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PROXIMINALITY IN GENERALIZED DIRECT SUMS

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We consider proximinality and transitivity of proximinality for subspaces of finite codimension in generalized direct sums of Banach spaces. We give several examples of Banach spaces where proximinality is transitive among subspaces of finite codimension.

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1. Introduction. Let X be a Banach space and let Y be a closed subspace of X. We recall that Y is said to be a proximinal subspace of X if for any $x \in X$ there exists a $y \in Y$ such that $d(x,Y) = \|x-y\|$.

In the first part of the paper, we study proximinal subspaces of finite codimension in generalized direct sums of Banach spaces (a concept due to Veselý [9], see below for the definition). Our motivation comes from some recent work of Indumathi [4] where she considered these questions for c_0-direct sums of a family of Banach spaces and proved the following.

Theorem 1.1. *Let $X = (\oplus X_\lambda)_{c_0}$ where each X_λ is a Banach space for each $\lambda \in \Lambda$. Let Y be a closed subspace of X of finite codimension n in X. Then Y is proximinal in X if and only if the following two conditions hold for every basis $\{f_i : 1 \le i \le n\}$ of $Y^\perp$, where $f_i = (f_{i,\lambda})_{\lambda\in\Lambda}$ for $1 \le i \le n$:*

(i) *for every i, $1 \le i \le n$, $f_{i,\lambda}$ is nonzero only for finite number of indices λ,*

(ii) *$Y_\lambda = \cap_{i=1}^{n} \ker f_{i,\lambda}$ is proximinal in X_λ for each $\lambda \in \Lambda$.*

In the present paper, we prove an analogue of the above result for generalized direct sums. We next consider transitivity of proximinality among subspaces of finite codimension in c_0-direct sums. Here the motivation comes from [6] where transitivity was established among finite codimensional subspaces of c_0. We give several new examples of spaces where transitivity of proximinality holds among subspaces of finite codimension answering [3, Question 2] in the affirmative. We give a partial positive answer to this question in c_0-direct sums. For a Banach space X, let $NA(X)$ denote the set of norm attaining elements of X^*. We recall from [3] that X is said to be an $R(1)$-space if $Y \subset X$ is of finite codimension and $Y^\perp \subset NA(X)$ implies Y is proximinal. We show that this property is preserved by c_0-direct sums but not by ℓ_1-direct sums. We only consider these topics for Banach spaces and leave the formulations in the more general setting of locally convex spaces open. An interested reader can consult the monograph [5] for a comprehensive treatment of proximinality in locally convex spaces.

We now define generalized direct sums. Let Λ be a nonempty set. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of normed linear spaces. By e_λ, we denote the characteristic function of the singleton $\{\lambda\} \subset \Lambda$, that is, $e_\lambda(\lambda') = \delta_{\lambda\lambda'}$.

Let Y be a linear space, $\Lambda_0 \subset \Lambda$, $y \in Y^\Lambda$. We denote by $y|_{\Lambda_0}$, the element of Y^Λ defined by

$$y|_{\Lambda_0} = \begin{cases} y(\lambda) & \lambda \in \Lambda_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Hence $y|_{\Lambda_0}$ is the canonical projection of y onto the subspace of functions whose support is contained in Λ_0 and $Y^{\Lambda_0} = \{y \in Y^\Lambda \mid \operatorname{supp}(y) \subset \Lambda_0\}$.

By a sequence space on Λ, we mean a normed linear space (V, γ) such that V is a linear subspace of $\mathbb{R}^\Lambda$.

Definition 1.2. Let (V, γ) be a sequence space on Λ such that γ is monotone on the nonnegative elements on V. Denote by $(\oplus X_\lambda)_V$ the linear space

$$(\oplus X_\lambda)_V = \left\{x \in [\cup X_\lambda]^\Lambda : x(\lambda) \in X_\lambda \ \forall \lambda \in \Lambda, \|x(\cdot)\| \in V\right\} \tag{1.2}$$

equipped with the norm $\|x\|_V = \gamma(\|x(\cdot)\|)$ where $\|x(\cdot)\|$ means the function $\lambda \to \|x(\lambda)\|_{X_\lambda}$.

