

## THE PSEUDODIFFERENTIAL OPERATOR $A(x, D)$

R. S. PATHAK and S. PATHAK

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The pseudodifferential operator (p.d.o.)  $A(x, D)$ , associated with the Bessel operator  $d^2/dx^2 + (1 - 4\mu^2)/4x^2$ , is defined. Symbol class  $H_{\rho, \delta}^m$  is introduced. It is shown that the p.d.o. associated with a symbol belonging to this class is a continuous linear mapping of the Zemanian space  $H_\mu$  into itself. An integral representation of p.d.o. is obtained. Using Hankel convolution  $L_{\sigma, \alpha}^p$ -norm continuity of the p.d.o. is proved.

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**1. Introduction.** The theory of the Hankel transformation

$$(h_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x) dx, \quad \mu \geq -\frac{1}{2}, \quad (1.1)$$

where  $0 < y < \infty$  and  $J_\mu$  is the Bessel function of the first kind and of order  $\mu$ , has been extended by Zemanian [14] to distributions belonging to  $H'_\mu$ , the dual of the test function space  $H_\mu$ , consisting of all complex-valued smooth functions  $\phi$  defined on  $I = (0, \infty)$  and satisfying

$$y_{m,k}^\mu(\phi) = \sup_{0 < x < \infty} \left| x^m \left( x^{-1} \frac{d}{dx} \right)^k [x^{-\mu-1/2} \phi(x)] \right| < \infty \quad (1.2)$$

for each pair of nonnegative integers  $m$  and  $k$ . Pseudodifferential operators associated with a numerical valued symbol  $a(x, y)$  were investigated by R. S. Pathak and Pandey [8, 9] and by R. S. Pathak and S. Pathak [10].

We will use the notation and terminology of [4, 7, 11, 12, 14]. The differential operators  $N_\mu$ ,  $M_\mu$ , and  $S_\mu$  are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+3/2} \left( x^{-1} \frac{d}{dx} \right) x^{-\mu-1/2}, \quad (1.3)$$

$$M_\mu = M_{\mu,x} = x^{-\mu+1/2} \left( x^{-1} \frac{d}{dx} \right) x^{\mu+1/2}, \quad (1.4)$$

$$S_\mu = S_{\mu,x} = M_\mu N_\mu = (2\mu + 2) x^{\mu+1/2} \left( x^{-1} \frac{d}{dx} \right) x^{-\mu-1/2} \\ + x^{\mu+5/2} \left( x^{-1} \frac{d}{dx} \right)^2 x^{-\mu-1/2}. \quad (1.5)$$

From [14, page 139] and [5, page 948] we know the following relations for any  $\phi \in H_\mu$ :

$$h_{\mu+1}(N_\mu\phi) = -y h_\mu\phi, \tag{1.6}$$

$$h_\mu(S_\mu\phi) = -y^2 h_\mu\phi, \tag{1.7}$$

$$\left(x^{-1} \frac{d}{dx}\right)^k (x^{-\mu-1/2}\theta\phi) = \sum_{v=0}^k \binom{k}{v} \left(x^{-1} \frac{d}{dx}\right)^v \theta \left(x^{-1} \frac{d}{dx}\right)^{k-v} (x^{-\mu-1/2}\phi), \tag{1.8}$$

$$S_{\mu,x}^r \phi(x) = \sum_{j=0}^r b_j x^{2j+\mu+1/2} \left(x^{-1} \frac{d}{dx}\right)^{r+j} (x^{-\mu-1/2}\phi(x)), \tag{1.9}$$

where the  $b_j$  are constants depending only on  $\mu$ .

Next, we define the space  $L_\sigma^p(I)$ ,  $1 \leq p < \infty$ , as the space of those real-valued measurable functions on  $I$  for which

$$\|f\|_{L_\sigma^p} = \left[ \int_0^\infty |f(x)|^p d\sigma(x) \right]^{1/p} < \infty, \tag{1.10}$$

where

$$d\sigma(x) = (2^\mu \Gamma(\mu+1))^{-1} x^{2\mu+1} dx. \tag{1.11}$$

Let  $\Delta(x, y, z)$  denote the area of a triangle with sides  $x, y$ , and  $z$ , if such a triangle exists. For fixed  $\mu \geq -1/2$ , set

$$D(x, y, z) = 2^{3\mu-1/2} (\pi)^{-1/2} [\Gamma(\mu+1)]^2 \left[ \Gamma\left(\mu + \frac{1}{2}\right) \right]^{-1} (xyz)^{-2\mu} [\Delta(x, y, z)]^{2\mu-1}, \tag{1.12}$$

if  $\Delta$  exists, and zero otherwise. Then  $D(x, y, z) \geq 0$  and that  $D(x, y, z)$  is symmetric in  $x, y, z$ . From [13, page 411], we know that

$$j(xt)j(yt) = \int_0^\infty j(z)D(x, y, z)d\sigma(z), \tag{1.13}$$

where

$$j(x) = 2^\mu \Gamma(\mu+1) x^{-\mu} J_\mu(x). \tag{1.14}$$

From (1.13) it follows that

$$J_\mu(xt)J_\mu(yt) = (2^\mu \Gamma(\mu+1))^{-1} \int_0^\infty (xyt)^\mu z^{-\mu} J_\mu(z)D(x, y, z)d\sigma(z). \tag{1.15}$$

