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GER-TYPE AND HYERS-ULAM STABILITIES FOR THE FIRST-ORDER LINEAR DIFFERENTIAL OPERATORS OF ENTIRE FUNCTIONS

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Dedicated to Professor Seiji Watanabe on his 60th birthday (Kanreki).

Let h be an entire function and T_h a differential operator defined by $T_h f = f' + hf$. We show that T_h has the Hyers-Ulam stability if and only if h is a nonzero constant. We also consider Ger-type stability problem for $|1 - f'/hf| \le \varepsilon$.

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1. Introduction. The first result, which we now call the Hyers-Ulam stability (HUS), is due to Hyers [4] who gave an answer to a question posed by Ulam (cf. [11, Chapter VI] and [12]) in 1940 concerning the stability of homomorphisms: for what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?

An answer to the above problem has been given as follows. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$, the set of all real numbers, for each fixed $x \in E_1$. If there exist $\theta \ge 0$ and $p \in \mathbb{R} \setminus \{1\}$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \theta\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E_1$, then there is a unique linear mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \le 2\theta\|x\|^p/|2-2^p|$ for every $x \in E_1$. Hyers [4] obtained the result for $p = 0$. Then Rassias [7] generalized the above result of Hyers to the case where $0 \le p < 1$, while the proof given in [7] also works for $p < 0$. Gajda [2] solved the problem for $1 < p$ and also gave an example that a similar result does not hold for $p = 1$ (cf. [8]).

In connection with the stability of exponential functions, Alsina and Ger [1] remarked that the differential equation $y' = y$ has the HUS. More explicitly, suppose I is an open interval, $\varepsilon > 0$, and $f : I \to \mathbb{R}$ is a differentiable function such that $|f'(t) - f(t)| \le \varepsilon$ for all $t \in I$. Then, there is a differentiable function $g : I \to \mathbb{R}$ such that $g' = g$ and $|f(t) - g(t)| \le 3\varepsilon$ for all $t \in I$. The third and first authors of this paper along with Miyajima [10] considered the Banach-space-valued differential equation $y' = \lambda y$, where λ is a complex constant. Then they proved the HUS of $y' = \lambda y$ under the condition that $\operatorname{Re}\lambda \ne 0$. Though, they treated the result as the stability of the operator $D - I_d$, where D denotes the ordinary differential operator and I_d the identity. Some stability results of other differential equations (or operators) are also known (cf. [5, 6, 9]).

Taking the group structure of $\mathbb{C}\setminus\{0\}$ into account, Ger and Šemrl [3] considered the inequality

$$\left|\frac{f(x+y)}{f(x)f(y)}-1\right| \le \theta \quad (x,y \in S) \tag{1.2}$$

for a mapping $f : S \to \mathbb{C}\setminus\{0\}$, where $(S,+)$ is a semigroup and $\mathbb{C}$ is the set of all complex numbers. If $0 \le \theta < 1$ and if $(S,+)$ is a cancellative abelian semigroup, then they proved that there is a unique function $g : S \to \mathbb{C}\setminus\{0\}$ such that $g(x+y) = g(x)g(y)$ for all $x,y \in S$ and that

$$\max\left\{\left|\frac{f(x)}{g(x)}-1\right|,\left|\frac{g(x)}{f(x)}-1\right|\right\} \le \sqrt{1+\frac{1}{(1-\theta)^2}-2\sqrt{\frac{1+\theta}{1-\theta}}} \tag{1.3}$$

for all $x \in S$. The stability phenomena of this kind is called Ger-type stability.

Throughout this paper, $H(\mathbb{C})$ stands for the set of all entire functions. Let $h \in H(\mathbb{C})$ and $T_h : H(\mathbb{C}) \to H(\mathbb{C})$ be a linear differential operator defined by

$$T_h f(z) = f'(z)+h(z)f(z) \quad (f \in H(\mathbb{C}),\ z \in \mathbb{C}). \tag{1.4}$$

Definition 1.1. The operator T_h is said to have the HUS if and only if there exists a constant $K \ge 0$ with the following property: to each $\varepsilon \ge 0$ and $f,g \in H(\mathbb{C})$ satisfying $\sup_{z\in\mathbb{C}} |T_h f(z) - g(z)| \le \varepsilon$, there exists an $f_0 \in H(\mathbb{C})$ such that $T_h f_0 = g$ and $\sup_{z\in\mathbb{C}} |f(z) - f_0(z)| \le K\varepsilon$. Such K is called an HUS constant for T_h. If, in addition, the minimum of all such K's exists, then it is called *the* HUS constant for T_h.

