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PRECISE LIM SUP BEHAVIOR OF PROBABILITIES OF LARGE DEVIATIONS FOR SUMS OF I.I.D. RANDOM VARIABLES

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Let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued i.i.d. random variables and let $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. In this paper, we study the probabilities of large deviations of the form $P(S_n > tn^{1/p})$, $P(S_n < -tn^{1/p})$, and $P(|S_n| > tn^{1/p})$, where $t > 0$ and $0 < p < 2$. We obtain precise asymptotic estimates for these probabilities under mild and easily verifiable conditions. For example, we show that if $S_n/n^{1/p} \xrightarrow{P} 0$ and if there exists a nonincreasing positive function $\phi(x)$ on $[0,\infty)$ which is regularly varying with index $\alpha \leq -1$ such that $\limsup_{x\to\infty} P(|X| > x^{1/p})/\phi(x) = 1$, then for every $t > 0$, $\limsup_{n\to\infty} P(|S_n| > tn^{1/p})/(n\phi(n)) = t^{p\alpha}$.

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1. Introduction. Throughout this paper, let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued independent and identically distributed (i.i.d.) random variables and, as usual, let $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$ denote their partial sums. If

$$\frac{S_n - a_n}{b_n} \xrightarrow{P} 0, \tag{1.1}$$

where $\{a_n;\ n \geq 1\}$ and $\{b_n > 0;\ n \geq 1\}$ are sequences of constants, then the probabilities of the form

$$P(S_n - a_n > tb_n), \qquad P(S_n - a_n < -tb_n), \qquad P(|S_n - a_n| > tb_n), \tag{1.2}$$

where $t > 0$, are called *probabilities of large deviations* of the sums $\{S_n;\ n \geq 1\}$. The study of large deviations started with Cramér [9] and Chernoff [7]. They showed that if

$$M(t) \equiv E(e^{tX}) < \infty \quad \forall t \in \mathbb{R}, \tag{1.3}$$

then

(i) for every closed set $\mathbf{A} \subseteq \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{\log P(n^{-1}S_n \in \mathbf{A})}{n} \leq -\Lambda(\mathbf{A}), \tag{1.4}$$

(ii) for every open set $\mathbf{A} \subseteq \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{\log P(n^{-1}S_n \in \mathbf{A})}{n} \geq -\Lambda(\mathbf{A}), \tag{1.5}$$

where, for $x \in \mathbb{R}$ and $\mathbf{A} \subseteq \mathbb{R}$,

$$\lambda(x) = \sup_{t\in\mathbb{R}} \left(tx - \log M(t)\right), \qquad \Lambda(\mathbf{A}) = \inf_{x\in\mathbf{A}} \lambda(x). \tag{1.6}$$

This fundamental result is what we call the *large deviation principle* for partial sums $\{S_n;\ n \geq 1\}$. Donsker and Varadhan [11] and Bahadur and Zabell [1] established a large deviation principle for sums of i.i.d. Banach space-valued random variables. Bolthausen [2] extended the Cramér-Chernoff-Donsker-Varadhan-Bahadur-Zabell large deviation principle when the laws of the random variables converge weakly and satisfy a uniform exponential integrability condition. As an application of the Bolthausen large deviation principle, Li et al. [20] established a large deviation principle for bootstrapped sample means. In another direction, under the Cramér condition, which asserts that there exists a positive constant C such that

$$M(t) \equiv E\left(e^{tX}\right) < \infty \quad \text{for } |t| < C, \tag{1.7}$$

