A GAUSS-KUZMIN-LÉVY THEOREM FOR A CERTAIN CONTINUED FRACTION

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We consider an interval map which is a generalization of the well-known Gauss transformation. In particular, we prove a result concerning the asymptotic behavior of the distribution functions of this map.

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1. Introduction. In 1800, Gauss studied the following problem. In modern notation, it reads as follows. Write $x \in [0, 1)$ as a regular continued fraction

$$\frac{1}{a_1 + \frac{1}{a_2 + \dots}} := [a_1, a_2, \dots]_G. \quad (1.1)$$

Here, $a_k \in \{1, 2, \dots\}$. Define the map (now known as the *Gauss transformation* or the *continued fraction map*) $T_G : [0, 1) \rightarrow [0, 1)$ as follows: for $x = 0$, $T_G 0 := 0$; for $x \neq 0$, we have

$$T_G x = T_G [a_1, a_2, \dots]_G := [a_2, a_3, \dots]_G. \quad (1.2)$$

Note that T_G removes the first digit and the first level of $x = [a_1, a_2, \dots]_G$. By T_G , we generate a sequence of x_k in the unit interval with

$$x_k = T_G^k x_0. \quad (1.3)$$

Assuming that the “seed,” x_0 , is a random real number uniformly distributed in the unit interval, define the distribution function

$$F_k(t) = \text{probability that } x_k \leq t, \quad \text{for } 0 \leq t \leq 1. \quad (1.4)$$

In his notebook, Gauss remarked, after some numerical computations, that “they $[F_k]$ come out so complicated that no hope appears to be left.” (See Knuth [17, page 346]).

Twelve years later, in a letter he wrote to Laplace, Gauss stated, without proof, that

$$\lim_{k \rightarrow \infty} F_k(x) = \frac{\log(1+x)}{\log 2}. \quad (1.5)$$

However, he was unable to describe the behavior of F_k for a large but finite k . At the time, Gauss considered the study of F_k a problem he could not resolve to his satisfaction.

A century later, a proof was finally provided by Kuzmin [18]. In the same paper, he actually proved that, for large k , one has

$$F_k(x) = \frac{\log(1+x)}{\log 2} + r_n(x), \quad (1.6)$$

where $r_n(x) = O(q^{\sqrt{n}})$, and $0 < q < 1$. Around the same time, Lévy [19], by using a different method that employs probabilistic notions, proved that

$$r_n(x) = O(q^n). \quad (1.7)$$

Note that in the 60's, Szűsz [27] was able to prove the same result by using Kuzmin's approach. The asymptotic behavior of $F_k(x)$ was finally resolved in 1974 by Wirsing [30] and a complete solution to Gauss's problem was found a few years later by Babenko [1]. For detailed introductions, see [6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 23, 25]. For various extensions and generalizations, see [4, 5, 9, 20, 21, 22, 24, 26, 28, 29]. See also the monographs [15, 25]. Just as in the original Gauss-Kuzmin-Lévy Problem, the hard part of these generalizations often involves finding the explicit expressions of the distribution functions. Finally, we remark that the Gauss transformation has strong ties with chaos theory [2, 3].

In this note, we consider generalization of the Gauss transformation and prove an analogous result. Write $x \in [0, 1)$ as

$$\frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1 + \dots}} := [a_1, a_2, \dots]. \quad (1.8)$$

Here, a_k are natural numbers (see the next section for details) and one should think of a_k as the digits of x . Define the *generalized Gauss transformation* $T : [0, 1) \rightarrow [0, 1)$ as follows: for $x = 0$, $T0 := 0$; for $x \neq 0$, we have

$$Tx = T[a_1, a_2, \dots] := [a_2, a_3, \dots]. \quad (1.9)$$

Here, T removes the first digit and the first level of $x = [a_1, a_2, \dots]$. Note that this is the same as, for $x = [a_1, a_2, \dots] \neq 0$,

$$Tx = \frac{2^{-a_1}}{x} - 1. \quad (1.10)$$

Likewise, we can generate a sequence of x_k in the unit interval with

$$x_k = T^k x_0. \quad (1.11)$$

Assuming that the “seed,” x_0 , is a random real number uniformly distributed in the unit interval, define the distribution function

$$G_k(t) = \text{probability that } x_k \leq t, \quad \text{for } 0 \leq t \leq 1. \quad (1.12)$$

Note that $G_0(t) = t$. Our main result is the following theorem.

