FINITE ELEMENT LEAST-SQUARES METHODS FOR A COMPRESSIBLE STOKES SYSTEM

KEEHWAN KIM, SANGDONG KIM, and SANGSIK SHIN

Received 25 April 2002

The least-squares functional related to a *vorticity* variable or a *velocity flux* variable is considered for two-dimensional compressible Stokes equations. We show ellipticity and continuity in an appropriate product norm for each functional.

2000 Mathematics Subject Classification: 65F10, 65F30.

1. Introduction. Let Ω be a convex polygonal domain in $\mathbb{R}^2$. Consider the stationary compressible Stokes equations with *zero* boundary conditions for the *velocity* $\mathbf{u} = (u_1, u_2)^t$ and *pressure* p as follows:

$$\begin{aligned} -\mu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla p &= g \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the symbols Δ , ∇ , and $\nabla \cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta \mathbf{u}$ is the vector of components Δu_i); the number μ is a viscous constant; $\mathbf{f}$ is a given vector function; $\boldsymbol{\beta} = (U, V)^t$ is a given C^1 function. The system (1.1) may be obtained by linearizing the steady-state barotropic compressible viscous Navier-Stokes equations without an ambient flow (see [8, 9] for more detail). Since the continuity equation is of hyperbolic type containing a convective derivative of p , we further assume that the boundary condition for the pressure is given on the inlet of the boundary where the characteristic function $\boldsymbol{\beta}$ points into Ω , that is,

$$p = 0 \quad \text{on } \Gamma_{\text{in}}, \tag{1.2}$$

where $\Gamma_{\text{in}} = \{(x, y) \in \partial\Omega \mid \boldsymbol{\beta} \cdot \mathbf{n} < 0\}$ with the outward unit normal $\mathbf{n}$ to $\partial\Omega$. Hence the boundary $\partial\Omega$ consists of Γ_{in} and Γ_{out} where $\Gamma_{\text{out}} = \{(x, y) \mid \boldsymbol{\beta} \cdot \mathbf{n} \geq 0\}$. There was a study on a mixed finite element theory for a compressible Stokes system (see, e.g., [8]), but there are a few trials dealing with a compressible Stokes system like (1.1) using least-squares method. Some papers focused on a H^{-1} least-squares method (see, e.g., [6, 9]). Least-squares approach was developed for the incompressible Stokes and Navier-Stokes equations in [1, 2, 7]. The purpose of this paper is to apply the philosophy of first-order system least-squares (FOSLS) methodology developed in [5] to a compressible stationary Stokes system. We consider two basic first-order systems. The first one is induced by a *vorticity* variable, and the second one is induced by a *velocity flux* variable

which is further extended to the system's associated curl and trace equations. This extended system is not a system of first order but a mixture system of first- and second-order equations due to the continuity equation $\nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla p = g$. In order to provide ellipticity for each functional, we assume the H^1 and H^2 regularity assumptions for the compressible Stokes equations. As usual in FOSLS approach, we first show that the H^{-1} and L^2 FOSLS functional is elliptic in the product norm $\|\mathbf{w}\| + \|\mathbf{u}\|_1 + \|q\| + \|p\|_{0,\beta}$ for the functional involving vorticity variable and $\|\mathbf{U}\| + \|\mathbf{u}\|_1 + \|p\|$ for the functional involving flux variable. We also show that the extended functional related to *velocity flux* variable is elliptic in the product norm $\|\mathbf{U}\|_1 + \|\mathbf{u}\|_1 + \|p\|_{1,\beta}$. Then we provide the error estimates for using finite element methods. The outline of the paper is as follows. In [Section 2](#), we discuss least-squares system and other preliminaries. The continuity and ellipticity of least-squares functionals are discussed in [Section 3](#). These can be done by employing regularity estimates for (1.1). The finite element approximations are briefly discussed in [Section 4](#).