Petrov [23] obtained asymptotic expansions for the following probabilities of large deviations:

$$P\left(S_n \geq nE(X) + n^{1/2}x\right), \qquad P\left(S_n \leq nE(X) - n^{1/2}x\right), \tag{1.8}$$

where $x \geq 0$ and $x = o(n^{1/2})$. Let $\sigma^2 = \operatorname{Var}(X)$. It follows from the Petrov asymptotic expansions [23] that if

$$\frac{b_n}{\sqrt{n}} \longrightarrow \infty, \qquad \frac{b_n}{n} \longrightarrow 0, \tag{1.9}$$

then

$$\begin{aligned} \limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{S_n - nE(X)}{b_n} \in \mathbf{A}\right) &\leq -\frac{\sigma^2}{2} \inf_{x\in\mathbf{A}} x^2 \quad \text{for } \mathbf{A} \text{ closed}, \\ \liminf_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{S_n - nE(X)}{b_n} \in \mathbf{A}\right) &\geq -\frac{\sigma^2}{2} \inf_{x\in\mathbf{A}} x^2 \quad \text{for } \mathbf{A} \text{ open}. \end{aligned} \tag{1.10}$$

Borovkov and Mogul'skiĭ [4], Chen [5, 6], de Acosta [10], and Ledoux [18] obtained versions of (1.10) in a Banach space setting under various conditions. There have been a great number of investigations on the probabilities of large deviations for sums of independent random variables. Surveys of these investigations can be found in [3, 16, 25, 26, 27, 28].

Clearly, (1.4), (1.5), and (1.10) provide the rate, in a certain sense, at which the convergence in probability takes place in the weak law of large numbers (WLLN). However, it can be shown that the Cramér condition (1.7) is necessary for the classical large deviation principle (1.4)-(1.5) to hold. Under some mild conditions, the convergence rates of the law of large numbers have been studied by many authors. There is a large literature of investigation on this topic; see, for example, Stout [29], Petrov [25, 26], and references in these three books. The introduction of [19] provides a concise summary of this topic. For example, assume $h(x)$ is positive, nondecreasing, and slowly varying

at infinity, and let $0 < p < 2$ and $r \geq 0$. Heyde and Rohatgi [14] showed that

$$P\left(\frac{|S_n|}{n^{1/p}} \geq \varepsilon\right) = o\left(\frac{1}{n^r h(n)}\right) \quad \forall \varepsilon > 0 \tag{1.11}$$

if and only if

$$\begin{gathered} E(X) = 0 \quad \text{if } 1 < p < 2, \\ \lim_{n\to\infty} n^{1-(1/p)} E\left(X I_{\{|X| \leq n^{1/p}\}}\right) = 0 \quad \text{if } 0 < p \leq 1, \\ P\left(|X| > n^{1/p}\right) = o\left(\frac{1}{n^{r+1} h(n)}\right). \end{gathered} \tag{1.12}$$

A version of Heyde-Rohatgi's result in a Banach space setting was obtained by Li [21].

It is natural to ask if it is possible to establish a Cramér-Chernoff-type large deviation principle if the random variable X is not assumed to satisfy the Cramér condition (1.7). While the Cramér condition (1.7) is of course weaker than (1.3), it implies that $E(|X|^r) < \infty$ for all $r > 0$. In this paper, we will answer this question in the positive by presenting, under mild and easily verifiable conditions, precise asymptotic estimates for the probabilities of large deviations of the form

$$P(S_n > tn^{1/p}), \qquad P(S_n < -tn^{1/p}), \qquad P(|S_n| > tn^{1/p}), \tag{1.13}$$

where $t > 0$ and $0 < p < 2$. This will be accomplished by Theorems 2.1, 2.2, and 2.3.

2. Main results. We now state our main results. Proofs will be given in Section 4.

Theorem 2.1 (two-sided large deviation probabilities). *Let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued i.i.d. random variables, and let $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. If, for some $0 < p < 2$,*

$$\frac{S_n}{n^{1/p}} \xrightarrow{P} 0, \tag{2.1}$$

and if there exists a positive nonincreasing function $\phi(x)$ on $[0,\infty)$ which is regularly varying with index $\alpha \leq -1$ such that

$$\lim_{x\to\infty} x\phi(x) = 0, \qquad \limsup_{x\to\infty} \frac{P(|X| > x^{1/p})}{\phi(x)} = 1, \tag{2.2}$$

then

$$\limsup_{n\to\infty} \frac{P(|S_n| > tn^{1/p})}{n\phi(n)} = t^{p\alpha} \quad \forall t > 0. \tag{2.3}$$