Let $\pi : \mathbb{R}^\Lambda \mapsto [0, +\infty]$ be a norm on $\mathbb{R}^\Lambda$ which is finite on the elements with finite support. By $S_\pi(\Lambda)$, we denote the linear space $S_\pi(\Lambda) = \{\xi \in \mathbb{R}^\Lambda : \pi(\xi) < +\infty\}$ equipped with the norm π.

Definition 1.3. A norm $\pi : \mathbb{R}^\Lambda \mapsto [0, +\infty]$ will be called

(i) *proper* if it is finite on the elements with finite support,

(ii) *finitely determined* if for every $\xi \in \mathbb{R}^\Lambda$,

$$\pi(\xi) = \sup\{\pi(\xi|_{\Lambda_0}) : \Lambda_0 \text{ is a finite subset of } \Lambda\}, \tag{1.3}$$

(iii) *monotonic* if $\pi(\xi) \le \pi(\eta)$ whenever $|\xi| \le |\eta|$ $\xi, \eta \in \mathbb{R}^\Lambda$,

(iv) *dual norm of a sequence space on* Λ if there exists (V, γ) sequence space on Λ (as defined above), containing basic vectors e_λ as unit vectors and such that its dual V^* is isometric with $S_\pi(\Lambda)$ and the isometric correspondence between $v^* \in V^*$ and $w \in S_\pi(\Lambda)$ is given by

$$v^*(\xi) = \sum_{\lambda \in \Lambda} \xi(\lambda) w(\lambda) \quad \text{where } \xi \in V. \tag{1.4}$$

When $V = S_\pi(\Lambda)$ we will write $(\oplus X_\lambda)_\pi$ instead of $(\oplus X_\lambda)_V$.

Example 1.4. Let $1 \le p \le +\infty$. Let $\pi : \mathbb{R}^\Lambda \mapsto [0, +\infty]$ be the classical ℓ^p-norm. Then π is monotonic, proper, and finitely determined, and we have $S_\pi(\Lambda) = \ell^p(\Lambda)$, $(\oplus X_\lambda)_\pi = (\oplus X_\lambda)_{\ell^p}$.

Each classical ℓ^p-norm is a dual norm of a sequence space on Λ, with the predual V given by

$$V = \begin{cases} c_0(\Lambda) & \text{if } p = 1, \\ \ell^q(\Lambda) & \text{if } 1 < p < +\infty,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \ell^1(\Lambda) & \text{if } p = +\infty. \end{cases} \tag{1.5}$$

The following lemma was proved in [9].

Lemma 1.5. *Let $\pi : \mathbb{R}^\Lambda \mapsto [0, +\infty]$ be a norm which is monotonic, proper, and finitely determined. Let X_λ be a Banach space for every $\lambda \in \Lambda$. Then*

(a) *$(\oplus X_\lambda)_\pi$ and $S_\pi(\Lambda)$ are Banach spaces,*

(b) *if π is a dual norm of a sequence space on Λ, then the space $(\oplus X_\lambda^*)_\pi$ is isometric to a dual space.*

2. Proximinality in generalized direct sums. We need the following theorems of Garkavi (see [7, pages 94–95]).

Theorem 2.1. *Let Y be a closed subspace of finite codimension in a normed linear space X. Then Y is proximinal if and only if for each $\Phi \in (Y^\perp)^*$, there exists $x \in X$ such that $\|\Phi\| = \|x\|$ and $\Phi(f) = f(x)$ for all $f \in Y^\perp$.*

The following result is easy to deduce from the above theorem.

Theorem 2.2. *Let X be a normed linear space and Y a closed subspace of finite codimension in X. Then Y is proximinal in X if and only if every closed subspace $Z \supseteq Y$ of X is proximinal in X.*

As an immediate consequence one has that if a finite codimensional subspace $Y \subset X$ is proximinal, then $Y^\perp \subset NA(X)$.

We extend Indumathi's result on c_0-direct sums which is mentioned in the last section, to generalized direct sums. We now introduce the notations that we are going to use in this result.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Let (V, γ) be a sequence space on Λ and π a dual norm of a sequence space on Λ such that $X^* = (\oplus X_\lambda^*)_\pi$ where $X = (\oplus X_\lambda)_V$. We recall from Definition 1.3(iv) that this in particular means that V has the canonical basis vectors as unit vectors. Let $Y \subset X$ be a closed subspace of finite codimension. Let $f_i = (f_{i,\lambda})_{\lambda \in \Lambda}$ where $i = 1, \dots, n$, be in $Y^\perp$ such that $Y = \cap_{i=1}^n \ker f_i$ and let $Z_\lambda = \cap_{i=1}^n \ker f_{i,\lambda}$. Assume that each Z_λ is proximinal in corresponding X_λ. Now we have the following.