2. Least-squares system for compressible Stokes equations, and other preliminaries. For the development of least-squares theory, we will adopt the notation introduced in [5] and introduce the necessary definitions in this section. A new independent variable related to the 4-vector function of gradients of the displacement vectors, u_i , $i = 1, 2$ will be given. It will be convenient to view the original n -vector functions as column vectors and the new 4-vector functions as either block column vectors or matrices. The velocity variable $\mathbf{u} = (u_1, u_2)^t$ is a column vector with scalar components u_i , so that the gradient $\nabla \mathbf{u}^t$ is a matrix with columns ∇u_i . For a function $\mathbf{U}$ with 2-vector components U_i

$$\mathbf{U} = \nabla \mathbf{u}^t = (\mathbf{U}_1, \mathbf{U}_2) = (U_{ij})_{2 \times 2}, \quad (2.1)$$

which is a matrix with entries $U_{ij} = \partial u_j / \partial x_i$, $1 \leq i, j \leq 2$. Then we can define the *trace* operator tr as

$$\text{tr} \mathbf{U} = \sum_{i=1}^n U_{ii}. \quad (2.2)$$

Let, for $\mathbf{v} \in L^2(\Omega)^2$,

$$\begin{aligned} \nabla \times \mathbf{v} &:= \text{curl} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, & \nabla \cdot \mathbf{v} &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \\ \nabla^\perp \mathbf{v}^t &= (\nabla^\perp v_1, \nabla^\perp v_2) = \begin{pmatrix} \partial_y v_1 & \partial_y v_2 \\ -\partial_x v_1 & -\partial_x v_2 \end{pmatrix}, \\ \mathbf{n} \times \mathbf{v} &= -n_2 v_1 + n_1 v_2. \end{aligned} \quad (2.3)$$

Define the curl as

$$\nabla \times \mathbf{U} = (\nabla \times \mathbf{U}_1, \nabla \times \mathbf{U}_2), \quad (2.4)$$

and the divergence as

$$(\nabla \cdot \mathbf{U})^t = (\nabla \cdot \mathbf{U}_1, \nabla \cdot \mathbf{U}_2)^t. \quad (2.5)$$

We also define the tangential operator $\mathbf{n} \times$ componentwise

$$\mathbf{n} \times \mathbf{U} = (\mathbf{n} \times \mathbf{U}_1, \mathbf{n} \times \mathbf{U}_2). \quad (2.6)$$

The inner products and norms on the block column vector functions are defined in the natural componentwise way; for example,

$$\|\mathbf{U}\|^2 = \sum_{i=1}^2 \|\mathbf{U}_i\|^2 = \sum_{i,j=1}^2 \|U_{ij}\|^2. \quad (2.7)$$

We use standard notations and definitions for the Sobolev spaces $H^s(\Omega)^n$, associated inner products $(\cdot, \cdot)_s$, and respective norms $\|\cdot\|_s$, $s \geq 0$. When $s = 0$, $H^0(\Omega)^n$ is the usual $L^2(\Omega)^n$, in which case the norm and inner product will be denoted by $\|\cdot\|_0 = \|\cdot\|$ and $(\cdot, \cdot)$, respectively. The space $H_0^s(\Omega)$ is the set of functions in $H^s(\Omega)$ vanishing on the boundaries. From now on, we will omit the superscript n and Ω if the dependence of vector norms on dimension is clear by context. We use $H_0^{-1}(\Omega)$ to denote the dual spaces of $H_0^1(\Omega)$ with norm defined by

$$\|\phi\|_{-1} = \sup_{\psi \in H_0^1(\Omega)} \frac{(\phi, \psi)}{\|\psi\|_1}. \quad (2.8)$$

Define the product spaces $H_0^s(\Omega)^2$ and $L^2(\Omega)^2$ in usual way with standard product norms. Let

$$H(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}. \quad (2.9)$$

Define a space

$$Q_k(\Omega) = \{q \in L^2(\Omega) : (\|q\|_k^2 + \|\boldsymbol{\beta} \cdot \nabla q\|_k^2)^{1/2} < \infty\}, \quad (2.10)$$

where k is either 1 or 0, which is a Hilbert space with norm

$$\|q\|_{k, \boldsymbol{\beta}} = (\|q\|_k^2 + \|\boldsymbol{\beta} \cdot \nabla q\|_k^2)^{1/2}. \quad (2.11)$$

We frequently use the notation constant C_Ω to denote that it depends on Ω only, but it may be a different constant. If a constant depends on another variable, we specify it in each place. Throughout this paper, we assume the following regularity.