Let  $f \in L_\sigma^1(I)$ , then its associated function  $f(x, y)$  is defined by

$$f(x, y) = \int_0^\infty f(z)D(x, y, z)d\sigma(z), \quad 0 < x, y < \infty, \tag{1.16}$$

and the Hankel convolution of  $f$  and  $g$  is defined by

$$f\#g(x) = \int_0^\infty f(x, y)g(y)d\sigma(y), \quad 0 < x < \infty. \tag{1.17}$$

From [9, page 102] we know that if  $f(x) \in L_\sigma^q(I)$ ,  $g(x) \in L_\sigma^p(I)$  with  $p > 1$ ,  $q > 1$ , and  $1/p + 1/q > 1$ , then

$$\|f\#g\|_r \leq \|f\|_q \|g\|_p, \tag{1.18}$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 > 0. \tag{1.19}$$

Many algebraic and topological properties of the generalized Hankel convolution have been given by Betancor and his associates [1, 2, 3, 6].

We also note the differentiation formula

$$\left(x^{-1} \frac{d}{dx}\right)^m x^{-\mu} J_\mu(xt) = (-t)^m x^{-\mu-m} J_{\mu+m}(xt). \tag{1.20}$$

In this paper, a general class  $H_{\rho, \delta}^m$  of symbols associated with the differentiation formula (1.20) and similar to the Hörmander class  $S_{\rho, \delta}^m$  is introduced. For  $\rho = 1/2$  and  $\delta = 0$ , the class  $H_{\rho, \delta}^m$  reduces to the symbol class  $H_0^m$  studied by R. S. Pathak and Pandey [8]. An integral representation for  $A(x, D)$  is given when the symbol belongs to the class  $H_{\rho, \delta}^m$ . It is shown that  $A(x, D)$  is a continuous linear mapping of Zemanian's space  $H_\mu$  into itself. The space  $L_{\sigma, \alpha}^p$  is defined and an  $L_{\sigma, \alpha}^p$ -boundedness result is also obtained.

## 2. The pseudodifferential operator $A(x, D)$

**DEFINITION 2.1.** Let  $a(x, \xi)$  be a complex-valued smooth function belonging to the space  $C^\infty(I \times I)$ , where  $I = (0, \infty)$ , and let its derivatives satisfy certain growth conditions such as (2.3). The pseudodifferential operator (p.d.o.)  $A(x, D)$ , associated with the symbol  $a(x, \xi)$ , is defined by

$$A(x, D)f(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) a(x, \xi) (h_\mu f)(\xi) d\xi, \tag{2.1}$$

where

$$(h_\mu f)(\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) f(x) dx, \quad \mu \geq -\frac{1}{2}. \tag{2.2}$$

**DEFINITION 2.2.** The function  $a(x, \xi) : C^\infty(I \times I) \rightarrow \mathbb{C}$  belongs to the symbol class  $H_{\rho, \delta}^m$ ,  $m \in \mathbb{R}$ ,  $0 \leq \delta \leq \rho \leq 1$ , if and only if for all  $\alpha, \beta \in \mathbb{N}_0$ , there exists  $C_{\alpha, \beta, m} > 0$  such that

$$\left| \left(\xi^{-1} \frac{d}{d\xi}\right)^\alpha \left(x^{-1} \frac{d}{dx}\right)^\beta a(x, \xi) \right| \leq C_{\alpha, \beta, m} (1 + \xi^2)^{m/2 - \rho\alpha - \delta\beta}. \tag{2.3}$$

A few examples of elements of  $H_{\rho,\delta}^m$  are given as

- (i)  $(1 + x^2 + \xi^2)^{m/2}$ ,  $m < 0$ ,
- (ii)  $\sin x^2 (1 + x^2 + \xi^2)^{m/2}$ ,  $m < 0$ ,
- (iii)  $e^{-x^2} (1 + x^2 + \xi^2)^{m/2}$ ,  $m < 0$ .

We give a proof for example (i). We have

$$\begin{aligned}
 a(x, \xi) &= (1 + x^2 + \xi^2)^{m/2}, \quad m < 0, \\
 \left(x^{-1} \frac{d}{dx}\right)^\beta (1 + x^2 + \xi^2)^{m/2} &= D_{m,\beta} (1 + x^2 + \xi^2)^{m/2-\beta},
 \end{aligned}
 \tag{2.4}$$

so that

$$\begin{aligned}
 &\left| \left(\xi^{-1} \frac{d}{d\xi}\right)^\alpha \left(x^{-1} \frac{d}{dx}\right)^\beta (1 + x^2 + \xi^2)^{m/2} \right| \\
 &= \left| D_{m,\alpha,\beta} (1 + x^2 + \xi^2)^{m/2-\beta-\alpha} \right| \leq D_{m,\alpha,\beta} (1 + \xi^2)^{m/2-\rho\alpha-\delta\beta} \quad (m < 0)
 \end{aligned}
 \tag{2.5}$$

for  $0 \leq \delta \leq 1$  and  $0 \leq \rho \leq 1$ .