Theorem 2.3. *Every finite codimensional closed subspace Y of the above form is proximinal in X for every dual norm π if and only if all but finitely many X_i's are $\{0\}$.*

Proof. First suppose that all but finitely many X_λ's are trivial spaces. Let $Y \subset X$ be of finite codimension. By our assumption, Z_λ is proximinal in respective X_λ. Since all but finitely many X_λ's are trivial, f_i's have only finitely many nonzero terms. Consider $A = \cup_{i=1}^n \{\lambda \in \Lambda : f_{i,\lambda} \neq 0\}$. We have that $|A| < \infty$.

Let G be a subspace of X such that $G = \{x_{|_A} : x \in X\}$ and $Z_\lambda = \cap_{i=1}^n \ker f_{i\lambda}$ for $\lambda \in A$. For $\lambda \in A$, each Z_λ is a proximinal subspace of finite codimension in X_λ. Further if $Z = (\oplus Z_\lambda)_V$, then we show that Z is a proximinal subspace of finite codimension in G. Let $g = (g_\lambda)$ be in G. For every g_λ we have z_λ^0 in Z_λ such that $\|g_\lambda - z_\lambda^0\| = d(g_\lambda, Z_\lambda)$. Let $z^0 = (z_\lambda^0) \in Z$. Now $\|g - z^0\|_V = \gamma(\|g_\lambda - z_\lambda^0\|) \le \gamma(\|g_\lambda - z_\lambda\|)$ for $z = (z_\lambda) \in Z$, by monotonicity of γ. Thus Z is proximinal in G. Set $Y_0 = \cap_{i=1}^n \{x_{|_A} : x = (x_\lambda) \in X$ and $\sum_{\lambda \in A} f_{i\lambda}(x_\lambda) = 0\}$. Then Y_0 is a subspace of G, $Z \subset Y_0 \subset G$. Now by Theorem 2.2, we conclude that Y_0 is proximinal in G. Again using the monotonicity of the norm γ it can be easily seen that this implies that Y is proximinal in X.

Conversely we assume that every finite codimensional closed subspace $Y \subset X$ of the form considered earlier is proximinal in X. Suppose infinitely many X_λ's are nontrivial. Then as in [4], we give an example of a subspace Y of X of codimension 2 such that Y is not proximinal in X.

Construction of the example. Let γ be a norm which is not equal to the c_0 norm. In particular we take $\gamma = \ell_1$. Assume without loss of generality that $\Lambda = \mathbb{N}$ and $X = \ell_1$. Then $X^* = \ell_\infty$. Take $f_1 = (1, 0, 3/4, 4/5, \dots, n/(n+1), \dots)$ and $f_2 = (0, 1, 3/4, 4/5, \dots, n/(n+1), \dots)$ in $NA(X) = \{(\alpha_n) \in \ell_\infty :$ there exists $n_0 \in \mathbb{N}$ such that $\|(\alpha_n)\|_\infty = |\alpha_{n_0}|\}$. Let $x = (1, 1, 0, 0, \dots)$. Then $f_1(x) = 1$ and $f_2(x) = 1$.

Since f_1 and f_2 are in $NA(X)$, $\ker f_1$ and $\ker f_2$ are proximinal hyperplanes of X. Let $Y = \cap_{i=1}^2 \ker f_i$ so that $\dim(X/Y) = 2$. We show that Y is not proximinal in X. Clearly $d(x, Y) = \|x|_{Y^\perp}\| \ge 1$.