THEOREM 1.1. *Let $c^{-1} = \log(4/3)$. Then, there exist*

$$G_k(t) = c \log \left(1 + \frac{t}{2+t} \right) + O(q^k), \quad (1.13)$$

where $0 < q < 1$.

In [Section 2](#), we set up the necessary machinery, and in [Section 3](#), we prove [Theorem 1.1](#).

2. Preliminaries. In this section, first, we show in [Lemma 2.1](#) that $x \in [0, 1)$ can be written in the form of [\(1.8\)](#).

LEMMA 2.1. *For all $x \in [0, 1)$, there exist integers $a_k \in \{0, 1, 2, \dots\}$ such that*

$$x = \frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1 + \dots}}. \quad (2.1)$$

PROOF. For any $x \in [0, 1)$, we can find a natural number a_1 such that

$$\frac{1}{2^{a_1+1}} < x \leq \frac{1}{2^{a_1}}. \quad (2.2)$$

This implies, for some $p \in [0, 1)$,

$$x = (1-p)2^{-a_1} + \frac{p}{2}2^{-a_1} = \left(1 - \frac{p}{2}\right)2^{-a_1}. \quad (2.3)$$

Defining $x_1 \in [0, 1)$ by $x_1 = p/(2-p)$, we can write x as

$$x = \frac{2^{-a_1}}{1 + x_1}. \quad (2.4)$$

Since $x_1 \in [0, 1)$, we can repeat the same iteration and obtain

$$x = \frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1 + \dots}}. \quad (2.5)$$

Thus, the lemma is proven. $\square$

We remark that, from the above proof, it follows that the digits $\{a_1(x), a_2(x), \dots\}$ are related by

$$a_k(x) = a_1(T^{k-1}x), \quad (2.6)$$

where $a_1(x) = m$ if $x \in (2^{-m-1}, 2^{-m}]$.

Next, we want to prove the convergence of expansion of the type of [\(2.1\)](#). Define $[a_1, a_2, \dots, a_n]$, the convergent of x , by truncating the expansion on the right-hand side of [\(2.1\)](#). We want to show

$$x = \lim_{n \rightarrow \infty} [a_1, a_2, \dots, a_n]. \quad (2.7)$$

To this end, define integer-valued functions $P_n(x)$, $Q_n(x)$ by the following:

$$\begin{aligned} P_k(x) &= 2^{a_k(x)} P_{k-1}(x) + 2^{a_{k-1}(x)} P_{k-2}(x), \quad k \geq 2, \\ Q_k(x) &= 2^{a_k(x)} Q_{k-1}(x) + 2^{a_{k-1}(x)} Q_{k-2}(x), \quad k \geq 1, \end{aligned} \quad (2.8)$$

with $a_0(x) = 0$, $P_0(x) = 0$, $P_1(x) = 1$, $Q_{-1}(x) = 0$, and $Q_0(x) = 1$.

Standard induction arguments show that

$$\frac{2^{-a_1}}{1} + \frac{2^{-a_2}}{1} + \cdots + \frac{2^{-a_k}}{1+t} = \frac{P_k + t2^{a_k}P_{k-1}}{Q_k + t2^{a_k}Q_{k-1}}, \quad 0 \leq t \leq 1, \quad (2.9)$$

$$P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x) = (-1)^n 2^{a_1} \cdots 2^{a_{n-1}}. \quad (2.10)$$

Note that the left-hand side of (2.9) is a compact notation of continued fractions of the type of (1.8) with k levels.

By combining Lemma 2.1 and (2.9), we have, for $x \in [0, 1]$,

$$x = \frac{P_n(x) + t2^{a_n(x)}P_{n-1}(x)}{Q_n(x) + t2^{a_n(x)}Q_{n-1}(x)}, \quad (2.11)$$

where $t = T^n x$. Taking $t = 0$ in (2.11) gives

$$[a_1, \dots, a_n] = \frac{P_n(x)}{Q_n(x)}. \quad (2.12)$$

By combining (2.10) and (2.12), we have

$$|x - [a_1, \dots, a_n]| = \frac{2^{a_1} \cdots 2^{a_n}}{Q_n(t^{-1}Q_n + 2^{a_n}Q_{n-1})}, \quad (2.13)$$

where $t = T^n x$. Note that this equation, which measures the difference between x and its convergent, is the key ingredient of the following estimate.