**THEOREM 2.3.** *Let the symbol  $a(x, \xi) \in H_{\rho,\delta}^m$ . Then for  $\mu \geq -1/2$ , the p.d.o.  $A(x, D)$  is a continuous linear mapping of  $H_\mu$  into  $H_\mu$ .*

**PROOF.** Let  $\Phi(x) = A(x, D)f(x)$ ,  $f \in H_\mu$ . Then using definitions (1.3) and (2.1), we have

$$N_\mu \Phi(x) = x^{\mu+3/2} \left(x^{-1} \frac{d}{dx}\right) x^{-\mu-1/2} \Phi(x).
 \tag{2.6}$$

By induction, we get

$$\begin{aligned}
 &N_{\mu+k-1} \cdots N_\mu \Phi(x) \\
 &= x^{\mu+k+1/2} \left(x^{-1} \frac{d}{dx}\right)^k x^{-\mu-1/2} \Phi(x) \\
 &= x^{\mu+k+1/2} \left(x^{-1} \frac{d}{dx}\right)^k x^{-\mu-1/2} \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) a(x, \xi) (h_\mu f)(\xi) d\xi.
 \end{aligned}
 \tag{2.7}$$

Using formula (1.8), we get

$$\begin{aligned}
 &N_{\mu+k-1} \cdots N_\mu \Phi(x) \\
 &= \sum_{r=0}^k \binom{k}{r} x^{\mu+k+1/2} \int_0^\infty \left(x^{-1} \frac{d}{dx}\right)^{k-r} x^{-\mu} J_\mu(x\xi) \\
 &\quad \times \left(x^{-1} \frac{d}{dx}\right)^r a(x, \xi) \xi^{1/2} (h_\mu f)(\xi) d\xi.
 \end{aligned}
 \tag{2.8}$$

Next, applying (1.20), we have

$$\begin{aligned}
 & N_{\mu+k-1} \cdots N_{\mu} \Phi(x) \\
 &= \sum_{r=0}^k \binom{k}{r} x^{\mu+k+1/2} \int_0^{\infty} (-\xi)^{k-r} x^{-\mu-k+r} J_{\mu+k-r}(x\xi) \\
 &\quad \times \left(x^{-1} \frac{d}{dx}\right)^r a(x, \xi) \xi^{1/2} (h_{\mu} f)(\xi) d\xi \\
 &= \sum_{r=0}^k \binom{k}{r} \int_0^{\infty} x^r (x\xi)^{1/2} J_{\mu+k-r}(x\xi) \\
 &\quad \times \left(x^{-1} \frac{d}{dx}\right)^r a(x, \xi) (-\xi)^{k-r} (h_{\mu} f)(\xi) d\xi.
 \end{aligned} \tag{2.9}$$

Using formula (1.6), we get

$$\begin{aligned}
 & (-x)(N_{\mu+k-1} \cdots N_{\mu} \Phi(x)) \\
 &= A_0 \int_0^{\infty} (x\xi)^{1/2} J_{\mu+k+1}(x\xi) N_{\mu+k} [a(x, \xi) (-\xi)^k (h_{\mu} f)(\xi)] d\xi \\
 &+ A_1 \int_0^{\infty} x^{3/2} \xi^{1/2} J_{\mu+k}(x\xi) N_{\mu+k-1} \\
 &\quad \times \left[ \left(x^{-1} \frac{d}{dx}\right) a(x, \xi) (-\xi)^{k-1} (h_{\mu} f)(\xi) \right] d\xi \\
 &+ A_2 \int_0^{\infty} x^{5/2} \xi^{1/2} J_{\mu+k-1}(x\xi) N_{\mu+k-2} \\
 &\quad \times \left[ \left(x^{-1} \frac{d}{dx}\right)^2 a(x, \xi) (-\xi)^{k-2} (h_{\mu} f)(\xi) \right] d\xi + \cdots \\
 &+ A_k \int_0^{\infty} x^{k+1/2} \xi^{1/2} J_{\mu+1}(x\xi) N_{\mu} \left[ \left(x^{-1} \frac{d}{dx}\right)^k a(x, \xi) (h_{\mu} f)(\xi) \right] d\xi,
 \end{aligned} \tag{2.10}$$

$$\begin{aligned}
 & (-x^2)(N_{\mu+k-1} \cdots N_{\mu} \Phi(x)) \\
 &= A_0 \int_0^{\infty} (x\xi)^{1/2} J_{\mu+k+2}(x\xi) N_{\mu+k+1} N_{\mu+k} [a(x, \xi) (-\xi)^k (h_{\mu} f)(\xi)] d\xi \\
 &+ A_1 \int_0^{\infty} x^{3/2} \xi^{1/2} J_{\mu+k+1}(x\xi) N_{\mu+k} N_{\mu+k-1} \\
 &\quad \times \left[ \left(x^{-1} \frac{d}{dx}\right) a(x, \xi) (-\xi)^{k-1} (h_{\mu} f)(\xi) \right] d\xi \\
 &+ A_2 \int_0^{\infty} x^{5/2} \xi^{1/2} J_{\mu+k}(x\xi) N_{\mu+k-1} N_{\mu+k-2} \\
 &\quad \times \left[ \left(x^{-1} \frac{d}{dx}\right)^2 a(x, \xi) (-\xi)^{k-2} (h_{\mu} f)(\xi) \right] d\xi + \cdots \\
 &+ A_k \int_0^{\infty} x^{k+1/2} \xi^{1/2} J_{\mu+2}(x\xi) N_{\mu+1} N_{\mu} \left[ \left(x^{-1} \frac{d}{dx}\right)^k a(x, \xi) (h_{\mu} f)(\xi) \right] d\xi.
 \end{aligned}$$