We now claim that $d(x, Y) = 1$. Select x_n in $\mathbb{R}$ such that $x_1 = 1 = x_2$ and for $n \ge 3$, put $x_n = -(n+1)/n$. Define y_k in X for $k \ge 3$ by

$$y_k(n) = \begin{cases} x_n & \text{if } n \in \{1, 2, k\}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

Then $f_i(y_k) = 0$ for $i = 1, 2$ and so $y_k \in Y$ for all k. Further $\|x - y_k\| = (k+1)/k \to 1$ as $k \to \infty$. Hence $d(x, Y) = 1$. Now if there is a $y_0 \in Y$ such that $\|x - y_0\| = d(x, Y) = 1$, then $f_1(x - y_0) = 1 = \|x - y_0\|$ and $f_2(x - y_0) = 1 = \|x - y_0\|$. Since the first equality implies $y_0 = e_1$, clearly the second equality cannot hold. Thus Y is not proximinal. $\square$

We next show that any proximinal subspace of finite codimension in $X = (\oplus X_\lambda)_{c_0}$ is also a proximinal subspace of $W = (\oplus X_\lambda)_{\ell_\infty}$. The proof uses ideas similar to the ones given above and the well-known facts, X is a proximinal subspace of W (this can be seen by verifying the "3-ball property" and concluding proximinality as in [2, Proposition II.1.1]) and that if Z_1 and Z_2 are two proximinal subspaces in X_1 and X_2, then $Z_1 \oplus_{\ell_\infty} Z_2$ is proximinal in $X_1 \oplus_{\ell_\infty} X_2$.

Corollary 2.4. *Let $X = (\oplus X_\lambda)_{c_0}$ and let $W = (\oplus X_\lambda)_{\ell_\infty}$. Let Y be a proximinal subspace of finite codimension in X. Then Y is proximinal in W.*

Proof. We follow the notations used during the proof of the above theorem. Let $X_1 = (\oplus X_{\lambda, \lambda \in A})_{\ell_\infty}$ and let $X_2 = (\oplus X_{\lambda, \lambda \notin A})_{c_0}$. Then one can see as in the above proof Y_0

is proximinal in X_1. Also as remarked above, X_2 is proximinal in $(\oplus X_{\lambda \notin A})_{\ell_\infty}$. It is easy to see that $Y = Y_0 \oplus_{\ell_\infty} X_2$. Therefore Y is proximinal in $X_1 \oplus_{\ell_\infty} (\oplus X_{\lambda \notin A})_{\ell_\infty} = W$. □

Remark 2.5. Let Y be a proximinal subspace of finite codimension in $X = (\oplus X_\lambda)_{c_0}$. Let Y_1 be the finite codimensional subspace in $W = (\oplus X_\lambda)_{\ell_\infty}$ obtained by intersecting the kernals of the same functionals that determine Y. Then by Theorem 2.3 we have that Y_1 is a proximinal subspace of W. We also get from the above corollary that Y is proximinal in Y_1.

Our next result substantially improves on the above corollary when the component spaces are scalars. We retain the notations used above.

Proposition 2.6. *Let $Y \subset \ell_\infty(I)$ be a subspace of finite codimension determined by finitely supported functionals in $\ell_1(I)$. Then Y is proximinal, under the canonical embedding in $\ell_\infty^{**}(I)$.*

Proof. We have that Y is a weak*-closed subspace and hence proximinal in $\ell_\infty(I)$. Let $X_1 = (\oplus \mathbb{R}_{\lambda,\lambda \in A})_{\ell_\infty}$ and $X_2 = (\oplus \mathbb{R}_{\lambda,\lambda \notin A})_{\ell_\infty}$. We have as before $Y = Y_0 \oplus_{\ell_\infty} X_2$ and $\ell_\infty(I) = X_1 \oplus_{\ell_\infty} X_2$. We next recall the well-known fact that any space of continuous functions on a compact set is proximinal in its bidual, see [8]. Thus X_2 which can be identified as the space of continuous functions on the Stone-Cech compactification of the index set, is proximinal in its bidual. Also X_1 is a finite-dimensional space. Therefore Y is proximinal in $X_1 \oplus_{\ell_\infty} X_2^{**} = \ell_\infty^{**}(I)$. □

We now consider transitivity of proximinality among subspaces of finite codimension. We use the notation $Y \overset{p}{\subset} X$ to indicate that Y is a proximinal subspace of X.

Definition 2.7. A Banach space X is said to be a P-space (Pollul space) if proximinality is transitive for subspaces of finite codimension, that is, $Y \overset{p}{\subset} Z \overset{p}{\subset} X$, and the fact that both Y and Z are of finite codimension implies $Y \overset{p}{\subset} X$.