ASSUMPTION 1. Assume that μ and $\boldsymbol{\beta}$ are such that (1.1) has a unique solution which satisfies the following a priori estimate:

$$\|\nabla \mathbf{u}^t\|_k^2 + \|p\|_k^2 \leq C_0(\mu, \Omega) (\|-\mu \Delta \mathbf{u} + \nabla p\|_{k-1}^2 + \|\nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla p\|_k^2), \quad (2.12)$$

where k is either 0 or 1; $C_0 := C_0(\mu, \Omega)$ is a constant depending on μ , $\boldsymbol{\beta}$, and Ω . Note that one may find (2.12) for $k = 1$ in [10, Theorem 1.3] for $\boldsymbol{\beta} = (1, 0)^t$ and one may get (2.12)

for $k = 0$ by following the arguments in [10, Section 3]. In fact, using triangle inequality and the assumption (2.12), one may get the improved a priori estimates:

$$\|\nabla \mathbf{u}^t\|_k^2 + \|\mathbf{p}\|_{k,\beta}^2 \leq C_0(\mu, \Omega) (\|-\mu \Delta \mathbf{u} + \nabla \mathbf{p}\|_{k-1}^2 + \|\nabla \cdot \mathbf{u} + \beta \cdot \nabla \mathbf{p}\|_k^2), \quad (2.13)$$

where k is 1 or 0 and $C_0 := C_0(\mu, \Omega)$ is a constant depending on μ , β , and Ω .

2.1. Velocity-vorticity-pressure formulation. Note that

$$\nabla^\perp (\nabla \times \mathbf{u}) = -\Delta \mathbf{u} + \nabla (\nabla \cdot \mathbf{u}). \quad (2.14)$$

As in [4] for Stokes equations, introducing the vorticity variable $w = \nabla \times \mathbf{u}$, the first equation of the compressible Stokes equations (1.1) using the second equation of (1.1) is

$$\mu \nabla^\perp w - \mu \nabla \cdot \mathbf{q} + \nabla p = \mathbf{f}. \quad (2.15)$$

By setting $\mathbf{q} = \nabla \cdot \mathbf{u}$, the equivalent first-order system is now

$$\begin{aligned} w - \nabla \times \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{q} - \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mu \nabla^\perp w - \mu \nabla \mathbf{q} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{q} + \beta \cdot \nabla p &= g && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \\ p &= 0 && \text{on } \Gamma_{\text{in}}. \end{aligned} \quad (2.16)$$

2.2. Velocity-flux-pressure formulation. As in [5] for Stokes equations, introducing the velocity flux variable $\mathbf{U} = \nabla \mathbf{u}^t$, the compressible Stokes equations (1.1) may be written as the following equivalent first-order system:

$$\begin{aligned} \mathbf{U} - \nabla \mathbf{u}^t &= \mathbf{0} && \text{in } \Omega, \\ -\mu (\nabla \cdot \mathbf{U})^t + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \beta \cdot \nabla p &= g && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \\ p &= 0 && \text{on } \Gamma_{\text{in}}. \end{aligned} \quad (2.17)$$

We consider the following extended equivalent system for (2.17):

$$\begin{aligned} \mathbf{U} - \nabla \mathbf{u}^t &= \mathbf{0} && \text{in } \Omega, \\ -\mu (\nabla \cdot \mathbf{U})^t + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + \beta \cdot \nabla p &= g && \text{in } \Omega, \\ \nabla \times \mathbf{U} &= \mathbf{0} && \text{in } \Omega, \\ \nabla (\text{tr} \mathbf{U}) + \nabla (\beta \cdot \nabla p) &= \nabla g && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \\ \mathbf{n} \times \mathbf{U} &= \mathbf{0} && \text{on } \partial\Omega, \\ p &= 0 && \text{on } \Gamma_{\text{in}}. \end{aligned} \quad (2.18)$$

3. Least-squares functionals. The main objective in this section is to establish ellipticity and continuity of least-squares functionals based on (2.16), (2.17), and (2.18) in appropriate Sobolev spaces.