In this paper, we first consider the HUS of the differential operator T_h. Then we show that T_h has the HUS if and only if $h \in H(\mathbb{C})$ is a nonzero constant function. Moreover, we give the HUS constant for T_h. Finally, we consider the Ger-type stability problem of the differential equation $y' = \lambda y$. To be more explicit, suppose $\varepsilon \ge 0$ and $f \in H(\mathbb{C})$ satisfies

$$\sup_{z\in\mathbb{C}}\left|\frac{f'(z)}{\lambda f(z)}-1\right| \le \varepsilon. \tag{1.5}$$

Does there exist $K \ge 0$ such that

$$\sup_{z\in\mathbb{C}}\left|\frac{f(z)}{ce^{\lambda t}}-1\right| \le K\varepsilon \quad \text{or} \quad \sup_{z\in\mathbb{C}}\left|\frac{ce^{\lambda t}}{f(z)}-1\right| \le K\varepsilon \tag{1.6}$$

holds for some $c \in \mathbb{C}\setminus\{0\}$? To this problem, we give a negative answer: the Ger-type stability does not hold in general. Moreover, we show that the solution $f \in H(\mathbb{C})$ to the differential equation $y' = \lambda y$ is only the function which satisfies both (1.5) and (1.6).

2. The HUS for T_h. For simplicity, we write $\int_0^z f(\zeta)d\zeta$ for $\int_0^1 f(zt)z\,dt$, where $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$. We associate to each $h \in H(\mathbb{C})$ a function $\tilde{h}$ defined by

$$\tilde{h}(z) = \exp\int_0^z h(\zeta)d\zeta \quad (z \in \mathbb{C}). \tag{2.1}$$

Let $h \in H(\mathbb{C})$. Throughout this section, $T_h : H(\mathbb{C}) \to H(\mathbb{C})$ denotes a linear differential operator defined by (1.4). Suppose $f, g \in H(\mathbb{C})$. Then note that $T_h f = g$ if and only if f is of the form

$$f(z) = \frac{1}{\tilde{h}(z)}\left\{f(0) + \int_0^z g(\zeta)\tilde{h}(\zeta)d\zeta\right\} \quad (z \in \mathbb{C}). \tag{2.2}$$

Lemma 2.1. *Suppose $h \in H(\mathbb{C})$ is not a constant function, $f \in H(\mathbb{C})$, and*

$$0 < \sup_{z\in\mathbb{C}} |T_h f(z)| < \infty. \tag{2.3}$$

Then

$$\sup_{z\in\mathbb{C}} \left| f(z) - \frac{c}{\tilde{h}(z)} \right| = \infty \tag{2.4}$$

for every $c \in \mathbb{C}$.

Proof. By hypothesis, $T_h f$ is a bounded entire function, and so $T_h f$ must be constant, say $c_0 \in \mathbb{C} \setminus \{0\}$ by Liouville's theorem. Hence, by (2.2), f is of the form

$$f(z) = \frac{1}{\tilde{h}(z)}\left\{f(0) + c_0\int_0^z \tilde{h}(\zeta)d\zeta\right\} \quad (z \in \mathbb{C}). \tag{2.5}$$

Suppose $\sup_{z\in\mathbb{C}} |f(z) - c_1/\tilde{h}(z)| < \infty$ for some $c_1 \in \mathbb{C}$. Another application of Liouville's theorem yields the existence of a constant $c_2 \in \mathbb{C}$ such that $c_2 = f - c_1/\tilde{h}$, and therefore (2.5) gives

$$c_2\tilde{h}(z) = f(0) - c_1 + c_0\int_0^z \tilde{h}(\zeta)d\zeta \quad (z \in \mathbb{C}). \tag{2.6}$$

By differentiating both sides of (2.6) with respect to z, we obtain

$$c_2 h\tilde{h} = c_0\tilde{h}, \tag{2.7}$$

and hence

$$c_2 h = c_0. \tag{2.8}$$

Since h is not constant, this implies that $c_2 = 0$. Thus, $f = c_1/\tilde{h}$, and hence $T_h f = 0$ (see (2.2)), which contradicts $0 < \sup_{z\in\mathbb{C}} |T_h f(z)|$. □