By induction, we have

$$\begin{aligned}
 & (-x)^n (N_{\mu+k-1} \cdots N_{\mu} \Phi(x)) \\
 &= A_0 \int_0^{\infty} (x\xi)^{1/2} J_{\mu+k+n}(x\xi) \\
 &\quad \times N_{\mu+k+n-1} \cdots N_{\mu+k} [a(x, \xi) (-\xi)^k (h_{\mu}f)(\xi)] d\xi \\
 &+ A_1 \int_0^{\infty} x^{3/2} \xi^{1/2} J_{\mu+k+n-1}(x\xi) N_{\mu+k+n-2} \cdots N_{\mu+k-1} \\
 &\quad \times \left[ \left( x^{-1} \frac{d}{dx} \right) a(x, \xi) (-\xi)^{k-1} (h_{\mu}f)(\xi) \right] d\xi + \cdots \\
 &+ A_k \int_0^{\infty} x^{k+1/2} \xi^{1/2} J_{\mu+n}(x\xi) N_{\mu+n-1} \cdots N_{\mu} \\
 &\quad \times \left[ \left( x^{-1} \frac{d}{dx} \right)^k a(x, \xi) (h_{\mu}f)(\xi) \right] d\xi.
 \end{aligned} \tag{2.11}$$

Therefore,

$$\begin{aligned}
 & (-x)^n (N_{\mu+k-1} \cdots N_{\mu} \Phi(x)) \\
 &= \sum_{r=0}^k \binom{k}{r} \int_0^{\infty} x^{r+1/2} \xi^{1/2} J_{\mu+k-r+n}(x\xi) N_{\mu+k-r+n-1} \cdots N_{\mu+k-r} \\
 &\quad \times \left[ \left( x^{-1} \frac{d}{dx} \right)^r a(x, \xi) (-\xi)^{k-r} (h_{\mu}f)(\xi) \right] d\xi,
 \end{aligned} \tag{2.12}$$

which in view of formula (2.7) can be written as

$$\begin{aligned}
 & (-x)^n x^{\mu+k+1/2} \left( x^{-1} \frac{d}{dx} \right)^k x^{-\mu-1/2} \Phi(x) \\
 &= \sum_{r=0}^k \binom{k}{r} (-1)^{k-r} \int_0^{\infty} x^{r+1/2} \xi^{1/2} J_{\mu+k-r+n}(x\xi) \xi^{\mu+k-r+n+1/2} \\
 &\quad \times \left( \xi^{-1} \frac{d}{d\xi} \right)^n \left[ \xi^{-\mu-1/2} \left( x^{-1} \frac{d}{dx} \right)^r a(x, \xi) (h_{\mu}f)(\xi) \right] d\xi \\
 &= \sum_{r=0}^k \binom{k}{r} (-1)^{k-r} \int_0^{\infty} (x\xi)^{-\mu-k+r} J_{\mu+k-r+n}(x\xi) x^{\mu+k+1/2} \\
 &\quad \times \xi^{2(\mu+k-r)+n+1} \left( \xi^{-1} \frac{d}{d\xi} \right)^n \\
 &\quad \times \left[ \xi^{-\mu-1/2} \left( x^{-1} \frac{d}{dx} \right)^r a(x, \xi) (h_{\mu}f)(\xi) \right] d\xi.
 \end{aligned} \tag{2.13}$$

Putting  $\mu + k - r = \lambda$ , we get

$$\begin{aligned}
 & (-x)^n \left(x^{-1} \frac{d}{dx}\right)^k x^{-\mu-1/2} \Phi(x) \\
 &= \sum_{r=0}^k \binom{k}{r} (-1)^{k-r} \int_0^\infty (x\xi)^{-\lambda} J_{\lambda+n}(x\xi) \xi^{2\lambda+n+1} \left(\xi^{-1} \frac{d}{d\xi}\right)^n \\
 &\quad \times \left[\xi^{-\mu-1/2} \left(x^{-1} \frac{d}{dx}\right)^r a(x, \xi) (h_\mu f)(\xi)\right] d\xi.
 \end{aligned} \tag{2.14}$$

Now an application of formula (1.8) yields

$$\begin{aligned}
 & \left| x^n \left(x^{-1} \frac{d}{dx}\right)^k x^{-\mu-1/2} \Phi(x) \right| \\
 & \leq \sum_{r=0}^k \binom{k}{r} \int_0^\infty |(x\xi)^{-\lambda} J_{\lambda+n}(x\xi)| \xi^{2\lambda+n+1} \\
 & \quad \times \sum_{v=0}^n \binom{n}{v} \left| \left(\xi^{-1} \frac{d}{d\xi}\right)^v \left(x^{-1} \frac{d}{dx}\right)^r a(x, \xi) \right| \\
 & \quad \times \left| \left(\xi^{-1} \frac{d}{d\xi}\right)^{n-v} \xi^{-\mu-1/2} (h_\mu f)(\xi) \right| d\xi.
 \end{aligned} \tag{2.15}$$