3.1. Velocity, vorticity, and pressure. The first-order least-squares functional corresponding to (2.16) is

$$G_0(w, \mathbf{u}, q, p; \mathbf{f}, g) = \|\mu \nabla^\perp w - \mu \nabla q + \nabla p - \mathbf{f}\|_{-1,0}^2 + \|q + \boldsymbol{\beta} \cdot \nabla p - g\|^2 + \|w - \nabla \times \mathbf{u}\|^2 + \|q - \nabla \cdot \mathbf{u}\|^2. \quad (3.1)$$

Define

$$M_0(w, \mathbf{u}, q, p) = \|w\|^2 + \|\mathbf{u}\|_1^2 + \|q\|^2 + \|p\|_{0,\boldsymbol{\beta}}^2, \quad (3.2)$$

and let

$$\mathcal{V}_0 = L^2(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega) \times Q_0(\Omega). \quad (3.3)$$

The FOSLS variational problem for the compressible Stokes equations corresponding to (2.16) is to minimize the quadratic functional G_0 over $\mathcal{V}_0$: find $(w, \mathbf{u}, q, p) \in \mathcal{V}_0$ such that

$$G_0(w, \mathbf{u}, q, p; \mathbf{f}, g) = \inf_{(z, \mathbf{v}, r, s) \in \mathcal{V}_0} G_0(z, \mathbf{v}, r, s; \mathbf{f}, g). \quad (3.4)$$

THEOREM 3.1. *Under the assumption (2.12), there are two positive constants c and C , dependent on δ and Ω , such that for all $(w, \mathbf{u}, q, p) \in \mathcal{V}_0$,*

$$cM_0(w, \mathbf{u}, q, p) \leq G_0(w, \mathbf{u}, q, p; \mathbf{0}, 0) \leq CM_0(w, \mathbf{u}, q, p). \quad (3.5)$$

PROOF. Upper bound in (3.5) is a simple consequence of the triangle inequality and Cauchy-Schwarz inequality. For any $(w, \mathbf{u}, q, p) \in \mathcal{V}_0$, using (2.13), triangle inequality, and $(\cdot)$, we have

$$\begin{aligned} \|\nabla \mathbf{u}^t\|^2 + \|p\|_{0,\boldsymbol{\beta}}^2 &\leq C_0 (\|-\mu \Delta \mathbf{u} + \nabla p\|_{-1,0}^2 + \|\nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla p\|^2) \\ &\leq C_0 (\|\mu \nabla^\perp w - \mu \nabla q + \nabla p\|_{-1,0}^2 + \mu^2 \|\nabla^\perp (w - \nabla \times \mathbf{u})\|_{-1,0}^2 \\ &\quad + \mu^2 \|\nabla (\nabla \cdot \mathbf{u} - q)\|_{-1,0}^2 + \|q + \boldsymbol{\beta} \cdot \nabla p\|^2) \\ &\leq \hat{C}_0 G_1(\mathbf{U}, \mathbf{u}, p; \mathbf{0}, 0), \end{aligned} \quad (3.6)$$

where $\hat{C}_0$ is a constant that depends on μ , $\boldsymbol{\beta}$, and Ω . Using (3.6), we have

$$(w, w) = (w - \nabla \times \mathbf{u}, w) + (\nabla \times \mathbf{u}, w) \leq CG_0^{1/2}(w, \mathbf{u}, q, p) \|w\|, \quad (3.7)$$

where C is a constant depending on Ω and the Poincare constant. Now, cancelling $\|w\|$ on both sides and squaring the remainder, we have

$$\|w\|^2 \leq CG_0(w, \mathbf{u}, q, p; \mathbf{0}, 0), \quad (3.8)$$

where C is a constant depending on Ω and the Poincare constant. Now, using (3.6), we have

$$\begin{aligned}(q, q) &= (q - \nabla \cdot \mathbf{u}, q) + (\nabla \cdot \mathbf{u}, q) \\ &\leq \|q - \nabla \cdot \mathbf{u}\| \|q\| + \|\nabla \cdot \mathbf{u}\| \|q\| \\ &\leq CG_0^{1/2}(w, \mathbf{u}, q, p) \|q\|,\end{aligned}\tag{3.9}$$

where C is a constant depending on Ω . Cancelling $\|q\|$ on both sides and squaring the remainder, we have

$$\|q\| \leq CG_0(w, \mathbf{u}, q, p).\tag{3.10}$$

Finally, combining (3.6), (3.8), and (3.10) yields the lower bound. This completes the proof. $\square$