Theorem 2.2 (one-sided large deviation probabilities). (i) *If, for some $0 < p < 2$, (2.1) holds, and if there exists a positive nonincreasing function $\phi_1(x)$ on $[0,\infty)$ which*

is regularly varying with index $\alpha \le -1$ *such that*

$$\lim_{x\to\infty} x\phi_1(x) = 0, \qquad \limsup_{x\to\infty} \frac{P(X > x^{1/p})}{\phi_1(x)} = 1,$$
$$\lim_{x\to\infty} \frac{xP(|X| > x^{1/p})}{(x\phi_1(x))^{1/m}} = 0 \quad \text{for some integer } m \ge 1, \tag{2.4}$$

then

$$\limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} = t^{p\alpha} \quad \forall t > 0. \tag{2.5}$$

(ii) *If, for some* $0 < p < 2$, *(2.1) holds and if there exists a positive nonincreasing function* $\phi_2(x)$ *on* $[0,\infty)$ *which is regularly varying with index* $\alpha \le -1$ *such that*

$$\lim_{x\to\infty} x\phi_2(x) = 0, \qquad \limsup_{x\to\infty} \frac{P(X < -x^{1/p})}{\phi_2(x)} = 1,$$
$$\lim_{x\to\infty} \frac{xP(|X| > x^{1/p})}{(x\phi_2(x))^{1/m}} = 0 \quad \text{for some integer } m \ge 1, \tag{2.6}$$

then

$$\limsup_{n\to\infty} \frac{P(S_n < -tn^{1/p})}{n\phi_2(n)} = t^{p\alpha} \quad \forall t > 0. \tag{2.7}$$

Theorem 2.3. *If, for some* $0 < p < 2$, *(2.1) holds and the conditions (2.2), (2.4), or (2.6), respectively, are satisfied with* "lim" *in place of* "lim sup," *then*

$$\lim_{n\to\infty} \frac{P(t_1 n^{1/p} < |S_n| \le t_2 n^{1/p})}{n\phi(n)} = t_1^{p\alpha} - t_2^{p\alpha} \quad \forall 0 < t_1 < t_2 \le \infty, \tag{2.8}$$
$$\lim_{n\to\infty} \frac{P(t_1 n^{1/p} < S_n \le t_2 n^{1/p})}{n\phi_1(n)} = t_1^{p\alpha} - t_2^{p\alpha} \quad \forall 0 < t_1 < t_2 \le \infty, \tag{2.9}$$

or

$$\lim_{n\to\infty} \frac{P(-t_2 n^{1/p} \le S_n < -t_1 n^{1/p})}{n\phi_2(n)} = t_1^{p\alpha} - t_2^{p\alpha} \quad \forall 0 < t_1 < t_2 \le \infty, \tag{2.10}$$

respectively.

Example 2.4. Consider a sequence of real-valued i.i.d. random variables $\{X, X_n;\ n \ge 1\}$ with the following common probability density function:

$$f(x) = \frac{1}{x^3} I_{\{x\ge 1\}} + \frac{3}{2x^4} I_{\{x\le -1\}}, \quad -\infty < x < \infty. \tag{2.11}$$

Then (1.7) fails but

$$E(X) = 0.25, \qquad E(|X|^p) < \infty \quad \forall 0 < p < 2. \tag{2.12}$$

Thus, by the Marcinkiewicz-Zygmund strong law of large numbers (SLLN) (see, e.g., Chow and Teicher [8, page 125]),

$$\frac{S_n - 0.25n}{n^{1/p}} \longrightarrow 0 \text{ a.s.} \tag{2.13}$$

Note that for $0 < p < 2$ and $x \geq 1$,

$$\begin{gathered} P(X > x) = 0.5x^{-2}, \qquad P(X < -x) = 0.5x^{-3}, \\ P(|X| > x) = 0.5(x^{-2} + x^{-3}) \end{gathered} \tag{2.14}$$

whence, for all large x,

$$\begin{aligned} P(X - 0.25 > x^{1/p}) &= 0.5(0.25 + x^{1/p})^{-2} \sim 0.5x^{-2/p}, \\ P(X - 0.25 < -x^{1/p}) &= 0.5(x^{1/p} - 0.25)^{-3} \sim 0.5x^{-3/p}, \\ P(|X - 0.25| > x^{1/p}) &= 0.5(0.25 + x^{1/p})^{-2} + 0.5(x^{1/p} - 0.25)^{-3} \sim 0.5x^{-2/p}. \end{aligned} \tag{2.15}$$