Theorem 2.2. *If $h \in H(\mathbb{C})$, then each of the following statements implies the other:*
(a) *h is a nonzero constant function,*
(b) *T_h has the HUS.*

Proof. (a)⇒(b). Suppose h is a nonzero constant function, say $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $\tilde{h}(z) = e^{\lambda z}$ for $z \in \mathbb{C}$. Suppose $\varepsilon \ge 0$ and $f, g \in H(\mathbb{C})$ satisfy $\sup_{z\in\mathbb{C}} |T_h f(z) - g(z)| \le \varepsilon$. Then there exists a $c_0 \in \mathbb{C}$ such that $T_h f - g = c_0$ by Liouville's theorem. Put

$$u(z) = e^{-\lambda z}\left\{\int_0^z g(\zeta)e^{\lambda\zeta}d\zeta\right\} \quad (z \in \mathbb{C}). \tag{2.9}$$

Then $T_h u = g$, and so $T_h(f-u) = c_0$, $|c_0| \le \varepsilon$. Hence, by (2.2), f is of the form

$$\begin{aligned} f(z) &= u(z) + \frac{1}{\tilde{h}(z)}\left\{f(0) - u(0) + c_0 \int_0^z \tilde{h}(\zeta)d\zeta\right\} \\ &= \frac{c_0}{\lambda} + u(z) + \left(f(0) - u(0) - \frac{c_0}{\lambda}\right)e^{-\lambda z} \end{aligned} \tag{2.10}$$

for all $z \in \mathbb{C}$. Put

$$f_0(z) = u(z) + \left(f(0) - u(0) - \frac{c_0}{\lambda}\right)e^{-\lambda z} \quad (z \in \mathbb{C}), \tag{2.11}$$

then $T_h f_0 = g$ and

$$|f(z) - f_0(z)| = \left|\frac{c_0}{\lambda}\right| \le \frac{\varepsilon}{|\lambda|} \tag{2.12}$$

for every $z \in \mathbb{C}$ so that T_h has the HUS with an HUS constant $1/|\lambda|$.

(b)⇒(a). Put

$$f_1(z) = \frac{1}{\tilde{h}(z)} \int_0^z \tilde{h}(\zeta)d\zeta \quad (z \in \mathbb{C}). \tag{2.13}$$

Then we obtain $T_h f_1 = 1$. Let $K < \infty$ be an HUS constant for T_h. Since T_h has the HUS, there is an $f_2 \in H(\mathbb{C})$, such that $T_h f_2 = 0$ and

$$\sup_{z \in \mathbb{C}} |f_1(z) - f_2(z)| \le K. \tag{2.14}$$

Note that f_2 is of the form $f_2(z) = f_2(0)/\tilde{h}(z)$ for all $z \in \mathbb{C}$, since $T_h f_2 = 0$. Lemma 2.1, applied to f_1, yields that h is a constant function. If h were 0, then (2.13) would be written in the form $f_1(z) = z$ for $z \in \mathbb{C}$, and hence from (2.14), $\sup_{z\in\mathbb{C}} |z - f_2(0)| \le K < \infty$, which is a contradiction. Thus, we conclude that h is a nonzero constant function. $\square$

Theorem 2.3. *Suppose $\lambda \in \mathbb{C} \setminus \{0\}$, $f, g \in H(\mathbb{C})$, and $\sup_{z\in\mathbb{C}} |T_\lambda f(z) - g(z)| < \infty$. Then there exists a unique $f_0 \in H(\mathbb{C})$ such that $T_\lambda f_0 = g$ and*

$$\sup_{z \in \mathbb{C}} |f(z) - f_0(z)| < \infty. \tag{2.15}$$

Furthermore, $1/|\lambda|$ is the HUS constant for T_λ.

Proof. The existence of such a function $f_0 \in H(\mathbb{C})$ is proved by Theorem 2.2, and so we need to show only the uniqueness. Suppose $f_1 \in H(\mathbb{C})$ and $f_2 \in H(\mathbb{C})$ satisfy $T_\lambda f_j = g$ and

$$\sup_{z \in \mathbb{C}} |f(z) - f_j(z)| < \infty \tag{2.16}$$

for $j = 1, 2$. Since $T_\lambda f_j = g$,

$$f_j(z) = e^{-\lambda z}\left\{f_j(0) + \int_0^z g(\zeta)e^{\lambda\zeta}d\zeta\right\} \quad (z \in \mathbb{C}) \tag{2.17}$$

for $j = 1,2$, and hence

$$f_1(z) - f_2(z) = \big(f_1(0) - f_2(0)\big)e^{-\lambda z} \quad \forall z \in \mathbb{C}. \tag{2.18}$$

It follows from (2.16) that $f_1 - f_2$ is constant by Liouville's theorem. Therefore, $f_1(0) = f_2(0)$ by (2.18), which implies that $f_1 = f_2$, proving the uniqueness.