3.2. Velocity, flux, and pressure. The first-order least-squares functional corresponding to (2.17) is

$$\begin{aligned}G_1(\mathbf{U}, \mathbf{u}, p; \mathbf{f}, g) &= \|-\mu(\nabla \cdot \mathbf{U})^t + \nabla p - \mathbf{f}\|_{-1,0}^2 \\ &\quad + \|\nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla p - g\|^2 + \|\mathbf{U} - \nabla \mathbf{u}^t\|^2.\end{aligned}\tag{3.11}$$

The extended least-squares functional corresponding to (2.18) is

$$\begin{aligned}G_3(\mathbf{U}, \mathbf{u}, q, p; \mathbf{f}, g) &= \|\mathbf{U} - \nabla \mathbf{u}^t\|^2 + \|-\mu(\nabla \cdot \mathbf{U})^t + \nabla p - \mathbf{f}\|^2 + \|\nabla \times \mathbf{U}\|^2 \\ &\quad + \|\nabla \cdot \mathbf{u} + \boldsymbol{\beta} \cdot \nabla - g\|^2 + \|\nabla \operatorname{tr} \mathbf{U} + \nabla(\boldsymbol{\beta} \cdot \nabla p)\|^2.\end{aligned}\tag{3.12}$$

Define

$$\begin{aligned}M_1(\mathbf{U}, \mathbf{u}, p) &= \|\mathbf{U}\|^2 + \|\mathbf{u}\|_1^2 + \|p\|_{0,\boldsymbol{\beta}}^2, \\ M_2(\mathbf{U}, \mathbf{u}, q, p) &= \|\mathbf{U}\|_1^2 + \|\mathbf{u}\|_1^2 + \|p\|_{1,\boldsymbol{\beta}}^2.\end{aligned}\tag{3.13}$$

Let

$$\mathbf{V}_0 = \{\mathbf{U} \in H^1(\Omega)^4 : \mathbf{n} \times \mathbf{U} = \mathbf{0} \text{ on } \partial\Omega\}.\tag{3.14}$$

Define

$$\begin{aligned}\mathcal{V}_1 &= L^2(\Omega)^4 \times H_0^1(\Omega)^2 \times Q_0(\Omega), \\ \mathcal{V}_2 &= \mathbf{V}_0 \times H_0^1(\Omega)^2 \times Q_1(\Omega).\end{aligned}\tag{3.15}$$

The least-squares variational problem for the compressible Stokes equations corresponding to (2.17) or (2.18) is to minimize the quadratic functional G_i over $\mathcal{V}_i$: find $(\mathbf{U}, \mathbf{u}, p) \in \mathcal{V}_i$ such that

$$G_i(\mathbf{U}, \mathbf{u}, p; \mathbf{f}, g) = \inf_{(\mathbf{V}, \mathbf{v}, r) \in \mathcal{V}_i} G_i(\mathbf{V}, \mathbf{v}, r; \mathbf{f}, g) \quad \text{for } i = 1, 2.\tag{3.16}$$

THEOREM 3.2. *Under the assumption (2.12), there are two positive constants c and C , dependent on μ , β , and Ω , such that for all $(\mathbf{U}, \mathbf{u}, p) \in \mathcal{V}_1$,*

$$cM_1(\mathbf{U}, \mathbf{u}, p) \leq G_1(\mathbf{U}, \mathbf{u}, p; \mathbf{0}, 0) \leq CM_1(\mathbf{U}, \mathbf{u}, p). \quad (3.17)$$

PROOF. Upper bound in (3.17) is a simple consequence of the triangle inequality and Cauchy-Schwarz inequality. To limit arguments, it is enough to show that lower bound in (3.17) holds for $\tilde{\mathcal{V}} = H(\operatorname{div}; \Omega)^2 \times H_0^1(\Omega)^2 \times Q(\Omega)$. Using (2.12) and triangle inequality, we have

$$\begin{aligned} & \|\nabla \mathbf{u}^t\|^2 + \|p\|_{0,\beta}^2 \\ & \leq C_0 (\|-\mu \Delta \mathbf{u} + \nabla p\|_{-1,0}^2 + \|\nabla \cdot \mathbf{u} + \beta \cdot \nabla p\|^2) \\ & \leq C_0 (\|-\mu(\nabla \cdot \mathbf{U})^t + \nabla p\|_{-1,0}^2 + \mu^2 \|\nabla \cdot (\mathbf{U} - \nabla \mathbf{u}^t)\|^2_{-1,0} + \|\nabla \cdot \mathbf{u} + \beta \cdot \nabla p\|^2) \\ & \leq \hat{C}_0 G_1(\mathbf{U}, \mathbf{u}, p; \mathbf{0}, 0), \end{aligned} \quad (3.18)$$