Well-known examples of P-spaces are c_0 space, reflexive spaces. Also the space of compact operators $\mathcal{K}(\ell_2)$ on the Hilbert space ℓ_2 is a P-space. To see this, we note that we have from [1, Lemma 4.2] that $NA(\mathcal{K}(\ell_2))$ is a linear space. Also from [1, Theorem 5.3] we know that $\mathcal{K}(\ell_2)$ is an $R(1)$-space. It thus follows from [3, Corollary 5] that $\mathcal{K}(\ell_2)$ is a P-space. This answers [3, Question 2]. See [3] for more general results on transitivity.

The following lemma gives a way of giving more examples of P-spaces.

Lemma 2.8. *Let X be a P-space and let $Y \subset X$ be a proximinal subspace of finite codimension. Then Y is a P-space.*

Proof. Let $Z_1 \overset{p}{\subset} Z_2 \overset{p}{\subset} Y \overset{p}{\subset} X$, where both Z_1 and Z_2 are finite codimensional subspaces of Y. Since X is a P-space, Z_2 is proximinal in X. Using the same reasoning this time with Z_2 and X, we see that Z_1 is proximinal in X and hence in Y. Thus Y is a P-space. □

To motivate the results that we will be proving next we give the details of transitivity of proximinality for c_0. The proof we present here is simpler than the one in [3, 6].

Lemma 2.9. *Let $X = c_0 = (\oplus\mathbb{R})_{c_0}$. Let $Z \overset{p}{\subset} Y \overset{p}{\subset} X$ and $\dim(X/Y) = n < \infty$, $\dim(Y/Z) = m < \infty$. Then $Z \overset{p}{\subset} X$.*

Proof. We will give the proof when $n = 1$ and given the nature of proximinal subspaces of finite codimension, the arguments are similar when $n > 1$. Let $f \in NA(X)$ be such that $Y = \ker f$. Clearly f has only finitely many nonzero terms.

Let $Z \overset{p}{\subset} Y$, $\dim(Y/Z) = m$. Let $f_1, f_2, \ldots, f_m \in Z^\perp \subset Y^*$ be such that

$$Z = \{y \in Y : f_i(y) = 0 \ \forall i, \ 1 \le i \le m\}. \tag{2.2}$$

Here $f_1, \ldots, f_m \in NA(Y)$. Let $\tilde{f}_1, \ldots, \tilde{f}_m$ be norm-preserving extensions of $f_1, \ldots, f_m$ onto X. Then

$$Z = \{x \in X : f(x) = 0, \ \tilde{f}_i(x) = 0 \ \forall i, \ 1 \le i \le m\}. \tag{2.3}$$

As $f_1, \ldots, f_m$ are norm-attaining functionals on Y, $\tilde{f}_1, \ldots, \tilde{f}_m$ are also norm attaining on X, which implies $Z \overset{p}{\subset} X$. □

The following corollary can also be deduced from some of our later results. However we prefer to present a proof here using above ideas.

Corollary 2.10. *Let $X = (\oplus X_\lambda)_{c_0}$, where each X_λ is a reflexive Banach space. Then X is a P-space. Any proximinal subspace of finite codimension in X is also a P-space.*

Proof. We suppose that Y and Z are closed subspaces of X such that $Z \overset{p}{\subset} Y \overset{p}{\subset} X$ with $\dim(X/Y) = n < \infty$ and $\dim(Y/Z) = m < \infty$. Then as before we can write Y and Z as $Y = \cap_{i=1}^{n} \ker f_i$ and $Z = \cap_{i=1}^{n+m} \ker f_i$ where $f_1, f_2, \ldots, f_{n+m} \in X^*$. Let $Z_\lambda = \cap_{i=1}^{n+m} \ker f_{i,\lambda} \subset Y_\lambda = \cap_{i=1}^{n} \ker f_{i,\lambda} \subset X_\lambda$. Then Z_λ is proximinal in Y_λ as well as in X_λ by reflexivity. So by Theorem 1.1, Z is proximinal in X which completes the proof. The second assertion follows from Lemma 2.8. □

As seen in the above proof in the general case the main difficulty is to prove the proximinality of Z_λ in X_λ. We now give a positive result for the validity of transitivity in the case of c_0-direct sums. To state the result we need the following notation.