We show that $1/|\lambda|$ is the HUS constant for T_λ. Indeed, $1/|\lambda|$ is an HUS constant by (2.12). Conversely, let K be an arbitrary HUS constant for T_λ, and put

$$f_2(z) = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda z} \quad (z \in \mathbb{C}). \tag{2.19}$$

A simple calculation shows that $f_2'(z) + \lambda f_2(z) = 1$ for all $z \in \mathbb{C}$, and hence $\sup_{z\in\mathbb{C}} |T_\lambda f_2(z)| = 1$. Then, there exists an $f_3 \in H(\mathbb{C})$ such that $T_\lambda f_3 = 0$ and $\sup_{z\in\mathbb{C}} |f_2(z) - f_3(z)| \le K$. Since $|f_2(z) + \lambda^{-1}e^{-\lambda z}| = 1/|\lambda|$ for $z \in \mathbb{C}$, the uniqueness implies that $f_3(z) = -\lambda^{-1}e^{-\lambda z}$, which proves $1/|\lambda| \le K$. Thus, $1/|\lambda|$ is the HUS constant for T_λ. □

3. Stability for the Ger-type differential inequality. In this section, we consider the Ger-type stability problem. First, we show that the Ger-type stability does not hold in general. Indeed, the following proposition is true.

Proposition 3.1. *For $\lambda \in \mathbb{C}\setminus\{0\}$ and $\varepsilon > 0$, there exists an $f \in H(\mathbb{C})$ with the following properties:*

$$\begin{gathered} \sup_{z\in\mathbb{C}} \left| \frac{f'(z)}{\lambda f(z)} - 1 \right| \le \varepsilon, \\ \sup_{z\in\mathbb{C}} \left| \frac{f(z)}{ce^{\lambda z}} - 1 \right| = \sup_{z\in\mathbb{C}} \left| \frac{ce^{\lambda z}}{f(z)} - 1 \right| = \infty \quad \forall c \in \mathbb{C}\setminus\{0\}. \end{gathered} \tag{3.1}$$

Proof. We associate to each $\lambda \in \mathbb{C}\setminus\{0\}$ and $\varepsilon > 0$ a function f defined by

$$f(z) = e^{(\lambda + |\lambda|\varepsilon)z} \quad (z \in \mathbb{C}). \tag{3.2}$$

As above, we obtain

$$f'(z) = (\lambda + |\lambda|\varepsilon)f(z) \quad (z \in \mathbb{C}), \tag{3.3}$$

so that

$$\left| \frac{f'(z)}{\lambda f(z)} - 1 \right| = \varepsilon \quad \forall z \in \mathbb{C}. \tag{3.4}$$

If $c \in \mathbb{C}\setminus\{0\}$, then we have

$$\begin{gathered} \left| \frac{f(z)}{ce^{\lambda z}} - 1 \right| \ge \frac{1}{|c|}\left|e^{|\lambda|\varepsilon z}\right| - 1 \longrightarrow \infty \quad (\operatorname{Re} z \longrightarrow \infty), \\ \left| \frac{ce^{\lambda z}}{f(z)} - 1 \right| \ge |c|\left|e^{-|\lambda|\varepsilon z}\right| - 1 \longrightarrow \infty \quad (\operatorname{Re} z \longrightarrow -\infty), \end{gathered} \tag{3.5}$$

and so

$$\sup_{z\in\mathbb{C}}\left|\frac{f(z)}{ce^{\lambda z}}-1\right| = \sup_{z\in\mathbb{C}}\left|\frac{ce^{\lambda z}}{f(z)}-1\right| = \infty \quad \forall c\in\mathbb{C}\setminus\{0\}. \tag{3.6}$$

□

One might ask when the Ger-type stability is true. We give an answer to this question. If the Ger-type stability holds, then the function $f\in H(\mathbb{C})$ must be of the form $f(z)=f(0)e^{\lambda z}$. That is, the only solution to the differential equation $y'=\lambda y$ has the Ger-type stability.