Thus by Theorem 2.3, for any $0 < p < 2$, we have the following conclusions:

$$\begin{aligned} \lim_{n\to\infty} n^{(2/p)-1} P(S_n > 0.25n + tn^{1/p}) &= 0.5t^{-2} \quad \forall t > 0, \\ \lim_{n\to\infty} n^{(3/p)-1} P(S_n < 0.25n - tn^{1/p}) &= 0.5t^{-3} \quad \forall t > 0, \\ \lim_{n\to\infty} n^{(2/p)-1} P(|S_n - 0.25n| > tn^{1/p}) &= 0.5t^{-2} \quad \forall t > 0. \end{aligned} \tag{2.16}$$

Example 2.5. Let $0 < p < 2$ and let $\{X, X_n;\ n \geq 1\}$ be a sequence of real-valued symmetric i.i.d. random variables such that

$$P(|X| > x) \sim \frac{L(x^p)}{x^{\beta p}} \quad \text{as } x \longrightarrow \infty, \tag{2.17}$$

where $L(x)$ is a positive function on $[0, \infty)$ which is slowly varying as $x \to \infty$ and either $\beta = 1$ and $L(x) \to 0$ as $x \to \infty$ or $\beta > 1$. Then

$$xP(|X| > x^{1/p}) \sim \frac{L(x)}{x^{\beta-1}} \longrightarrow 0 \quad \text{as } x \longrightarrow \infty \tag{2.18}$$

and (2.1) then holds by the Klass and Teicher [17] WLLN analogue of Feller's [12] famous generalization of the Marcinkiewicz-Zygmund SLLN. Since $\phi(x) \equiv P(|X| > x^{1/p}) = L(x)/x^{\beta}$ is regularly varying with index $-\beta \leq -1$ and (2.2) holds with "lim" in place of "lim sup," it follows from Theorem 2.3 that

$$\lim_{n\to\infty} \frac{P(|S_n| > tn^{1/p})}{nL(n)/n^{\beta}} = \lim_{n\to\infty} \frac{n^{\beta-1} P(|S_n| > tn^{1/p})}{L(n)} = t^{-p\beta}. \tag{2.19}$$

It may be noted that $E(|X|^r) = \infty$ whenever $r > \beta p$ whence (1.7) fails. Moreover, $S_n/n^{1/p}$ does not necessarily converge to 0 a.s. To see this, suppose $\beta = 1$ and $L(x) \sim 1/\log x$

as $x \to \infty$. Then for x_0 sufficiently large,

$$\begin{aligned} E(|X|^p) &\geq \text{const.} \int_{x_0}^{\infty} \frac{x^{p-1}L(x^p)}{x^p}\,dx \\ &\geq \text{const.} \int_{x_0}^{\infty} \frac{1}{x\log x}\,dx \\ &= \infty. \end{aligned} \tag{2.20}$$

Thus by the (converse of the) Marcinkiewicz-Zygmund SLLN, $S_n/n^{1/p}$ does not converge to 0 a.s.

3. Auxiliary results. In this section, we collect some auxiliary results needed for the proofs of our main results. We need some additional notation. Let $\kappa_q(Y)$ denote a quantile of order q, $0 < q < 1$, for a real-valued random variable Y. The inequalities provided by (3.1) and (3.2) are extensions of the classical Lévy inequalities wherein $q = 1/2$.