Let $X = (\oplus X_\lambda)_{c_0}$, where X_λ is a Banach space for each $\lambda \in \Lambda$. Let Y and Z be closed subspaces of X such that $Z \subset Y \subset X$ with finite codimensions, that is, $\dim(X/Y) = n < \infty$ and $\dim(Y/Z) = m < \infty$. Then we can write Z as $Z = \cap_{i=1}^{n+m} \ker f_i$, where $f_i = (f_{i,\lambda}) \in Z^\perp \subset X^*$ for $1 \le i \le n+m$. Let $Z_\lambda = \cap_{i=1}^{n+m} \ker f_{i,\lambda}$ for $\lambda \in \Lambda$. Assume that if $z = (z_\lambda) \in Z$, then $z_\lambda \in Z_\lambda$ for every $\lambda \in \Lambda$. Then we have the following.

Proposition 2.11. *Suppose each X_λ in the above direct sum is a P-space. With the above assumption on Z, if $Z \overset{p}{\subset} Y \overset{p}{\subset} X$, then $Z \overset{p}{\subset} X$.*

Proof. We suppose that $Z \overset{p}{\subset} Y \overset{p}{\subset} X = (\oplus X_\lambda)_{c_0}$. Since $Z \overset{p}{\subset} Y$ and $Y \overset{p}{\subset} X$, there exists $f_i \in NA(X)$ for $1 \le i \le n+m$ such that $Y = \cap_{i=1}^{n} \ker f_i$ and $Z = \cap_{i=1}^{n+m} \ker f_i$ in X. Let $Z_\lambda = \cap_{i=1}^{n+m} \ker f_{i,\lambda} \subset \cap_{i=1}^{n} \ker f_{i,\lambda} \subset X_\lambda$.

We claim that Z_λ is proximinal in $Y_\lambda = \cap_{i=1}^n \ker f_{i,\lambda}$. Indeed let $y_\lambda \in Y_\lambda$. Now consider $\bar{y} = (0,\dots,0,y_\lambda,0,\dots)$. Then there exists $z^0 = (z_\lambda^0)_{\lambda\in\Lambda} \in Z$ such that $d(\bar{y},Z) = \|z^0 - \bar{y}\|$. By our assumption $z_\lambda^0 \in Z_\lambda$. We next show that z_λ^0 is a best approximation.

Let $z_\lambda \in Z_\lambda$ and consider $z = (0,\dots,0,z_\lambda,0,\dots)$, $z \in Z$. Now $\|y_\lambda - z_\lambda\| = \|\bar{y} - z\| \geq \|\bar{y} - z^0\| \geq \|y_\lambda - z_\lambda^0\|$ which implies that z_λ^0 is a best approximation.

Therefore Z_λ is proximinal in Y_λ for all $\lambda \in \Lambda$. Thus Z_λ is proximinal in X_λ by the transitivity property of X_λ. By Theorem 1.1, Z is proximinal in X. □

Conversely we have the following.

Proposition 2.12. *Let $X = (\oplus X_\lambda)_V$ be a P-space. Then X_λ is a P-space for each $\lambda \in \Lambda$.*

Proof. Fix $\lambda_0 \in \Lambda$ and let Z_{λ_0} and Y_{λ_0} be closed subspaces of X_{λ_0} such that $Z_{\lambda_0} \overset{p}{\subset} Y_{\lambda_0} \overset{p}{\subset} X_{\lambda_0}$ with $\dim(X_{\lambda_0}/Y_{\lambda_0}) = n < +\infty$ and $\dim(Y_{\lambda_0}/Z_{\lambda_0}) = m < +\infty$. Now consider $Y = (\oplus Y_\lambda)_V$ and $Z = (\oplus Z_\lambda)_V$ where

$$Y_\lambda = \begin{cases} X_\lambda & \text{if } \lambda \neq \lambda_0, \\ Y_{\lambda_0} & \text{if } \lambda = \lambda_0, \end{cases} \qquad Z_\lambda = \begin{cases} X_\lambda & \text{if } \lambda \neq \lambda_0, \\ Z_{\lambda_0} & \text{if } \lambda = \lambda_0. \end{cases} \tag{2.4}$$

Clearly Z and Y are subspaces of finite codimension in X. It is easy to see that $Z \overset{p}{\subset} Y \overset{p}{\subset} X$. Therefore $Z \overset{p}{\subset} X$ since X is a P-space.