Then using inequality (2.3) and the boundedness property of the Bessel function, we find that the right-hand side of (2.15) is bounded by

$$\begin{aligned}
 & \sum_{r=0}^k \sum_{v=0}^n \binom{k}{r} \binom{n}{v} A_{\lambda,n} C_{v,r,m} \int_0^\infty \xi^{2\lambda+n+1} (1+\xi^2)^{m/2-\rho v-\delta r} \\
 & \quad \times \left| \left(\xi^{-1} \frac{d}{d\xi}\right)^{n-v} \xi^{-\mu-1/2} (h_\mu f)(\xi) \right| d\xi \\
 & \leq \sum_{r=0}^k \sum_{v=0}^n \binom{k}{r} \binom{n}{v} A_{\lambda,n} C_{v,r,m} \int_0^\infty (1+\xi^2)^{\lambda+n/2+1/2+m/2-\rho v-\delta r} \\
 & \quad \times \left| \left(\xi^{-1} \frac{d}{d\xi}\right)^{n-v} \xi^{-\mu-1/2} (h_\mu f)(\xi) \right| d\xi.
 \end{aligned} \tag{2.16}$$

Let  $p$  be a positive integer greater than or equal to  $\mu + k + m/2 + n/2 + 1/2$ , then we can write

$$\begin{aligned}
 & \left| x^n \left(x^{-1} \frac{d}{dx}\right)^k x^{-\mu-1/2} \Phi(x) \right| \\
 & \leq \sum_{r=0}^k \sum_{v=0}^n \binom{k}{r} \binom{n}{v} A_{\lambda,n} C_{v,r,m} \\
 & \quad \times \int_0^\infty \frac{(1+\xi^2)^{p+1}}{(1+\xi^2)} \left| \left(\xi^{-1} \frac{d}{d\xi}\right)^{n-v} \xi^{-\mu-1/2} (h_\mu f)(\xi) \right| d\xi
 \end{aligned}$$

$$\begin{aligned} &\leq \sum_{r=0}^k \sum_{v=0}^n \sum_{s=0}^{p+1} \binom{k}{r} \binom{n}{v} \binom{p+1}{s} A_{\lambda,n} C_{v,r,m} \\ &\quad \times \int_0^\infty \xi^{2s} \left| \left( \xi^{-1} \frac{d}{d\xi} \right)^{n-v} \xi^{-\mu-1/2} (h_\mu f)(\xi) \right| \left| \frac{d\xi}{(1+\xi^2)} \right|. \end{aligned} \tag{2.17}$$

Thus

$$\gamma_{n,k}^\mu(\Phi) \leq \sum_{r=0}^k \sum_{v=0}^n \sum_{s=0}^{p+1} D_{\lambda,n,k,m} \gamma_{2s,n-v}^\mu(h_\mu f), \tag{2.18}$$

where  $D_{\lambda,n,k,m}$  is a positive constant. From (2.18), the continuity of  $A(x, D)$  follows. □

**3. An integral representation.** The function  $a_x(\xi)$ , associated with the symbol  $a(x, \xi)$  and defined by

$$a_x(\xi) = \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) [(x\eta)^{1/2} J_\mu(x\eta) a(x, \eta)] d\eta, \tag{3.1}$$

will play a fundamental role in our investigation. An estimate for  $a_x(\xi)$  is given by the following lemma.

**LEMMA 3.1.** *Let the symbol  $a(x, \eta) \in H_{\rho,\delta}^m$ . Then the function  $a_x(\xi)$  defined by (3.1) satisfies the inequality*

$$|a_x(\xi)| \leq E_{\mu,m,t} (1 + \xi^2)^{-t} (1 + x)^{\mu+4t+1/2}, \tag{3.2}$$

where  $E_{\mu,m,t}$  is a positive constant,  $\mu \geq -1/2$ , and  $m < -\mu - 3/2 - t$ .

**PROOF.** Using property (1.7), we can write

$$\begin{aligned} a_x(\xi) &= \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \frac{(1 - S_{\mu,\eta})^t}{(1 + \xi^2)^t} [(x\eta)^{1/2} J_\mu(x\eta) a(x, \eta)] d\eta \\ &= \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \sum_{r=0}^t (-1)^r \binom{t}{r} \frac{S_{\mu,\eta}^r}{(1 + \xi^2)^t} [(x\eta)^{1/2} J_\mu(x\eta) a(x, \eta)] d\eta. \end{aligned} \tag{3.3}$$

In view of (1.9), we have

$$\begin{aligned} a_x(\xi) &= \sum_{r=0}^t (-1)^r \binom{t}{r} \frac{1}{(1 + \xi^2)^t} \\ &\quad \times \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \sum_{j=0}^r b_j \eta^{2j+\mu+1/2} \left( \eta^{-1} \frac{d}{d\eta} \right)^{r+j} \\ &\quad \times [\eta^{-\mu-1/2} a(x, \eta) (x\eta)^{1/2} J_\mu(x\eta)] d\eta \end{aligned}$$



$$\begin{aligned}
 &= \sum_{r=0}^t \sum_{j=0}^r (-1)^r \binom{t}{r} b_j \frac{1}{(1+\xi^2)^t} \\
 &\quad \times \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \eta^{2j+\mu+1/2} \\
 &\quad \times \sum_{i=0}^{r+j} \binom{r+j}{i} \left(\eta^{-1} \frac{d}{d\eta}\right)^i a(x, \eta) \left(\eta^{-1} \frac{d}{d\eta}\right)^{r+j-i} \\
 &\quad \times \eta^{-\mu} J_\mu(x\eta) x^{1/2} d\eta.
 \end{aligned} \tag{3.4}$$