Lemma 3.1. *Let $\{V_k;\ 1 \leq k \leq n\}$ be a set of n real-valued independent random variables and set $U_0 = 0$ and $U_k = V_1 + \cdots + V_k$, $1 \leq k \leq n$. Then, for every q in the interval $(0,1)$ and every real x,*

$$P\Big(\max_{1\leq k\leq n}(U_k + \kappa_{1-q}(U_n - U_k)) > x\Big) \leq \frac{1}{q}P(U_n > x), \tag{3.1}$$

$$P\Big(\max_{1\leq k\leq n}(|U_k| - \kappa_q(|U_n - U_k|)) > x\Big) \leq \frac{1}{q}P(|U_n| > x), \tag{3.2}$$

$$P\Big(\max_{1\leq k\leq n}(V_k + \kappa_{1-q}(U_{k-1})) > x\Big) \leq \frac{1}{q}P\Big(\max_{1\leq k\leq n} U_k > x\Big), \tag{3.3}$$

$$P\Big(\max_{1\leq k\leq n}(|V_k| - \kappa_q(|U_{k-1}|)) > x\Big) \leq \frac{1}{q}P\Big(\max_{1\leq k\leq n} |U_k| > x\Big). \tag{3.4}$$

Proof. Note that we can put $\kappa_{1-q}(U_n - U_k) = -\kappa_q(U_k - U_n)$, $1 \leq k \leq n$, and so (3.1) is due to Petrov [24]; also see [26, Theorem 2.1]. Using almost the same argument as in the proof of (3.1) given by Petrov [24], (3.2) follows. Our proof of (3.3) is also a modification of Petrov's [24] proof of (3.1). We write

$$\begin{gathered} M_j = \max_{j\leq k\leq n}(V_k + \kappa_{1-q}(U_{k-1})), \quad j = 1,2,\ldots,n, \\ D_n = \{V_n + \kappa_{1-q}(U_{n-1}) > x\}, \\ D_j = \{M_{j+1} \leq x,\ V_j + \kappa_{1-q}(U_{j-1}) > x\}, \quad j = 1,2,\ldots,n-1, \\ E_j = \{U_{j-1} - \kappa_{1-q}(U_{j-1}) \geq 0\}, \quad j = 1,2,\ldots,n. \end{gathered} \tag{3.5}$$

We then have

$$\Big\{\max_{1\leq k\leq n}(V_k - \kappa_{1-q}(U_{k-1})) > x\Big\} = \{M_1 > x\} = \cup_{j=1}^{n} D_j \tag{3.6}$$

and, noting that $\{D_j;\ 1 \le j \le n\}$ is a disjoint collection of events,

$$P(M_1 > x) = \sum_{j=1}^{n} P(D_j). \tag{3.7}$$

Furthermore,

$$P(E_j) \ge q, \quad j = 1,2,\dots,n. \tag{3.8}$$

Note that the events D_j and E_j are independent and

$$D_j \cap E_j \subseteq \{U_j > x\} \subseteq \left\{\max_{1\le k\le n} U_k > x\right\}, \quad j = 1,2,\dots,n. \tag{3.9}$$

Hence

$$P\left(\max_{1\le k\le n} U_k > x\right) \ge P(\cup_{k=1}^{n}(D_k \cap E_k)) = \sum_{j=1}^{n} P(D_j \cap E_j) = \sum_{j=1}^{n} P(D_j)P(E_j). \tag{3.10}$$

Taking into account (3.7) and (3.8), we conclude that

$$P\left(\max_{1\le k\le n} U_k > x\right) \ge q\sum_{j=1}^{n} P(D_j) = qP(M_1 > x), \tag{3.11}$$

proving (3.3). Inequality (3.4) follows from the same procedure as above if $\{M_j;\ 1 \le j \le n\}$, $\{D_j;\ 1 \le j \le n\}$, and $\{E_j;\ 1 \le j \le n\}$ are defined, respectively, by

$$\begin{gathered} M_j = \max_{j\le k\le n}(|V_k| - \kappa_q(|U_{k-1}|)), \quad j = 1,2,\dots,n, \\ D_n = \{|V_n| - \kappa_q(|U_{n-1}|) > x\}, \\ D_j = \{M_{j+1} \le x,\ |V_j| - \kappa_q(|U_{j-1}|) > x\}, \quad j = 1,2,\dots,n-1, \\ E_j = \{|U_{j-1}| - \kappa_q(|U_{j-1}|) \le 0\}, \quad j = 1,2,\dots,n. \end{gathered} \tag{3.12}$$

The lemma is proved. □

The following result due to Li [22] is a generalization of the Hoffmann-Jørgensen [15] inequalities.

