and employing definitions (2.12) and (3.1), we reach the spherical form of the flow fields. Indeed, by means of the definition of the quantities

$$\begin{aligned} &\gamma = \frac{a}{b}, \quad \gamma < 1, \\ &K = 2 - 3\gamma + 3\gamma^5 - 2\gamma^6, \end{aligned} \tag{4.9}$$

where a, b are the radii of the concentric spheres, formula (3.8) for the velocity field yields

$$\begin{aligned}\mathbf{v}(\mathbf{r}) = \boldsymbol{U} + \boldsymbol{\Omega}\times\mathbf{r} \\ +\hat{\mathbf{r}}\frac{U}{K}P_1(\zeta)\left[-(3\gamma^5+2)+\gamma^5\left(\frac{r}{a}\right)^2-\left(\frac{a}{r}\right)^3+(2\gamma^5+3)\left(\frac{a}{r}\right)\right] \\ +\hat{\boldsymbol{\zeta}}\frac{U}{2K}P_1^1(\zeta)\left[2(3\gamma^5+2)-4\gamma^5\left(\frac{r}{a}\right)^2-\left(\frac{a}{r}\right)^3-(2\gamma^5+3)\left(\frac{a}{r}\right)\right],\end{aligned} \tag{4.10}$$

where it is calculated within the domain which is limited between the two spheres: $\mathbf{r}\in\Omega(\mathbb{R}^3)$. The total pressure field, which is provided by (3.9), is taken to be

$$\mathrm{P}(\mathbf{r}) = \mathrm{P}_0 + \frac{\mu U}{aK}P_1(\zeta)\left[10\gamma^5\left(\frac{r}{a}\right)+(2\gamma^5+3)\left(\frac{a}{r}\right)^2\right], \quad \mathbf{r}\in\Omega(\mathbb{R}^3), \tag{4.11}$$

while for the vorticity field (3.10), it is confirmed that

$$\boldsymbol{\omega}(\mathbf{r}) = \boldsymbol{\Omega} + \hat{\boldsymbol{\varphi}}\frac{U}{2aK}P_1^1(\zeta)\left[-5\gamma^5\left(\frac{r}{a}\right)+(2\gamma^5+3)\left(\frac{a}{r}\right)^2\right], \quad \mathbf{r}\in\Omega(\mathbb{R}^3). \tag{4.12}$$

If we continue to focus on the spherical coordinate system, the stress tensor (3.11) is written as

$$\begin{aligned}\tilde{\boldsymbol{\Pi}}(\mathbf{r}) = -p(\mathbf{r})\tilde{\mathbf{I}} + \frac{\mu U}{aK}\Bigg\{2\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}P_1(\zeta)\left[3\left(\frac{a}{r}\right)^4-(2\gamma^5+3)\left(\frac{a}{r}\right)^2+2\gamma^5\left(\frac{r}{a}\right)\right] \\ -(\hat{\boldsymbol{\zeta}}\otimes\hat{\boldsymbol{\zeta}}+\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}})P_1(\zeta)\left[3\left(\frac{a}{r}\right)^4+2\gamma^5\left(\frac{r}{a}\right)\right] \\ +3(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\zeta}}+\hat{\boldsymbol{\zeta}}\otimes\hat{\mathbf{r}})P_1^1(\zeta)\left[\left(\frac{a}{r}\right)^4-\gamma^5\left(\frac{r}{a}\right)\right]\Bigg\}\end{aligned} \tag{4.13}$$

for $\mathbf{r}\in\Omega(\mathbb{R}^3)$, where the unit dyadic $\tilde{\mathbf{I}}$ is given by (3.12) and the thermodynamic pressure is connected with the total pressure field (4.11) via formula (2.9):

$$p(\mathbf{r}) = \mathrm{P}(\mathbf{r}) - \rho g h, \quad \mathbf{r}\in\Omega(\mathbb{R}^3). \tag{4.14}$$

Of course, the arbitrary constant pressure P_0 and the arbitrary height of reference h are appropriately chosen depending upon the physical requirements.

Hence, the Stokes flow fields for the non-axisymmetric Happel problem have been calculated in the closed forms provided by (4.10)–(4.14).

