

FIXED POINT CHARACTERIZATION OF LEFT AMENABLE LAU ALGEBRAS

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The present paper deals with the concept of left amenability for a wide range of Banach algebras known as Lau algebras. It gives a fixed point property characterizing left amenable Lau algebras \mathcal{A} in terms of left Banach \mathcal{A} -modules. It also offers an application of this result to some Lau algebras related to a locally compact group G , such as the Eymard-Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$, and the measure algebra $M(G)$. In particular, it presents some equivalent statements which characterize amenability of locally compact groups.

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1. Introduction. A *Lau algebra* is a complex Banach algebra \mathcal{A} which is the unique predual of a von-Neumann algebra \mathcal{M} and the identity element of \mathcal{M} is a multiplicative linear functional on \mathcal{A} . The subject of this large class of Banach algebras originated with a paper published in 1983 by Lau [3], in which he referred to them as “ F -algebras.” Later on, in his useful monograph, Pier [11] introduced the name “Lau algebra.”

As pointed out in [3], such an \mathcal{M} is not necessarily unique as a dual von-Neumann algebra of \mathcal{A} , although it is plainly unique as the dual space of \mathcal{A} . We will endow the dual space \mathcal{A}^* of \mathcal{A} with the structure of a fixed von-Neumann algebra whose identity element is a multiplicative linear functional on \mathcal{A} .

The main development of the theory of Lau algebras concerns some notions of amenability; however, the theory also deals with some other aspects of analysis on Banach algebras. See [2, 4, 5, 6, 7, 8, 9]. In the pioneering paper [3], Lau introduced and studied a concept of amenability for Lau algebras called left amenability. In the same paper, he obtained several properties characterizing left amenable Lau algebras.

In this paper, we establish a fixed point characterization of left amenable Lau algebras. We also give an application of this result to the following important Lau algebras related to a locally compact group G : the Eymard-Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$, and the measure algebra $M(G)$, as defined in [10].

2. Preliminaries. Let \mathcal{A} be a Lau algebra. By a *left Banach \mathcal{A} -module*, we mean a Banach space X equipped with a bounded bilinear map from $\mathcal{A} \times X$ into X , denoted by $(a, \xi) \mapsto a \cdot \xi$ ($a \in \mathcal{A}$, $\xi \in X$) such that

$$a \cdot (b \cdot \xi) = (ab) \cdot \xi \quad (a, b \in \mathcal{A}, \xi \in X). \quad (2.1)$$

A *right Banach \mathcal{A} -module* is defined similarly. A *two-sided Banach \mathcal{A} -module* is a left and right Banach \mathcal{A} -module such that $(a \cdot \xi) \cdot b = a \cdot (\xi \cdot b)$ for all $a, b \in \mathcal{A}$ and $\xi \in X$.

The dual space X^* of a left (resp., right) Banach \mathcal{A} -module X becomes a right (resp., left) Banach \mathcal{A} -module with

$$\langle \xi^* \cdot a, \xi \rangle = \langle \xi^*, a \cdot \xi \rangle \quad (\text{resp., } \langle a \cdot \xi^*, \xi \rangle = \langle \xi^*, \xi \cdot a \rangle) \quad (2.2)$$

for all $\xi \in X$, $\xi^* \in X^*$, and $a \in \mathcal{A}$.

Now, let X be a left Banach \mathcal{A} -module, and denote by $\mathcal{B}(X^{**})$ the Banach space of bounded linear operators on the second dual space X^{**} of X . By *weak* operator topology* on $\mathcal{B}(X^{**})$, we mean the locally convex topology of $\mathcal{B}(X^{**})$ determined by the family $\{q(\xi^{**}, \xi^*) : \xi^{**} \in X^{**}, \xi^* \in X^*\}$ of seminorms on $\mathcal{B}(X^{**})$, where

$$q(\xi^{**}, \xi^*)(T) = |\langle T\xi^{**}, \xi^* \rangle| \quad \forall T \in \mathcal{B}(X^{**}). \quad (2.3)$$