LEMMA 2.2. *For all $x \in [0, 1]$, there exists $|x - [a_1, \dots, a_n]| \leq (1/2)^n$.*

REMARKS 2.3. This lemma implies (2.7).

PROOF. By using (2.13) and the fact that $t^{-1} \geq 1$, we have

$$|x - [a_1, \dots, a_n]| \leq \frac{2^{a_1} \cdots 2^{a_n}}{Q_n(Q_n + 2^{a_n}Q_{n-1})} := s_n. \quad (2.14)$$

We claim that

$$s_n \leq \frac{1}{2} s_{n-1}. \quad (2.15)$$

Indeed,

$$s_n \leq \frac{1}{2} \left(\frac{2^{a_1} \cdots 2^{a_{n-1}}}{Q_n Q_{n-1}} \right) \leq \frac{1}{2} \left(\frac{2^{a_1} \cdots 2^{a_{n-1}}}{Q_{n-1}(Q_{n-1} + 2^{a_{n-1}}Q_{n-2})} \right) = \frac{1}{2} s_{n-1}. \quad (2.16)$$

Note that, in obtaining the first inequality, we have used the fact that $Q_n \geq 2^{a_n} Q_{n-1}$. In obtaining the second inequality, we have used the fact that $Q_n \geq Q_{n-1} + 2^{a_{n-1}} Q_{n-2}$.

This proves the claim. By direct computation, we have $s_1 = 2^{-1-a_1} \leq 2^{-1}$. This, with (2.15), shows that $s_n \leq (1/2)^n$ and so the lemma is proven. $\square$

A few remarks are as follows. First, it is clear that every irrational $x \in [0, 1)$ has a unique expansion of the type of (2.1).

Second, we note that some particular cases of this type of continued fractions have been studied before. For example, by setting $q = 1/2$ and $a_k = k$, the right-hand side of (1.8) gives the well-known continued fraction of Rogers and Ramanujan:

$$\frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}. \quad (2.17)$$

Another example is the beautiful result due to Davison [7]. Let $a_k = F_k$, where F_k is the k th Fibonacci number. Davison showed that

$$\frac{2^{-F_1}}{1 + \frac{2^{-F_2}}{1 + \frac{2^{-F_3}}{1 + \dots}}} = \frac{1}{2} \sum_{n \geq 1} 2^{-\lfloor n\phi \rfloor}, \quad (2.18)$$

where ϕ is the Golden Ratio and $\lfloor \cdot \rfloor$ denotes the floor function.

Third, we give an example. In terms of the continued fraction of the type of (1.8), we have

$$\pi - 3 = \frac{2^{-2}}{1 + \frac{2^{-0}}{1 + \frac{2^{-1}}{1 + \dots}}} = [2, 0, 1, 0, 0, 0, 1, 1, 1, 6, \dots]. \quad (2.19)$$

Here, in the first equality, we gave only the first three digits. In the second equality where we used the compact notation in (1.8), we gave the first ten digits.

Next, we prove the following lemma.

LEMMA 2.4. *Let $c^{-1} = \log(4/3)$. The invariant probability density of the map T is given by*

$$\rho(t) = \frac{c}{(1+t)(2+t)}. \quad (2.20)$$

REMARK 2.5. As expected, the integral of ρ is the first term of the right-hand side of (1.13), that is,

$$\int_0^t \rho(s) ds = c \log \left(1 + \frac{t}{2+t} \right). \quad (2.21)$$

PROOF. To this end, we need to show that $\rho(t)$ defined in (2.20) is an eigenfunction of eigenvalue 1 of the Perron-Frobenius operator (see, e.g., [14, 15, 25])

$$L_T \rho(t) = \sum_{s \in T^{-1}(t)} \frac{\rho(s)}{|T'(s)|}. \quad (2.22)$$

First, we note that

$$T^{-1}(t) = \left\{ \frac{2^{-k}}{1+t}; k = 0, 1, 2, \dots \right\}. \quad (2.23)$$