where $\hat{C}_0$ is a constant that depends on μ and Ω . Note that

$$(\mathbf{U}, \mathbf{U}) = (\mathbf{U} - \nabla \mathbf{u}^t, \mathbf{U}) + (\nabla \mathbf{u}^t, \mathbf{U}) \leq C(\|\mathbf{U} - \nabla \mathbf{u}^t\| \|\mathbf{U}\| + \|\mathbf{u}\|_1 \|\mathbf{U}\|), \quad (3.19)$$

where C is a constant depending on Ω . Now cancelling $\|\mathbf{U}\|$ on both sides, squaring the remainder, and using (3.19), we have

$$\|\mathbf{U}\|^2 \leq CG_1(\mathbf{U}, \mathbf{u}, p; \mathbf{0}, 0), \quad (3.20)$$

where C is a constant depending on μ , β , and Ω . Finally, combining (3.19) and (3.20) yields the lower bound. This completes the proof. $\square$

The following lemma is basically proved in [5, Lemma 3.2].

LEMMA 3.3. *Let $\phi = (\phi_1, \phi_2)^t$ and $\mathbf{q} = (q_1, q_2)^t$; if each $q_i \in H_0^1(\Omega) \cap H^2(\Omega)$ and each $\phi_i \in H^1(\Omega)$ is such that $\Delta \phi_i \in L^2(\Omega)$ and $\mathbf{n} \cdot \nabla \phi_i = 0$ on $\partial\Omega$, then*

$$|\nabla \cdot \mathbf{q} + \beta \cdot \nabla p|_1^2 \leq C_\Omega (|\nabla \cdot \mathbf{q} + \operatorname{tr} \nabla^\perp \phi^t + \beta \cdot \nabla p|_1^2 + \|\Delta \phi\|^2). \quad (3.21)$$

PROOF. Note that $\operatorname{tr}(\nabla^\perp \phi_1, \nabla^\perp \phi_2) = -\nabla \times \phi$,

$$\begin{aligned} |\nabla \cdot \mathbf{q} + \beta \cdot \nabla p|_1^2 & \leq 2(|\nabla \cdot \mathbf{q} - \nabla \times \phi + \beta \cdot \nabla p|_1^2 + |\nabla \times \phi|_1^2) \\ & \leq C(|\nabla \cdot \mathbf{q} + \operatorname{tr} \nabla^\perp \phi^t + \beta \cdot \nabla p|_1^2 + |\phi|_2) \\ & \leq C(|\nabla \cdot \mathbf{q} + \operatorname{tr} \nabla^\perp \phi^t + \beta \cdot \nabla p|_1^2 + \|\Delta \phi\|), \end{aligned} \quad (3.22)$$

where the constant C depends on Ω . $\square$

Due to the above lemma, one may get the following theorem.

THEOREM 3.4. *Under the assumption of (2.12), there are two positive constants c and C dependent on μ , β , and Ω such that for all $(\mathbf{U}, \mathbf{u}, p) \in \mathcal{V}_2$,*

$$cM_2(\mathbf{U}, \mathbf{u}, q, p) \leq G_2(\mathbf{U}, \mathbf{u}, q, p; \mathbf{0}, 0) \leq CM_2(\mathbf{U}, \mathbf{u}, q, p). \quad (3.23)$$

The proof of Theorem 3.4 comes immediately by following techniques similar to those in [5].