We denote by $\mathcal{P}(\mathcal{A}, X^{**})$ the closure of the set $\{\Lambda_a : a \in P_1(\mathcal{A})\}$ in the weak* operator topology of $\mathcal{B}(X^{**})$. Here, $P_1(\mathcal{A})$ is the set of all elements $a \in \mathcal{A}$ with norm one that induces positive functionals on the dual von-Neumann algebra \mathcal{A}^* , and $\Lambda_a \in \mathcal{B}(X^{**})$ is the operator of left action of $a \in P_1(\mathcal{A})$ on X^{**} ; that is,

$$\Lambda_a(\xi^{**}) = a \cdot \xi^{**} \quad \forall a \in P_1(\mathcal{A}), \xi^{**} \in X^{**}. \quad (2.4)$$

We remark that $P_1(\mathcal{A})$ with the multiplication of \mathcal{A} is a semigroup. This can be readily checked by using the interesting equality

$$P_1(\mathcal{A}) = \{a \in \mathcal{A} : \|a\| = \langle u, a \rangle = 1\}, \quad (2.5)$$

where u denotes the identity element of \mathcal{A}^* ; the latter equality follows at once from the fact that a bounded linear functional ϕ on a C^* -algebra with identity is positive if and only if $\|\phi\|$ is equal to the value of ϕ at the identity [12, Propositions 1.5.1 and 1.5.2]. In particular, the set $\{\Lambda_a : a \in P_1(\mathcal{A})\}$ is a subsemigroup of the semigroup $\mathcal{B}(X^{**})$ with the ordinary multiplication of linear operators, and as easily verified, so is its closure $\mathcal{P}(\mathcal{A}, X^{**})$ in the weak* operator topology of $\mathcal{B}(X^{**})$.

The second dual \mathcal{A}^{**} of \mathcal{A} is a Lau algebra with the first Arens product defined by the equations

$$\langle F \odot H, f \rangle = \langle F, Hf \rangle, \quad \langle Hf, a \rangle = \langle H, fa \rangle, \quad \langle fa, b \rangle = \langle f, ab \rangle \quad (2.6)$$

for all $F, H \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$; see [3, Proposition 3.2] (for $R = \mathcal{A}^*$ in the notation of [3]). Considering \mathcal{A} as a left Banach \mathcal{A} -module, we have the following lemma

LEMMA 2.1. *Let \mathcal{A} be a Lau algebra. Then, the mapping Φ defined by $\Phi(N)(F) = N \odot F$ for all $N \in P_1(\mathcal{A}^{**})$ and $F \in \mathcal{A}^{**}$ is a semigroup homomorphism of $P_1(\mathcal{A}^{**})$ onto $\mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$.*

PROOF. In view of [4, Lemma 2.1], the set of states in the predual of a von-Neumann algebra is weak* dense in the set of states in its dual space. In particular, $P_1(\mathcal{A})$ is weak* dense in $P_1(\mathcal{A}^{**})$. So, the bounded linear operator $\Phi(N)$ on \mathcal{A}^{**} lies in $\mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ for

all $N \in P_1(\mathcal{A}^{**})$; indeed, there is a net (b_β) in $P_1(\mathcal{A})$ converging to N in the weak* topology of \mathcal{A}^{**} , and hence for every $f \in \mathcal{A}$ and $F \in \mathcal{A}^{**}$,

$$\begin{aligned}\langle b_\beta \cdot F, f \rangle &= \langle b_\beta \odot F, f \rangle = \langle b_\beta, Ff \rangle \\ &\rightarrow \langle N, Ff \rangle = \langle N \odot F, f \rangle,\end{aligned}\tag{2.7}$$

which implies that the net $(\Lambda_{b_\beta}) \subseteq \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ converges to $\Phi(N)$ in the weak* operator topology of $\mathcal{B}(\mathcal{A}^{**})$. This shows that Φ is well defined. Since Φ is obviously a semigroup homomorphism, it remains to prove that Φ is onto. To that end, let $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$. Choose a net (a_γ) in $P_1(\mathcal{A})$ such that $\Lambda_{a_\gamma} \rightarrow \Lambda$ in the weak* operator topology, and let N be a weak* cluster point of (a_γ) in $P_1(\mathcal{A}^{**})$. Then, for each $f \in \mathcal{A}^*$ and $F \in \mathcal{A}^{**}$ we conclude from (2.4) and (2.6) that

$$\begin{aligned}\langle \Lambda F, f \rangle &= \lim_\delta \langle \Lambda_{a_\delta} F, f \rangle = \lim_\delta \langle a_\delta \cdot F, f \rangle \\ &= \lim_\delta \langle a_\delta \odot F, f \rangle = \lim_\delta \langle a_\delta, Ff \rangle \\ &= \langle N, Ff \rangle = \langle N \odot F, f \rangle,\end{aligned}\tag{2.8}$$

where (a_δ) is a subnet of (a_γ) converging to N in the weak* topology, that is, $\Lambda = \Phi(N)$ as required. \square