With this understood, we have, with $y = 1/2$,

$$\begin{aligned} L_T \rho(t) &= c \sum_{k=0}^{\infty} \frac{y^k}{(1+t)^2} \rho\left(\frac{y^k}{1+t}\right) \\ &= c \sum_{k=0}^{\infty} \frac{y^{k+1}}{(t+1+y^k)(t+1+y^{k+1})} \\ &= c \sum_{k=0}^{\infty} \frac{1}{t+1+y^{k+1}} - \frac{1}{t+1+y^k} \\ &= c \left(\frac{1}{t+1} - \frac{1}{t+2} \right) \\ &= \rho(t). \end{aligned} \quad (2.24)$$

This proves the lemma. $\square$

3. Proof of Theorem 1.1. Here, we follow [23, pages 152-155]; see also [10, 17, 25]. First, by following the same trick that is used in the original Gauss-Kuzmin-Lévy problem, one can show that $\{G_k(t)\}$, defined in (1.12), satisfy a Kuzmin-type equation

$$G_{k+1}(t) = \sum_{m=0}^{\infty} G_k(y^m) - G_k\left(\frac{y^m}{1+t}\right). \quad (3.1)$$

Just as in the original Gauss-Kuzmin-Lévy problem, it is easier to work with the derivative of $G_k(t)$. To this end, we observe that since the derivative of $G_0(t) = t$ is bounded in the unit interval, we can show by induction that the derivative of $G_k(t)$ is also bounded in the unit interval.

This allows us to differentiate (3.1) term-by-term, obtaining

$$G'_{k+1}(t) = \sum_{m=0}^{\infty} \frac{y^m}{(1+t)^2} G'_k\left(\frac{y^m}{1+t}\right). \quad (3.2)$$

Here, the prime denotes the derivative with respect to t . Next, we introduce $f_k(t)$ in such a way that

$$G'_k(t) = \frac{f_k(t)}{(1+t)(2+t)}. \quad (3.3)$$

In terms of $f_k(t)$, (3.2) can be written as

$$f_{k+1}(t) = \sum_{m=0}^{\infty} p_m(t) f_k\left(\frac{y^m}{1+t}\right), \quad (3.4)$$

where

$$p_m(t) = \frac{y^{m+1}(1+t)(2+t)}{(1+t+y^m)(1+t+y^{m+1})} = y^{m+1} + \frac{\Delta_m}{1+t+y^m} - \frac{\Delta_{m+1}}{1+t+y^{m+1}}. \quad (3.5)$$

Here, $\Delta_m := y^m - y^{2m}$. Note that it follows from the definition of $p_m(t)$ that it is manifestly nonnegative for all $t \in [0, 1)$ and for all natural numbers m . Also, we have used partial fraction decomposition in obtaining the second equality.

These formulae fit into the overall strategy as follows. Introduce a function $R_k(t)$ such that

$$G_k(t) = c \log \left(1 + \frac{t}{2+t} \right) + R_k \left(c \log \left(1 + \frac{t}{2+t} \right) \right). \quad (3.6)$$

Here, c is the constant in [Theorem 1.1](#). Because $G_k(0) = 0$ and $G_k(1) = 1$, we have $R_k(0) = R_k(1) = 0$. To prove the theorem, we have to show that

$$R_k = O(q^k), \quad (3.7)$$

where $0 < q < 1$. To achieve this goal, we proceed as follows. First, by comparing the derivatives of (3.3) and of (3.6), we obtain

$$R'_k \left(c \log \left(1 + \frac{t}{2+t} \right) \right) = \frac{(1+t)(2+t)}{c^2} f'_k(t). \quad (3.8)$$

Next, by using (3.4), we can show that (the details will be given below)

$$f'_k(t) = O(q^k) \quad (3.9)$$

for $0 < q < 1$. With (3.8), this implies $R'_k = O(q^k)$. Finally, by an interpolation formula

$$R_k(t) = -\frac{t(1-t)}{2} R''_k(\xi), \quad (3.10)$$

where $0 < \xi < 1$, we arrive at (3.7), and [Theorem 1.1](#) is proven. All we need to do is to prove (3.9), as promised.