Lemma 3.2. *Under the conditions of Lemma 3.1, for all $n \ge 1$ and all $x, y, z \ge 0$,*

$$P(U_n > x + y + z) \le P\left(\max_{1\le k\le n} V_k > x\right) + 4P(U_n > y - \mu_n^{(1)})P(U_n > z - \mu_n^{(2)}), \tag{3.13}$$

$$P(|U_n| > x + y + z) \le P\left(\max_{1\le k\le n} |V_k| > x\right) + 4P(|U_n| > y - \mu_n^{(1)})P(|U_n| > z - \mu_n^{(2)}), \tag{3.14}$$

where $\mu_n^{(1)} = \max_{1\le k\le n}\kappa_{1/2}(|U_n - U_k|)$ and $\mu_n^{(2)} = \max_{1\le k\le n}\kappa_{1/2}(|U_k|)$, $n \ge 1$.

4. Proofs of main results. We only give the proof of Theorem 2.2. The relations (2.3), (2.8), and (2.9) can be proved in the same vein. Replacing $\{X, X_n;\ n \geq 1\}$ by $\{-X, -X_n;\ n \geq 1\}$ in (2.5) and (2.9) yields, respectively, (2.7) and (2.10). The proof of Theorem 2.2 will be broken down into the following four steps.

Step 1. We show that (2.1) and (2.4) imply that there exists an integer $v \geq 1$ such that

$$P\left(\frac{|S_n|}{n^{1/p}} > \varepsilon\right) = O((n\phi_1(n))^{1/v}) \quad \forall \varepsilon > 0. \tag{4.1}$$

To see this, write

$$X_{n,i} = X_i I_{\{|X_i| \leq n^{1/p}\}}, \quad i = 1, 2, \ldots, n, \qquad T_n = \sum_{i=1}^{n} X_{n,i}, \quad n \geq 1. \tag{4.2}$$

Then it follows from (2.1) that (see, e.g., Chow and Teicher [8, page 359])

$$\lim_{n\to\infty} E\left(\frac{T_n}{n^{1/p}}\right) = \lim_{n\to\infty} n^{1-(1/p)} E(X I_{\{|X| \leq n^{1/p}\}}) = 0, \tag{4.3}$$

and (2.4) ensures that

$$P\left(\frac{|S_n - T_n|}{n^{1/p}} > \varepsilon\right) \leq nP(|X| > n^{1/p}) = o((n\phi_1(n))^{1/m}) \quad \forall \varepsilon > 0. \tag{4.4}$$

Now it follows from (2.4) and [13, Theorem 1(b), page 281] that

$$\begin{aligned}
P\left(\frac{|T_n - E(T_n)|}{n^{1/p}} > \varepsilon\right) &\leq \frac{2n}{\varepsilon^2 n^{2/p}} E(X^2 I_{\{|X| \leq n^{1/p}\}}) \\
&\leq \frac{4}{\varepsilon^2 n^{(2/p)-1}} \int_0^{n^{1/p}} x P(|X| > x)\, dx \\
&= \frac{4}{p\varepsilon^2 n^{(2/p)-1}} \int_0^{n} y^{1/p} P(|X| > y^{1/p}) y^{(1/p)-1}\, dy \\
&= O\left(n^{1-(2/p)} \int_0^{n} y^{(2/p)-2} (y\phi_1(y))^{1/m}\, dy\right) \\
&= O\left(n^{1-(2/p)} \int_0^{n} y^{(2/p)-2} (y\phi_1(y))^{1/v}\, dy\right) \\
&= O(n^{1-(2/p)} n^{(2/p)-2} (n\phi_1(n))^{1/v} n) \\
&= O((n\phi_1(n))^{1/v}),
\end{aligned} \tag{4.5}$$

where $v \geq m$ is an integer such that $2/p - 1 + (1+\alpha)/v > 0$. Thus, the relation (4.1) follows from (4.3), (4.4), and (4.5).