3. The main result. Let \mathcal{A} be a Lau algebra. A *derivation* from \mathcal{A} into the dual space X^* of a two-sided Banach \mathcal{A} -module X is a linear map D such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad \forall a, b \in \mathcal{A}.\tag{3.1}$$

The Lau algebra \mathcal{A} is called *left amenable* if for each two-sided Banach \mathcal{A} -module X with $a \cdot \xi = \xi$ ($a \in P_1(\mathcal{A})$, $\xi \in X$), every bounded derivation $D : \mathcal{A} \rightarrow X^*$ is inner; that is, there exists $\xi^* \in X^*$ with

$$D(a) = a \cdot \xi^* - \xi^* \cdot a \quad \forall a \in \mathcal{A}.\tag{3.2}$$

We now give the following fixed point characterization of left amenable Lau algebras. First, we recall that an element $M \in P_1(\mathcal{A}^{**})$ is called a *topological left invariant mean* on \mathcal{A}^* if $a \odot M = M$ for all $a \in P_1(\mathcal{A})$.

THEOREM 3.1. *Let \mathcal{A} be an arbitrary Lau algebra. Then, the following conditions are equivalent.*

- (a) \mathcal{A} is left amenable.
- (b) There exists $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$.
- (c) For each left Banach \mathcal{A} -module X , there exists $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ such that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$.

PROOF. (a) \Rightarrow (c). Suppose that \mathcal{A} is left amenable. Appealing to [3, Theorem 4.6], there exists a net (a_γ) in $P_1(\mathcal{A})$ such that

$$\|aa_\gamma - a_\gamma\| \rightarrow 0 \quad \forall a \in P_1(\mathcal{A}).\tag{3.3}$$

Now, recall that the operator algebra $\mathcal{B}(X^{**})$ can be identified with the dual space $(X^{**} \hat{\otimes} X^*)^*$ of the projective tensor product $X^{**} \hat{\otimes} X^*$ in a natural way; see, for example, [1, Corollary VIII.2.2]. In particular, the weak* operator topology of $\mathcal{B}(X^{**})$ coincides with the weak* topology of $(X^{**} \hat{\otimes} X^*)^*$ on bounded subsets of $\mathcal{B}(X^{**})$, and therefore $\mathcal{P}(\mathcal{A}, X^{**})$ is compact in the weak* operator topology of $\mathcal{B}(X^{**})$; indeed, as readily checked, $\|\Lambda\| \leq K$ for all $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$, where K is a constant satisfying

$$\|b \cdot \xi\| \leq K \|b\| \|\xi\| \quad \forall b \in \mathcal{A}, \xi \in X. \quad (3.4)$$

Since (Λ_{a_γ}) is contained in $\mathcal{P}(\mathcal{A}, X^{**})$, we may find $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ with $\|\Lambda\| \leq K$ and a subnet (a_δ) of (a_γ) such that $\Lambda_{a_\delta} \rightarrow \Lambda$ in the weak* operator topology of $\mathcal{B}(X^{**})$. For each $a \in P_1(\mathcal{A})$, we therefore have $\Lambda_a \Lambda_{a_\delta} \rightarrow \Lambda_a \Lambda$ in the weak* operator topology. Also, by (3.3) and (3.4) we have

$$\|\Lambda_a \Lambda_{a_\delta} - \Lambda_{a_\delta}\| \leq K \|aa_\delta - a_\delta\| \rightarrow 0. \quad (3.5)$$

It follows that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$.

(c) \Rightarrow (b). The implication is clear.