Again applying formula (1.20), we have

$$\begin{aligned}
 a_x(\xi) &= \sum_{r=0}^t \sum_{j=0}^r \sum_{i=0}^{r+j} (-1)^r \binom{t}{r} \binom{r+j}{i} b_j \frac{1}{(1+\xi^2)^t} \\
 &\quad \times \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \eta^{2j+\mu+1/2} \left(\eta^{-1} \frac{d}{d\eta}\right)^i \\
 &\quad \times a(x, \eta) (-1)^{r+j-i} x^{r+j-i} \eta^{-\mu-r-j+i} \\
 &\quad \times J_{\mu+r+j-i}(x\eta) x^{1/2} d\eta.
 \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}
 |a_x(\xi)| &\leq \sum_{r=0}^t \sum_{j=0}^r \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| \frac{x^{\mu+1/2+2(r+j-i)}}{(1+\xi^2)^t} \\
 &\quad \times \int_0^\infty |(\eta\xi)^{1/2} J_\mu(\eta\xi)| \eta^{2j+\mu+1/2} \left| \left(\eta^{-1} \frac{d}{d\eta}\right)^i a(x, \eta) \right| \\
 &\quad \times |(x\eta)^{-\mu-r-j+i} J_{\mu+r+j-i}(x\eta)| d\eta.
 \end{aligned} \tag{3.6}$$

Then, using inequality (2.3) and the boundedness property of the Bessel function, we have

$$\begin{aligned}
 |a_x(\xi)| &\leq \sum_{r=0}^t \sum_{j=0}^r \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| \frac{x^{\mu+1/2+2(r+j-i)}}{(1+\xi^2)^t} \\
 &\quad \times A_\mu B_{\mu,i,j,r} C_{i,m} \int_0^\infty \frac{\eta^{2j+\mu+1/2}}{(1+\eta^2)^{-m/2+\rho_i}} d\eta \\
 &\leq \sum_{r=0}^t \sum_{j=0}^r \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| A_\mu B_{\mu,i,j,r} C_{i,m} x^{\mu+2r+2j-2i+1/2} \\
 &\quad \times (1+\xi^2)^{-t} B\left(j + \frac{\mu}{2} + \frac{3}{4}, -\frac{m}{2} - \frac{\mu}{2} - \frac{3}{4} + \rho_i - j\right)
 \end{aligned} \tag{3.7}$$

for  $m < -\mu - 3/2 - t$ . Therefore there exists a constant  $E_{\mu,m,t}$  such that

$$|a_x(\xi)| \leq E_{\mu,m,t}(1 + \xi^2)^{-t}(1 + x)^{\mu+4t+1/2}. \tag{3.8}$$

□

**THEOREM 3.2.** For any symbol  $a(x, \xi) \in H_{\rho,\delta}^m$ , the associated operator  $A(x, D)$  can be represented by

$$A(x, D)f(x) = \int_0^\infty \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi) (h_\mu f)(\eta) d\eta d\xi, \quad f \in H_\mu(I), \tag{3.9}$$

where all the involved integrals are convergent for  $\mu \geq -1/2$  and  $m < -\mu - 3/2 - t$  with  $t > 1/2$ .

**PROOF.** Since  $a_x(\xi) = \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) [(x\eta)^{1/2} J_\mu(x\eta) a(x, \eta)] d\eta$ , by inversion, formally, we have

$$\int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi) d\xi = (x\eta)^{1/2} J_\mu(x\eta) a(x, \eta). \tag{3.10}$$

Therefore,

$$\begin{aligned} A(x, D)f(x) &= \int_0^\infty (x\eta)^{1/2} J_\mu(x\eta) a(x, \eta) (h_\mu f)(\eta) d\eta \\ &= \int_0^\infty (h_\mu f)(\eta) \left[ \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi) d\xi \right] d\eta \\ &= \int_0^\infty \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi) (h_\mu f)(\eta) d\eta d\xi, \end{aligned} \tag{3.11}$$

so that

$$|A(x, D)f(x)| \leq A_\mu \int_0^\infty |a_x(\xi)| \left( \int_0^\infty |(h_\mu f)(\eta)| d\eta \right) d\xi. \tag{3.12}$$

Since  $(h_\mu f)(\eta) \in H_\mu(I)$ , we have

$$|(h_\mu f)(\eta)| \leq C\eta^{\mu+1/2}(1 + \eta)^{-l} \quad \forall l > 0. \tag{3.13}$$

Now using the above estimate and (3.2), we obtain

$$\begin{aligned} &|A(x, D)f(x)| \\ &\leq A_\mu \int_0^\infty \int_0^\infty E_{\mu,m,t}(1 + \xi^2)^{-t}(1 + x)^{\mu+4t+1/2} C\eta^{\mu+1/2}(1 + \eta)^{-l} d\eta d\xi \\ &\leq E_{\mu,m,t}^1(1 + x)^{\mu+4t+1/2} \int_0^\infty (1 + \xi^2)^{-t} d\xi \int_0^\infty (1 + \eta)^{\mu+1/2-l} d\eta. \end{aligned} \tag{3.14}$$