First, we note that from (3.4), we have

$$f'_{k+1}(t) = \sum_{m \geq 0} p'_m(t) f_k \left(\frac{y^m}{1+t} \right) - \sum_{m \geq 0} p_m(t) \frac{y^m}{(1+t)^2} f'_k \left(\frac{y^m}{1+t} \right). \quad (3.11)$$

Our immediate goal is to express, by using (3.11), $f'_{k+1}(t)$ in terms of $f'_k(t)$. To this end, we need to rewrite the first sum in (3.11) as follows. By the second equality of (3.5) and partial summation, we have

$$\begin{aligned} \sum_{m \geq 0} p'_m(t) f_k \left(\frac{y^m}{1+t} \right) &= \sum_{m \geq 0} \left(\frac{\Delta_{m+1}}{(1+t+y^{m+1})^2} - \frac{\Delta_m}{(1+t+y^m)^2} \right) f_k \left(\frac{y^m}{1+t} \right) \\ &= \sum_{m \geq 0} \frac{\Delta_{m+1}}{(1+t+y^{m+1})^2} \left(f_k \left(\frac{y^m}{1+t} \right) - f_k \left(\frac{y^{m+1}}{1+t} \right) \right). \end{aligned} \quad (3.12)$$

This, with the application of the mean value theorem to the difference

$$f_k \left(\frac{y^m}{1+t} \right) - f_k \left(\frac{y^{m+1}}{1+t} \right), \quad (3.13)$$

enables us to rewrite (3.11) as

$$f'_{k+1}(t) = \sum_{m \geq 0} \frac{y^{m+1} \Delta_{m+1}}{(1+t)(1+t+y^{m+1})^2} f'_k(\theta_m) - \sum_{m \geq 0} p_m(t) \frac{y^m}{(1+t)^2} f'_k\left(\frac{y^m}{1+t}\right), \quad (3.14)$$

with

$$\frac{y^{m+1}}{1+t} < \theta_m < \frac{y^m}{1+t}. \quad (3.15)$$

Note that the right-hand side of (3.14) is written in terms of the derivative of $f_k(t)$ alone.

With the above understood, we proceed to compare the maximum of f'_{k+1} and that of f'_k . Let M_j be the maximum of $|f'_j(t)|$ on $t \in [0, 1]$. Then (3.14) implies

$$M_{k+1} \leq M_k \cdot \max_{t \in [0, 1]} \left| \sum_{m \geq 0} \frac{y^{m+1} \Delta_{m+1}}{(1+t)(1+t+y^{m+1})^2} + \sum_{m \geq 0} p_m(t) \frac{y^m}{(1+t)^2} \right|. \quad (3.16)$$

We estimate the sums on the right-hand side of (3.16). First, we note that each term in the first sum in (3.16) is bounded; precisely, we have

$$\frac{y^{m+1} \Delta_{m+1}}{(1+t)(1+t+y^{m+1})^2} \leq \frac{y^{2m+2}}{(1+y^{m+1})^2}. \quad (3.17)$$

Next, observe that the function (cf. the second sum in (3.16))

$$p_m(t) \frac{y^m}{(1+t)^2} \quad (3.18)$$

is decreasing for $t \geq 0$. Therefore, it attains its maximum at $t = 0$. This leads to

$$p_m(t) \frac{y^m}{(1+t)^2} \leq \frac{y^{2m}}{(1+y^m)(1+y^{m+1})} \leq \frac{y^{2m}}{(1+y^{m+1})^2}. \quad (3.19)$$

These two observations allow us to rewrite inequality (3.16) as

$$M_{k+1} \leq q M_k, \quad (3.20)$$

where

$$q := (1+y^2) \sum_{m \geq 0} \frac{y^{2m}}{(1+y^{m+1})^2} = 5 \sum_{m \geq 0} \frac{1}{(1+2^{m+1})^2}. \quad (3.21)$$

Since we have, for $m \geq 2$,

$$\frac{1}{(1+2^{m+1})^2} \leq \frac{1}{20} \left(\frac{1}{2}\right)^m, \quad (3.22)$$

therefore,

$$q \leq 5 \left(\frac{1}{9} + \frac{1}{25} + \frac{1}{20} \sum_{m \geq 2} \frac{1}{2^m} \right) = 0.880555 \dots < 1. \quad (3.23)$$

This, with (3.20), implies $f'_k(t) = O(q^k)$, that is, (3.9). This completes the proof of Theorem 1.1.

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