4. Finite element approximations. In this section, we provide the finite element approximation of the minimization of the least-squares functionals G_0 only. Note that an obvious modification in this section also provides the finite element error analysis for the least-squares functionals G_1 and G_2 . Let $T : H_0^{-1}(\Omega)^2 \rightarrow H_0^1(\Omega)^2$ be the solution operator ($\mathbf{u} = T\mathbf{f}$) for the following elliptic boundary value problem with zero boundary condition $-\Delta \mathbf{u} + \mathbf{u} = \mathbf{f}$ in Ω . It is well known that (see [3, Lemma 2.1])

$$(\mathbf{f}, T\mathbf{f}) = \|\mathbf{f}\|_{-1}^2 = \sup_{\boldsymbol{\phi} \in H_0^1(\Omega)^2} \frac{(\mathbf{f}, \boldsymbol{\phi})^2}{\|\boldsymbol{\phi}\|_1^2} \quad \forall \mathbf{f} \in H_0^{-1}(\Omega)^2. \quad (4.1)$$

Let $\mathcal{T}_h$ be a family of triangulations of Ω by standard finite element subdivisions of Ω into quasi-uniform triangles with $h = \max\{\text{diam}(K) : K \in \mathcal{T}_h\}$.

Let $\mathcal{V}_{0,h}$ be a finite-dimensional subspace of $\mathcal{V}_0$ with an approximation property such that for $(w, \mathbf{u}, q, p) \in \mathcal{V}_0$, there exists positive integers $l, m, n \geq 1$ and $s \geq 1$ satisfying

$$\begin{aligned} \inf_{w_h \in \mathcal{U}_h} \{ \|w - w_h\| + h \|w - w_h\|_1 \} &\leq Ch^r \|w\|_r, \\ \inf_{\mathbf{u}_h \in \mathcal{F}_{0,h}} \{ \|\mathbf{u} - \mathbf{u}_h\| + h \|\mathbf{u} - \mathbf{u}_h\|_1 \} &\leq Ch^{s+1} \|\mathbf{u}\|_{s+1}, \\ \inf_{q_h \in \mathcal{U}_h} \{ \|q - q_h\| + h \|q - q_h\|_1 \} &\leq Ch^r \|q\|_r, \\ \inf_{p_h \in \mathcal{W}_h} \{ \|p - p_h\| + h \|p - p_h\|_1 \} &\leq Ch^{k+1} \|p\|_{k+1}, \end{aligned} \quad (4.2)$$

where C is a positive integer. Then the finite element approximation of (3.4) is to find $(w_h, \mathbf{u}_h, q_h, p_h) \in \mathcal{V}_{0,h}$ which satisfies

$$G_0(w_h, \mathbf{u}_h, q_h, p_h; \mathbf{f}, g) = \inf_{(z_h, \mathbf{v}_h, r_h, s_h) \in \mathcal{V}_{0,h}} G_0(z_h, \mathbf{v}_h, r_h, s_h; \mathbf{f}, g). \quad (4.3)$$

From (4.1), we have

$$\begin{aligned} G_0(w, \mathbf{u}, q, p; \mathbf{0}, 0) &= (T(\mu \nabla^\perp w - \mu \nabla q + \nabla p), \mu \nabla^\perp w - \mu \nabla q + \nabla p) \\ &\quad + (q + \beta \cdot \nabla p, q + \beta \cdot \nabla p) + (q - \nabla \cdot \mathbf{u}, q - \nabla \cdot \mathbf{u}) \\ &\quad + (w - \nabla \times \mathbf{u}, w - \nabla \times \mathbf{u}). \end{aligned} \quad (4.4)$$

THEOREM 4.1. Suppose that the assumption in [Theorem 3.1](#) holds. Assume that $(w, \mathbf{u}, q, p) \in \mathcal{V}_0$ is the solution of the minimization problem for G_1 in (3.4) and $(w_h, \mathbf{u}_h, q_h, p_h)$ is the unique minimizer of G_0 over $\mathcal{V}_{0,h}$. Then

$$\begin{aligned} & \|w - w_h\|^2 + \|\mathbf{u} - \mathbf{u}_h\|_1^2 + \|q - q_h\|^2 + \|p - p_h\|_{0,\beta}^2 \\ & \leq C \inf_{(z_h, \mathbf{v}_h, r_h, s_h) \in \mathcal{V}_{0,h}} (\|w - z_h\|^2 + \|\mathbf{u} - \mathbf{v}_h\|_1^2 + \|q - r_h\|^2 + \|p - s_h\|_{0,\beta}^2). \end{aligned} \quad (4.5)$$