5. Conclusions. A method for solving 3D Stokes flow problems with Happel-type BCs was developed. Based on this method, we examined the flow in a spherical cell as a means of modeling flow through a swarm of spherical particles with the help of the Papkovich-Neuber differential representation, which offer solutions for such problems in spherical geometry. The important physical flow fields (velocity, total pressure, vorticity, and stress tensor) were presented in closed form after the imposition of the BCs.

The present work invoked a useful tool for dealing with non-axisymmetric problems, which is the representation theory. The freedom that 3D representations offer makes the solution of creeping flow problems within such domains feasible. Work under progress involves extension to *ellipsoidal* harmonic eigenfunctions for the Papkovich-Neuber representation and their Stokes flow counterparts for problems involving small ellipsoidal particles moving within Stokes fluids.

Appendix

In the interest of making this work complete and independent, we provide some useful material, which was used during the calculations.

We begin with the introduction of certain identities. Let u, v and $\mathbf{f}$, $\mathbf{g}$ denote two scalar and two vector fields, respectively. Then, if we define by $\tilde{\mathbb{S}}$ a dyadic, the basic identities used in this project concern the action of the gradient operator on the following expressions:

$$\begin{aligned}
\nabla \otimes (u\mathbf{f}) &= u\nabla \otimes \mathbf{f} + \nabla u \otimes \mathbf{f},\\
\nabla \cdot (u\mathbf{f}) &= u\nabla \cdot \mathbf{f} + \nabla u \cdot \mathbf{f},\\
\nabla \times (u\mathbf{f}) &= u\nabla \times \mathbf{f} + \nabla u \times \mathbf{f},\\
\nabla (\mathbf{f}\cdot\mathbf{g}) &= (\nabla \otimes \mathbf{f})\cdot\mathbf{g} + (\nabla\otimes\mathbf{g})\cdot\mathbf{f},\\
\nabla (uv) &= u\nabla v + v\nabla u,\\
\nabla \otimes (\tilde{\mathbb{S}}\cdot\mathbf{f}) &= (\nabla\otimes\tilde{\mathbb{S}})\cdot\mathbf{f} + (\nabla\otimes\mathbf{f})\cdot\tilde{\mathbb{S}}^{\top},\\
\nabla \otimes (\mathbf{f}\otimes\mathbf{g}) &= (\nabla\otimes\mathbf{f})\otimes\mathbf{g} + \left[\mathbf{f}\otimes(\nabla\otimes\mathbf{g})\right]^{213},
\end{aligned} \tag{A.1}$$

whereas $\tilde{\mathbb{S}}^{\top}$ is the inverted dyadic and the symbol $(\cdot)^{213}$ denotes the left transposition for a triadic.

The associated Legendre functions of the first kind [9] satisfy the recurrence relations

$$\begin{aligned}
(2n+1)\zeta P_n^m(\zeta) &= (n+m)P_{n-1}^m(\zeta) + (n-m+1)P_{n+1}^m(\zeta),\\
(2n+1)(1-\zeta^2)\frac{d}{d\zeta}P_n^m(\zeta) &= (n+1)(n+m)P_{n-1}^m(\zeta) - n(n-m+1)P_{n+1}^m(\zeta),\\
(2n+1)\sqrt{1-\zeta^2}P_n^m(\zeta) &= P_{n+1}^{m+1}(\zeta) - P_{n-1}^{m+1}(\zeta)\\
&= (n+m)(n+m-1)P_{n-1}^{m-1}(\zeta)\\
&\quad - (n-m+1)(n-m+2)P_{n+1}^{m-1}(\zeta)
\end{aligned} \tag{A.2}$$

for every $|\zeta| \le 1$ and $n \ge 0$, $m = 0, 1, \ldots, n$.