Step 2. Set $S_0 = 0$ and note that since the X_n, $n \geq 1$, are i.i.d.,

$$\mu_n^{(1)} = \max_{0 \leq k \leq n-1} \kappa_{1/2}(|S_n - S_k|) = \max_{1 \leq k \leq n} \kappa_{1/2}(|S_k|) = \mu_n^{(2)}, \quad n \geq 1, \tag{4.6}$$

and that (2.1) guarantees that

$$\lim_{n\to\infty} \frac{\mu_n^{(2)}}{n^{1/p}} = 0. \tag{4.7}$$

Then for every $\varepsilon > 0$, it follows from (3.13) and (2.4) that

$$\begin{aligned} P\left(\frac{S_n}{n^{1/p}} > \varepsilon\right) &\le P\left(\max_{1\le k\le n} X_k > \frac{\varepsilon}{3} n^{1/p}\right) + 4\left(P\left(\frac{S_n}{n^{1/p}} > \frac{\varepsilon}{3} - \frac{\mu_n^{(2)}}{n^{1/p}}\right)\right)^2 \\ &= O(n\phi_1(n)) + 4\left(P\left(\frac{S_n}{n^{1/p}} > \frac{\varepsilon}{3} - \frac{\mu_n^{(2)}}{n^{1/p}}\right)\right)^2. \end{aligned} \tag{4.8}$$

In view of (4.7), it follows by repeating $v-1$ times the above procedure for arriving at (4.8) that for every $\varepsilon > 0$ and all sufficiently small $\varepsilon_1 > 0$ and sufficiently large n,

$$\begin{aligned} P\left(\frac{S_n}{n^{1/p}} > \varepsilon\right) &\le O(n\phi_1(n)) + O\left(\left(P\left(\frac{S_n}{n^{1/p}} > \varepsilon_1\right)\right)^{2^v}\right) \\ &= O(n\phi_1(n)) + O\left((n\phi_1(n))^{2^v/v}\right) \quad \text{(by (4.1))} \\ &= O(n\phi_1(n)). \end{aligned} \tag{4.9}$$

Step 3. For every given $t > 0$, applying (3.13) again, it follows from (4.7) and (4.9) that for every $0 < \varepsilon < t$,

$$\begin{aligned} P\left(\frac{S_n}{n^{1/p}} > t\right) &\le P\left(\max_{1\le k\le n} X_k > (t-\varepsilon)n^{1/p}\right) + 4\left(P\left(\frac{S_n}{n^{1/p}} > \frac{\varepsilon}{2} - \frac{\mu_n^{(2)}}{n^{1/p}}\right)\right)^2 \\ &\le nP(X > (t-\varepsilon)n^{1/p}) + O((n\phi_1(n))^2). \end{aligned} \tag{4.10}$$

Since $\phi_1(x)$ is regularly varying with index $\alpha \le -1$ and since $\limsup_{x\to\infty} P(X > x^{1/p})/\phi_1(x) = 1$ and $\lim_{x\to\infty} x\phi_1(x) = 0$, it follows from (4.10) that

$$\limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} \le (t-\varepsilon)^{p\alpha}. \tag{4.11}$$

Letting $\varepsilon \downarrow 0$ yields

$$\limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} \le t^{p\alpha}. \tag{4.12}$$

Step 4. For $n \ge 1$ and $0 < q < 1$, write

$$g(n,q) = \min_{0\le k\le n-1} \kappa_{1-q}(S_k), \tag{4.13}$$

where $S_0 = 0$. Then we see from (2.1) that

$$\lim_{n\to\infty} \frac{g(n,q)}{n^{1/p}} = 0 \quad \forall 0 < q < 1. \tag{4.14}$$