PROOF. For convenience, let

$$\begin{aligned} [w, \mathbf{u}, q, p; z, \mathbf{v}, r, s] &= (T(\mu \nabla^\perp w - \mu \nabla q + \nabla p), \mu \nabla^\perp z - \mu \nabla r + \nabla s) \\ &\quad + (q + \beta \cdot \nabla p, r + \beta \cdot \nabla s) + (w - \nabla \times \mathbf{u}, z - \nabla \times \mathbf{v}) \\ &\quad + (q - \nabla \cdot \mathbf{u}, r - \nabla \cdot \mathbf{s}). \end{aligned} \quad (4.6)$$

Then, using (4.1), [Theorem 3.1](#), the orthogonality of the error $(w - w_h, \mathbf{u} - \mathbf{u}_h, q - q_h, p - p_h)$ to $\mathcal{V}_{0,h}$, with respect to the above inner product, and the Schwarz inequality, we have the conclusion. $\square$

From this theorem and approximate property of $\mathcal{V}_{0,h}$, we have

$$\begin{aligned} & \|w - w_h\|^2 + \|\mathbf{u} - \mathbf{u}_h\|_1^2 + \|q - q_h\|^2 + \|p - p_h\|_{0,\beta}^2 \\ & \leq C(h^{2l}\|w\|_l^2 + h^{2m}\|\mathbf{u}\|_{m+1}^2 + h^{2n}\|q\|_n^2 + h^{2s}\|p\|_{s+1}^2), \end{aligned} \quad (4.7)$$

where $(w, \mathbf{u}, q, p) \in (H^l(\Omega) \times H_0^m(\Omega)^2 \times H^{n+1}(\Omega)^2 \times H^s(\Omega)) \cap \mathcal{V}_0$ is the solution of the minimization problem for G_0 in (3.4) and $(w_h, \mathbf{u}_h, q_h, p_h)$ is the unique minimizer of G_0 over $\mathcal{V}_{0,h}$.

ACKNOWLEDGMENT. This work was supported by the Korea Research Foundation Grant KRF-2002-070-C00014.

REFERENCES

- [1] P. B. Bochev, Z. Cai, T. A. Manteuffel, and S. F. McCormick, *Analysis of velocity-flux first-order system least-squares principles for the Navier-Stokes equations. I*, SIAM J. Numer. Anal. **35** (1998), no. 3, 990–1009.
- [2] P. B. Bochev and M. D. Gunzburger, *Analysis of least squares finite element methods for the Stokes equations*, Math. Comp. **63** (1994), no. 208, 479–506.
- [3] J. H. Bramble, R. D. Lazarov, and J. E. Pasciak, *A least-squares approach based on a discrete minus one inner product for first order systems*, Math. Comp. **66** (1997), no. 219, 935–955.
- [4] Z. Cai, T. A. Manteuffel, and S. F. McCormick, *First-order system least squares for velocity-vorticity-pressure form of the Stokes equations, with application to linear elasticity*, Electron. Trans. Numer. Anal. **3** (1995), 150–159.
- [5] ———, *First-order system least squares for the Stokes equations, with application to linear elasticity*, SIAM J. Numer. Anal. **34** (1997), no. 5, 1727–1741.
- [6] Z. Cai and X. Ye, *A least-squares finite element approximation for the compressible Stokes equations*, Numer. Methods Partial Differential Equations **16** (2000), no. 1, 62–70.
- [7] B.-N. Jiang and C. L. Chang, *Least-squares finite elements for the Stokes problem*, Comput. Methods Appl. Mech. Engrg. **78** (1990), no. 3, 297–311.
- [8] R. B. Kellogg and B. Liu, *A finite element method for the compressible Stokes equations*, SIAM J. Numer. Anal. **33** (1996), no. 2, 780–788.

- [9] J. R. Kweon and R. B. Kellogg, *Compressible Navier-Stokes equations in a bounded domain with inflow boundary condition*, SIAM J. Math. Anal. **28** (1997), no. 1, 94–108.
- [10] ———, *Compressible Stokes problem on nonconvex polygonal domains*, J. Differential Equations **176** (2001), no. 1, 290–314.

Keehwan Kim: Department of Mathematics, Yeungnam University, Kyongsan 712-749, Korea
E-mail address: khkim@yu.ac.kr

Sangdong Kim: Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
E-mail address: skim@knu.ac.kr

Sangsik Shin: Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
E-mail address: sshin@wmail.knu.ac.kr