Theorem 3.2. *Suppose $\lambda\in\mathbb{C}\setminus\{0\}$, $\varepsilon>0$, and $f\in H(\mathbb{C})$ satisfies $f(z)\neq 0$ for all $z\in\mathbb{C}$ and (1.5) holds. Suppose*

$$\sup_{z\in\mathbb{C}}\left|\frac{f(z)}{ce^{\lambda z}}-1\right| \quad \text{or} \quad \sup_{z\in\mathbb{C}}\left|\frac{ce^{\lambda z}}{f(z)}-1\right| \tag{3.7}$$

is finite for some $c\in\mathbb{C}\setminus\{0\}$; then f is of the form $f(z)=f(0)e^{\lambda z}$ for all $z\in\mathbb{C}$.

Proof. It follows from (1.5) that $1-f'/\lambda f$ is constant, say $c_0\in\mathbb{C}$, by Liouville's theorem. Thus, $f'=(1-c_0)\lambda f$, and hence

$$f(z)=f(0)e^{(1-c_0)\lambda z} \quad (z\in\mathbb{C}). \tag{3.8}$$

Suppose that there is a $c_1\in\mathbb{C}\setminus\{0\}$ such that

$$\sup_{z\in\mathbb{C}}\left|\frac{f(z)}{c_1e^{\lambda z}}-1\right| < \infty. \tag{3.9}$$

From (3.8), it follows that

$$\sup_{z\in\mathbb{C}}\left|\frac{f(0)}{c_1}e^{-c_0\lambda z}-1\right| < \infty, \tag{3.10}$$

and hence c_0 must be 0, proving $f(z)=f(0)e^{\lambda z}$ for all $z\in\mathbb{C}$.

Similarly, we can treat the case where

$$\sup_{z\in\mathbb{C}}\left|\frac{c_2e^{\lambda z}}{f(z)}-1\right| < \infty \tag{3.11}$$

for some $c_2\in\mathbb{C}\setminus\{0\}$, and so the proof is omitted. □

Thus far, we have treated entire functions. Finally, we consider the Ger-type stability problem in the category of holomorphic functions on a bounded region.

Theorem 3.3. *Let $0\in\Omega$ be a bounded convex region of $\mathbb{C}$ and put $M=\sup_{z\in\Omega}|z|$. Suppose $\lambda\in\mathbb{C}\setminus\{0\}$, $0\le\varepsilon\le 1$, and $f:\Omega\to\mathbb{C}$ is holomorphic such that $f(z)\neq 0$ for all $z\in\Omega$ and*

$$\sup_{z\in\Omega}\left|\frac{f'(z)}{\lambda f(z)}-1\right| \le \varepsilon. \tag{3.12}$$

Then there are $K_\lambda > 0$ *and* $c \in \mathbb{C} \setminus \{0\}$ *such that*

$$\max\left\{\sup_{z\in\Omega}\left|\frac{f(z)}{ce^{\lambda z}}-1\right|,\sup_{z\in\Omega}\left|\frac{ce^{\lambda z}}{f(z)}-1\right|\right\} \le K_\lambda \varepsilon. \tag{3.13}$$

Proof. Put $g(z) = -1 + f'(z)/\lambda f(z)$ for $z \in \Omega$, and so

$$f'(z) = \lambda(1+g(z))f(z) \quad (z \in \Omega). \tag{3.14}$$

From (3.14), it follows that

$$f(z) = f(0)e^{\lambda z}\exp\int_0^z \lambda g(\zeta)\,d\zeta \tag{3.15}$$

for every $z \in \Omega$, and hence

$$\begin{aligned}\left|\frac{f(z)}{f(0)e^{\lambda z}}-1\right| &= \left|\exp\int_0^z \lambda g(\zeta)\,d\zeta - 1\right| \le \sum_{n=1}^{\infty}\frac{1}{n!}\left|\int_0^z \lambda g(\zeta)\,d\zeta\right|^n \\ &\le \sum_{n=1}^{\infty}\frac{|\lambda\varepsilon z|^n}{n!} \le \left(e^{|\lambda|M}-1\right)\varepsilon\end{aligned} \tag{3.16}$$

for all $z \in \Omega$. Similarly, we can show that

$$\sup_{z\in\Omega}\left|\frac{f(0)e^{\lambda z}}{f(z)}-1\right| \le \left(e^{|\lambda|M}-1\right)\varepsilon, \tag{3.17}$$

and so the proof is complete. □

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