(b) \Rightarrow (a). Suppose that (b) holds and choose an element Λ of $\mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$. Using Lemma 2.1, there exists an element M of $P_1(\mathcal{A}^{**})$ such that

$$\Lambda F = M \odot F \quad \forall F \in \mathcal{A}^{**}. \quad (3.6)$$

We therefore have $a \odot (M \odot F) = M \odot F$ for all $F \in \mathcal{A}^{**}$. It follows that for each $N \in P_1(\mathcal{A}^{**})$, $M \odot N$ is a topological left invariant mean on \mathcal{A}^* . Now, (a) follows from the fact that left amenability of \mathcal{A} is equivalent to the existence of a topological left invariant mean on \mathcal{A}^* ; see [3, Theorem 4.1]. \square

As an application of this result, we present the following descriptions of amenable locally compact groups. First, we recall that a locally compact group G is called *amenable* if there is a *left invariant mean* on the dual $L^\infty(G)$ of $L^1(G)$; that is a functional $m \in P_1(L^\infty(G)^*)$ such that $m({}_x g) = m(g)$ for all $g \in L^\infty(G)$ and $x \in G$, where $({}_x g)(y) = g(xy)$ for all $y \in G$. Examples of amenable groups include all commutative groups and all compact groups; refer to Pier [10] for details.

COROLLARY 3.2. *Let G be a locally compact group with left Haar measure λ . Each of the following properties characterizes the amenability of G .*

(i) *There exists $\Lambda \in \mathcal{P}(L^1(G), L^\infty(G)^*)$ such that $\Lambda_f \Lambda = \Lambda$ for all $f \in L^1(G)$ with $f \geq 0$ and $\int_G f d\lambda = 1$.*

(ii) *For each left Banach $L^1(G)$ -module X , there exists $\Lambda \in \mathcal{P}(L^1(G), X^{**})$ such that $\Lambda_f \Lambda = \Lambda$ for all $f \in L^1(G)$ with $f \geq 0$ and $\int_G f d\lambda = 1$.*

PROOF. The identity element of $L^\infty(G)$ is the constant function 1_G . So, the result follows from Theorem 3.1 and the fact that the amenability of G is equivalent to the left amenability of $L^1(G)$; see [3, Theorem 4.1] and [10, Theorem 4.19]. \square

COROLLARY 3.3. *A locally compact group G is amenable if and only if any of the following properties holds.*

- (i) *There exists $\Lambda \in \mathcal{P}(M(G), M(G)^{**})$ such that $\Lambda_\mu \Lambda = \Lambda$ for all $\mu \in M(G)$ with $\mu \geq 0$ and $\mu(G) = 1$.*
- (ii) *For each left Banach $M(G)$ -module X , there exists $\Lambda \in \mathcal{P}(M(G), X^{**})$ such that $\Lambda_\mu \Lambda = \Lambda$ for all $\mu \in M(G)$ with $\mu \geq 0$ and $\mu(G) = 1$.*

PROOF. The Lau algebra $M(G)$ is left amenable if and only if G is amenable; see [3, Corollary 4.3]. So, the result follows from Theorem 3.1 together with that the identity element of the dual W^* algebras of $M(G)$ is the functional defined by $u(\mu) = \mu(G)$ for all $\mu \in M(G)$. \square

We round up this paper by giving another consequence of our main result. Recall that $A(G)$ is spanned by functions with compact support in $P(G)$, and $B(G)$ is spanned by $P(G)$, where $P(G)$ denotes the set of all continuous positive definite functions on G ; see Pier [10] for details.

COROLLARY 3.4. *Let G be a locally compact group with identity e . Then, for each left Banach $A(G)$ -module X , there exists $\Lambda \in \mathcal{P}(A(G), X^{**})$ such that $\Lambda_\varphi \Lambda = \Lambda$ for all $\varphi \in A(G) \cap P(G)$ with $\varphi(e) = 1$. A similar result holds for $B(G)$.*

PROOF. Recall from [3] that the identity element of the dual W^* algebras of $A(G)$ is the functional defined by $u(\varphi) = \varphi(e)$ for all $\varphi \in A(G)$. Since $A(G)$ is a commutative Lau algebra, the result follows from Theorem 3.1 together with the fact that any commutative Lau algebra is left amenable; see [3, Example 1]. The proof of the second assertion is similar. \square

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