The above integrals are convergent since  $\mu \geq -1/2$ , and  $t$  can be chosen greater than  $1/2$  and  $l$  sufficiently large. □

**4.  $L^p_{\sigma, \alpha}$ -boundedness of  $A(x, D)$ .** In the proof of the boundedness result, we will need the following estimate for the Hankel transform of  $\eta^{\mu+1/2}a(x, \eta)$ . We write

$$A_x(z) = h_\mu(\eta^{\mu+1/2}a(x, \eta))(z). \tag{4.1}$$

**LEMMA 4.1.** *For  $1/2 < \rho < 1$  and  $\mu + m + 3/2 < 0$ , there exists a constant  $A_{\mu, m, \rho} > 0$  such that*

$$|A_x(z)| \leq A_{\mu, m, \rho}(1 + z^2)^{-1}. \tag{4.2}$$

**PROOF.** We have

$$A_x(z) = \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \eta^{\mu+1/2} a(x, \eta) d\eta. \tag{4.3}$$

Then using the property (1.7), we can write

$$A_x(z) = \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \frac{(1 - S_{\mu, \eta})}{(1 + z^2)} \eta^{\mu+1/2} a(x, \eta) d\eta. \tag{4.4}$$

Using (1.5), we get

$$\begin{aligned} A_x(z) = \frac{1}{(1 + z^2)} & \left[ \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \eta^{\mu+1/2} a(x, \eta) d\eta \right. \\ & - (2\mu + 2) \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \eta^{\mu+1/2} \left( \eta^{-1} \frac{d}{d\eta} \right) a(x, \eta) d\eta \\ & \left. - \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \eta^{\mu+5/2} \left( \eta^{-1} \frac{d}{d\eta} \right)^2 a(x, \eta) d\eta \right], \end{aligned} \tag{4.5}$$

so that

$$\begin{aligned} |A_x(z)| \leq \frac{1}{(1 + z^2)} & \left[ \int_0^\infty A_\mu \eta^{\mu+1/2} (1 + \eta^2)^{m/2} d\eta \right. \\ & + (2\mu + 2) \int_0^\infty A_\mu \eta^{\mu+1/2} (1 + \eta^2)^{m/2-\rho} d\eta \\ & \left. + \int_0^\infty A_\mu \eta^{\mu+5/2} (1 + \eta^2)^{m/2-2\rho} d\eta \right]. \end{aligned} \tag{4.6}$$

Since all the three integrals are convergent for  $\mu + m + 3/2 < 0$  and  $1/2 < \rho < 1$ , therefore there exists a constant  $A_{\mu, m, \rho}$  such that

$$|A_x(z)| \leq A_{\mu, m, \rho}(1 + z^2)^{-1}. \tag{4.7}$$

□

We will use the following subspace of  $H_\mu$ .

**DEFINITION 4.2.** The space  $L^p_{\sigma, \alpha}(0, \infty)$ ,  $\alpha \in \mathbb{R}$ ,  $1 \leq p \leq \infty$ , is defined as the set of all those elements  $f \in H_\mu(I)$  which satisfy

$$\|f\|_{L^p_{\sigma, \alpha}} = \|\xi^{-\alpha} f(\xi)\|_{L^p_\sigma}. \tag{4.8}$$

**THEOREM 4.3.** *Pseudodifferential operator  $A(x, D)$  is a continuous linear mapping from  $L^p_{\sigma, \mu+1/2}$  into  $L^r_{\sigma, \mu+1/2}$ , where  $1 \leq p \leq \infty$ ,  $q > (2\mu + 2)/(\mu + 5/2)$ , and  $1/r = 1/p + 1/q - 1$ .*

**PROOF.** From (3.9), we have

$$\begin{aligned} A(x, D)f(x) &= \int_0^\infty a_x(\xi) \left( \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) (h_\mu f)(\eta) d\eta \right) d\xi \\ &= \int_0^\infty a_x(\xi) f(\xi) d\xi. \end{aligned} \tag{4.9}$$

Using (3.1), this can be written as

$$A(x, D)f(x) = \int_0^\infty \left( \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) (x\eta)^{1/2} J_\mu(x\eta) a(x, \eta) d\eta \right) f(\xi) d\xi. \tag{4.10}$$

Next using relation (1.15), we can express it in the form

$$\begin{aligned} A(x, D)f(x) &= \int_0^\infty \left( \int_0^\infty (\eta\xi)^{1/2} (x\eta)^{1/2} (2^\mu \Gamma(\mu + 1))^{-1} \right. \\ &\quad \left. \times \int_0^\infty (x\xi\eta)^\mu z^{-\mu} J_\mu(z\eta) D(x, \xi, z) d\sigma(z) a(x, \eta) d\eta \right) f(\xi) d\xi \\ &= \int_0^\infty \int_0^\infty \left( \int_0^\infty (z\eta)^{1/2} J_\mu(z\eta) \eta^{\mu+1/2} a(x, \eta) d\eta \right) \\ &\quad \times x^{\mu+1/2} \xi^{-\mu-1/2} f(\xi) z^{-\mu-1/2} D(x, \xi, z) d\sigma(\xi) d\sigma(z) \\ &= x^{\mu+1/2} \int_0^\infty \int_0^\infty A_x(z) \xi^{-\mu-1/2} f(\xi) z^{-\mu-1/2} D(x, \xi, z) d\sigma(\xi) d\sigma(z). \end{aligned} \tag{4.11}$$