Now we claim that $Z_{\lambda_0} \overset{p}{\subset} X_{\lambda_0}$. Let $x_{\lambda_0} \in X_{\lambda_0}$. Consider $x_0 = (0,\dots,x_{\lambda_0},0,\dots) \in X$. Then there exists $Z_0 = (z_\lambda^0) \in Z$ such that $\|x_0 - z_0\| = d(x_0,Z)$. Now we show that $z_{\lambda_0}^0$ is a best approximation of $x_{\lambda_0}^0$ from Z_{λ_0}. Consider $z = (0,\dots,0,z_{\lambda_0},0,\dots)$ where $z_{\lambda_0} \in Z_{\lambda_0}$. Clearly $z \in Z$. Now $\|x_{\lambda_0} - z_{\lambda_0}\| = \|x_0 - z\| \geq \|x_{\lambda_0} - z_{\lambda_0}^0\|$ which implies that $z_{\lambda_0}^0$ is a best approximation of x_{λ_0} from Z_{λ_0}. Thus $Z_{\lambda_0} \overset{p}{\subset} X_{\lambda_0}$. Hence X_{λ_0} is a P-space. Since λ_0 is arbitrary, the conclusion follows. □

Remark 2.13. Identifying $\mathcal{K}(c_0)$ as $(\oplus \ell_1)_{c_0}$ we see that as ℓ_1 is not a P-space (see [3, page 137]), $\mathcal{K}(c_0)$ is not a P-space.

We now prove that being $R(1)$ is invariant under c_0-direct sums. Using the same arguments presented in the converse part of the following proposition, one can show that this part holds for generalized direct sums also.

Proposition 2.14. *Let $X = (\oplus_\lambda X_\lambda)_{c_0}$. Then X is an $R(1)$-space if and only if each X_λ is an $R(1)$-space.*

Proof. Let $Y \subset X$ be a closed subspace of finite codimension and $Y^\perp \subset NA(X)$. Then there exists $f_1,\dots,f_n \in X^*$ such that $Y = \cap_{i=1}^n \ker f_i$.

As $f_i \in NA(X)$, only finitely many coordinates of f_i's are nonzero for each i. Thus $f_i = (f_{i,\lambda})_{\lambda\in\Lambda}$, $f_{i,\lambda} \neq 0$ for finitely many $\lambda \in \Lambda$.

It is easy to see that if a functional in X^* with only finitely many nonzero coordinates attains its norm, then each nonzero component functional attains its norm. Thus since $Y^\perp \subset NA(X)$, following the notation of Theorem 1.1 we see that for any λ, $Y_\lambda^\perp \subset NA(X_\lambda)$. We thus have Y_λ is proximinal in X_λ since each X_λ is an $R(1)$-space. Now by Theorem 1.1, Y is proximinal which implies that X is an $R(1)$-space.

To see the converse it is enough to consider the case $X = X_1 \oplus_\infty X_2$ where X is an $R(1)$-space. Now let $Y \subset X_1$ be such that $Y^\perp \subset NA(X_1)$ and $Y^\perp = \text{span}\{f_1,\dots,f_k\}$. It is easy to see that $Y' = \text{span}\{(f_1,0),\dots,(f_k,0)\} \subset NA(X)$. Thus the preannihilator of Y' is a proximinal subspace of X. Hence Y is proximinal in X_1. Therefore X_1 is an $R(1)$-space. □

The following corollary allows us to give more examples of P-spaces. In particular it shows that $(\oplus\mathcal{K}(\ell_2))_{c_0}$ is a P-space. As mentioned before it also gives another proof of Corollary 2.10. We omit its easy proof.

Corollary 2.15. *Let $X = (\oplus X_\lambda)_{c_0}$, where each X_λ is an $R(1)$-space and $NA(X_\lambda)$ is a linear space. Then X has the same properties. In particular X is a P-space.*

Remark 2.16. Proposition 2.14 does not hold for generalized direct sums. It is not true even for ℓ_1-direct sums over infinite index. The same example presented in the proof of Theorem 2.3 works in this case also.

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