For every given $t > 0$, by applying Lemma 3.1, we have

$$\begin{aligned} P\Big(\max_{1\le k\le n} X_k > tn^{1/p}\Big) &\le P\Big(\max_{1\le k\le n} (X_k + \kappa_{1-q}(S_{k-1})) > tn^{1/p} + g(n,q)\Big) \\ &\le \frac{1}{q} P\Big(\max_{1\le k\le n} S_k > tn^{1/p} + g(n,q)\Big) \quad \text{(by (3.3))} \\ &\le \frac{1}{q} P\Big(\max_{1\le k\le n} (S_k + \kappa_{1-q}(S_{n-k})) > tn^{1/p} + 2g(n,q)\Big) \\ &= \frac{1}{q} P\Big(\max_{1\le k\le n} (S_k + \kappa_{1-q}(S_n - S_k)) > tn^{1/p} + 2g(n,q)\Big) \\ &\le \frac{1}{q^2} P(S_n > tn^{1/p} + 2g(n,q)) \quad \text{(by (3.1))} \end{aligned} \tag{4.15}$$

and this, together with (4.14), ensures that for every $\varepsilon > 0$,

$$\begin{aligned} \limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} &\ge \limsup_{n\to\infty} \frac{P(S_n > (t+\varepsilon)n^{1/p} + 2g(n,q))}{n\phi_1(n)} \\ &\ge q^2 \limsup_{n\to\infty} \frac{P(\max_{1\le k\le n} X_k > (t+\varepsilon)n^{1/p})}{n\phi_1(n)}. \end{aligned} \tag{4.16}$$

We now show that

$$\lim_{n\to\infty} \frac{P(\max_{1\le k\le n} X_k > tn^{1/p})}{nP(X > tn^{1/p})} = 1 \quad \forall t > 0. \tag{4.17}$$

Note that

$$\begin{aligned} \frac{P(\max_{1\le k\le n} X_k > tn^{1/p})}{nP(X > tn^{1/p})} &= \frac{1 - (1 - P(X > tn^{1/p}))^n}{nP(X > tn^{1/p})} \\ &\ge \frac{1 - \exp\{-nP(X > tn^{1/p})\}}{nP(X > tn^{1/p})} \\ &\longrightarrow 1 \end{aligned} \tag{4.18}$$

by L'Hospital's rule since (2.4) ensures that

$$xP(X > x^{1/p}) = x\phi_1(x)\frac{P(X > x^{1/p})}{\phi_1(x)} \longrightarrow 0 \quad \text{as } x \longrightarrow \infty. \tag{4.19}$$

Thus,

$$\liminf_{n\to\infty} \frac{P(\max_{1\le k\le n} X_k > tn^{1/p})}{nP(X > tn^{1/p})} \ge 1 \tag{4.20}$$

whereas

$$\limsup_{n\to\infty} \frac{P(\max_{1\le k\le n} X_k > tn^{1/p})}{nP(X > tn^{1/p})} \le 1 \tag{4.21}$$

is immediate thereby establishing (4.17). Since $\phi_1(x)$ is regularly varying with index $\alpha \le -1$ and $\limsup_{x\to\infty} P(X > x^{1/p})/\phi_1(x) = 1$, we have

$$\limsup_{n\to\infty} \frac{P(X > tn^{1/p})}{\phi_1(n)} = t^{\alpha p} \quad \forall t > 0. \tag{4.22}$$

Then it follows from (4.16), (4.17), and (4.22) that

$$\limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} \geq q^2(t+\varepsilon)^{\alpha p} \quad \forall 0 < q < 1,\ \forall \varepsilon > 0. \tag{4.23}$$

Letting $q \uparrow 1$ and $\varepsilon \downarrow 0$ gives

$$\limsup_{n\to\infty} \frac{P(S_n > tn^{1/p})}{n\phi_1(n)} \geq t^{\alpha p} \quad \forall t > 0, \tag{4.24}$$

which, when combined with (4.12), establishes (2.5).

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