An application of inequality (4.2) yields

$$\begin{aligned} &|x^{-\mu-1/2} A(x, D)f(x)| \\ &\leq A_{\mu, m, \rho} \int_0^\infty \int_0^\infty |\xi^{-\mu-1/2} f(\xi)| |z^{-\mu-1/2}| (1+z^2)^{-1} D(x, \xi, z) d\sigma(\xi) d\sigma(z). \end{aligned} \tag{4.12}$$

In view of definitions (1.16) and (1.17), the last expression can be expressed as a convolution, and for  $F(\xi) = |\xi^{-\mu-1/2} f(\xi)|$  and  $G(z) = |z^{-\mu-1/2}|(1+z^2)^{-1}$ , we have

$$|x^{-\mu-1/2} A(x, D)f(x)| \leq A_{\mu, m, \rho} (F\#G)(x), \tag{4.13}$$

so that

$$\left( \int_0^\infty |x^{-\mu-1/2} A(x, D)f(x)|^r d\sigma(x) \right)^{1/r} \leq A_{\mu, m, \rho} \left( \int_0^\infty (F\#G)^r(x) d\sigma(x) \right)^{1/r}. \tag{4.14}$$

We note that for  $q > (2\mu + 2)/(\mu + 5/2)$ ,  $G(z) \in L_\sigma^q(I)$ , and  $F(z) \in L_\sigma^p(I)$ ,  $1 \leq p \leq \infty$ , and using (1.18) finally we obtain

$$\|x^{-\mu-1/2}A(x, D)f(x)\|_{L_\sigma^r} \leq \|F\#G\|_{L_\sigma^r} \leq \|F\|_{L_\sigma^p} \|G\|_{L_\sigma^q}, \quad (4.15)$$

where  $1/r = 1/p + 1/q - 1 > 0$ . Therefore, in view of Definition 4.2, we get

$$\|A(x, D)f(x)\|_{L_\sigma^r, \mu+1/2} \leq \|f\|_{L_\sigma^p, \mu+1/2} \left\| (1+z^2)^{-1} \right\|_{L_\sigma^q, \mu+1/2}. \quad (4.16)$$

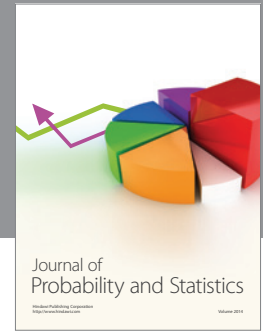
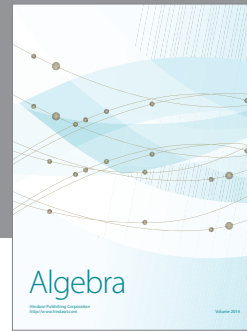
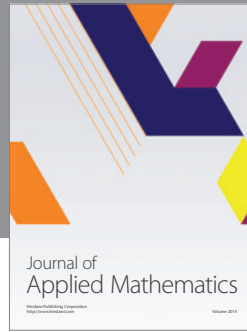
□

### REFERENCES

- [1] J. J. Betancor and I. Marrero, *The Hankel convolution and the Zemanian spaces  $\beta_\mu$  and  $\beta'_\mu$* , Math. Nachr. **160** (1993), 277-298.
- [2] ———, *Some properties of Hankel convolution operators*, Canad. Math. Bull. **36** (1993), no. 4, 398-406.
- [3] J. J. Betancor and L. Rodríguez-Mesa, *Hankel convolution on distribution spaces with exponential growth*, Studia Math. **121** (1996), no. 1, 35-52.
- [4] D. T. Haimo, *Integral equations associated with Hankel convolutions*, Trans. Amer. Math. Soc. **116** (1965), 330-375.
- [5] E. L. Koh and A. H. Zemanian, *The complex Hankel and I-transformations of generalized functions*, SIAM J. Appl. Math. **16** (1968), no. 5, 945-957.
- [6] I. Marrero and J. J. Betancor, *Hankel convolution of generalized functions*, Rend. Mat. Appl. (7) **15** (1995), no. 3, 351-380.
- [7] R. S. Pathak, *Integral Transforms of Generalized Functions and Their Applications*, Gordon and Breach Science Publishers, Amsterdam, 1997.
- [8] R. S. Pathak and P. K. Pandey, *A class of pseudo-differential operators associated with Bessel operators*, J. Math. Anal. Appl. **196** (1995), no. 2, 736-747.
- [9] ———, *Sobolev type spaces associated with Bessel operators*, J. Math. Anal. Appl. **215** (1997), no. 1, 95-111.
- [10] R. S. Pathak and S. Pathak, *Certain pseudo-differential operator associated with the Bessel operator*, Indian J. Pure Appl. Math. **31** (2000), no. 3, 309-317.
- [11] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1978.
- [12] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.
- [13] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1958.
- [14] A. H. Zemanian, *Generalized Integral Transformations*, Pure and Applied Mathematics, vol. 18, Interscience, New York, 1968.

R. S. Pathak: Department of Mathematics, Banaras Hindu University, Varanasi 221005, India  
E-mail address: [rspathak@banaras.ernet.in](mailto:rspathak@banaras.ernet.in)

S. Pathak: Department of Mathematics, Banaras Hindu University, Varanasi 221005, India



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