Furthermore, we have the relations

$$\begin{aligned}
\hat{\mathbf{r}} &= \zeta\hat{\mathbf{x}}_1 + \sqrt{1-\zeta^2}\cos\varphi\,\hat{\mathbf{x}}_2 + \sqrt{1-\zeta^2}\sin\varphi\,\hat{\mathbf{x}}_3, \\
\hat{\boldsymbol{\zeta}} &= -\sqrt{1-\zeta^2}\,\hat{\mathbf{x}}_1 + \zeta\cos\varphi\,\hat{\mathbf{x}}_2 + \zeta\sin\varphi\,\hat{\mathbf{x}}_3, \\
\hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}}_2 + \cos\varphi\,\hat{\mathbf{x}}_3
\end{aligned} \tag{A.3}$$

and their inverse

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \zeta\hat{\mathbf{r}} - \sqrt{1-\zeta^2}\,\hat{\boldsymbol{\zeta}}, \\
\hat{\mathbf{x}}_2 &= \sqrt{1-\zeta^2}\cos\varphi\,\hat{\mathbf{r}} + \zeta\cos\varphi\,\hat{\boldsymbol{\zeta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}, \\
\hat{\mathbf{x}}_3 &= \sqrt{1-\zeta^2}\sin\varphi\,\hat{\mathbf{r}} + \zeta\sin\varphi\,\hat{\boldsymbol{\zeta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{A.4}$$

for every $|\zeta| \le 1$ and $0 \le \varphi < 2\pi$.

References

[1] J. F. Brady and G. Bossis, *Stokesian dynamics*, Annu. Rev. Fluid Mech. **20** (1988), 111–157.

[2] G. Dassios, M. Hadjinicolaou, F. A. Coutelieris, and A. C. Payatakes, *Stokes flow in spheroidal particle-in-cell models with Happel and Kuwabara boundary conditions*, Internat. J. Engrg. Sci. **33** (1995), 1465–1490.

[3] G. Dassios, M. Hadjinicolaou, and A. C. Payatakes, *Generalized eigenfunctions and complete semiseparable solutions for Stokes flow in spheroidal coordinates*, Quart. Appl. Math. **52** (1994), no. 1, 157–191.

[4] G. Dassios, A. C. Payatakes, and P. Vafeas, *Interrelation between Papkovich-Neuber and Stokes general solutions of the Stokes equations in spheroidal geometry*, Quart. J. Mech. Appl. Math. **57** (2004), no. 2, 181–203.

[5] G. Dassios and P. Vafeas, *Comparison of differential representations for radially symmetric Stokes flow*, Abstr. Appl. Anal. **2004** (2004), no. 4, 347–360.

[6] N. Epstein and J. H. Masliyah, *Creeping flow through clusters of spheroids and elliptical cylinders*, Chem. Engrg. J. **3** (1972), 169–175.

[7] J. Happel, *Viscous flow in multiparticle systems: slow motion of fluids relative to beds of spherical particles*, AIChE J. **4** (1958), 197–201.

[8] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*, Martinus Nijholl Publishers, Dordrecht, 1986.

[9] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Publishing Company, New York, 1965.

[10] A. N. Kalarakis, V. N. Burganos, and A. C. Payatakes, *Galilean-invariant Lattice-Boltzmann simulation of liquid-vapor interface dynamics*, Phys. Rev. E **65** (2002), 1–13.

[11] S. Kuwabara, *The forces experienced by randomly distributed parallel circular cylinders or spheres in a viscous flow at small Reynolds numbers*, J. Phys. Soc. Japan **14** (1959), 527–532.

[12] P. Moon and D. E. Spencer, *Field Theory Handbook: Including Coordinate Systems, Differential Equations, and their Solutions*, 2nd ed., Springer-Verlag, Berlin, 1971.

[13] G. H. Neale and W. K. Nader, *Prediction of transport processes within porous media: creeping flow relative to a fixed swarm of spherical particles*, AIChE J. **20** (1974), 530–538.

[14] H. Neuber, *Ein neuer Ansatz zur Lösung räumblicher Probleme der Elastizitätstheorie*, Z. Angew. Math. Mech. **14** (1934), 203–212 (German).

[15] S. Uchida, *Abstract, slow viscous flow through a mass of particles*, Inst. Sci. Technol. Univ. Tokyo **3** (1949), 97 (Japanese), translated in Ind. Engng. Chem. **46** (1954), 1194–1195, by T. Motai.

[16] X. Xu and M. Wang, *General complete solutions of the equations of spatial and axisymmetric Stokes flow*, Quart. J. Mech. Appl. Math. **44** (1991), no. 4, 537–548.

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