MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Let $\Sigma^*(\alpha, \beta, k)$ be a class of meromorphic functions $f(z)$ with positive coefficients in $D = \{0 < |z| < 1\}$. The aim of the present note is to prove some properties for the class $\Sigma^*(\alpha, \beta, k)$.

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1. Introduction. Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the punctured disc $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1. A function $f(z) \in \Sigma$ is said to be meromorphically starlike of order α if it satisfies the following:

$$\operatorname{Re} \left(-\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in D) \quad (1.2)$$

for some α ($0 \leq \alpha < 1$). We say that $f(z)$ is in the class $\Sigma^*(\alpha)$ of such functions. Similarly a function $f(z) \in \Sigma$ is said to be meromorphically convex of order α if it satisfies the following:

$$\operatorname{Re} \left(-1 - \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in D) \quad (1.3)$$

for some α ($0 \leq \alpha < 1$). We say that $f(z)$ is in the class $\Sigma_c(\alpha)$ of such functions.

The class $\Sigma^*(\alpha)$ and various other subclasses of Σ have been studied rather extensively by Nehari and Netanyahu [9], Clunie [4], Pommerenke [11, 12], Miller [7], Royster [13], and others (cf., e.g., Bajpai [2], Goel and Sohi [6], Mogra et al. [8], Uralegaddi and Ganigi [15], Cho et al. [3], Aouf [1], and Uralegaddi and Somanatha [16, 17]; see also Duren [5, pages 29 and 137], and Srivastava and Owa [14, pages 86 and 429]).

Owa and Pascu [10] obtained some coefficient properties for the class $\Sigma^*(\alpha, k)$ which satisfies

$$\left| \frac{zf'(z)}{f(z)} + k \right| < \left| \frac{zf'(z)}{f(z)} + (2\alpha - k) \right| \quad (1.4)$$

for some α ($\alpha > 0$), k ($0 \leq k \leq 1$), and for all $z \in D$.

In this note, the above definition is extended and we obtain the extended coefficient inequalities.

The extended class of functions is defined as follows.

DEFINITION 1.1. A function $f(z) \in \Sigma$ is said to be a member of the class $\Sigma^*(\alpha, \beta, k)$ if it satisfies

$$\frac{1}{\beta} \left| \frac{zf'(z)}{f(z)} + k \right| < \left| \frac{zf'(z)}{f(z)} + (2\alpha - k) \right| \quad (1.5)$$

for some α ($\alpha > 0$), β ($0 < \beta \leq 1$), k ($0 \leq k \leq 1$), and for all $z \in D$.

2. Coefficient inequalities for functions. For the class $\Sigma^*(\alpha, k)$, Owa and Pascu [10] showed the following theorem.

THEOREM 2.1. Let the function $f(z)$ be defined by (1.1). If

$$\sum_{n=2}^{\infty} (n+k+|2\alpha+n-k|) |a_n| r^{n+1} \leq 2(1-\alpha) \quad (2.1)$$

for some k ($0 \leq k \leq 1$) and α ($0 \leq \alpha < 1$), then $f(z) \in \Sigma^*(\alpha, k)$.

Our first result for functions $f(z) \in \Sigma^*(\alpha, \beta, k)$ is given as the following theorem.

THEOREM 2.2. Let the function $f(z)$ be defined by (1.1). If

$$\sum_{n=2}^{\infty} (n+k+\beta|2\alpha+n-k|) |a_n| r^{n+1} \leq \beta(k+1-2\alpha)+1-k \quad (2.2)$$

for some k ($0 \leq k \leq 1$), α ($0 \leq \alpha < 1$), and β ($0 < \beta \leq 1$), then $f(z) \in \Sigma^*(\alpha, \beta, k)$.

PROOF. Using the same technique as in [10], we know that for $f \in \Sigma^*(\alpha, \beta, k)$,

$$\begin{aligned} & |zf'(z) + kf(z) - \beta|zf'(z) + (2\alpha - k)f(z)| \\ &= \left| (k-1)\frac{1}{z} + \sum_{n=0}^{\infty} (n+k)a_n z^n \right| - \beta \left| (2\alpha - k - 1)\frac{1}{z} + \sum_{n=0}^{\infty} (2\alpha + n - k)a_n z^n \right|. \end{aligned} \quad (2.3)$$

Therefore, applying the condition of the theorem, we have

$$\begin{aligned} & r|zf'(z) + kf(z) - \beta|zf'(z) + (2\alpha - k)f(z)| \\ & \leq (k-1) + \sum_{n=0}^{\infty} (n+k) |a_n| r^{n+1} - \beta(k+1-2\alpha) \frac{1}{r} \\ & \quad + \sum_{n=0}^{\infty} \beta|2\alpha + n - k| |a_n| r^{n+1} \\ & = k-1 - \beta(k+1-2\alpha) + \sum_{n=0}^{\infty} \{(n+k) + \beta|2\alpha + n - k|\} |a_n| r^{n+1} \\ & \leq 0, \end{aligned} \quad (2.4)$$

which shows that

$$\sum_{n=0}^{\infty} \{(n+k) + \beta|2\alpha+n-k|\} |a_n| r^{n+1} \leq \beta(k+1-2\alpha) + 1-k. \quad (2.5)$$

It follows from the above that

$$\frac{1}{\beta} \left| \frac{zf'(z)/f(z) + k}{zf'(z)/f(z) + (2\alpha - k)} \right| \leq 1. \quad (2.6)$$

The function $f(z)$ given by

$$f_n(z) = \frac{1}{z} + a_0 + \frac{\{\beta(k+1-2\alpha) + 1-k\} z^n}{(n+k) + \beta|2\alpha+n-k|} \quad (n \geq 1) \quad (2.7)$$

belongs to the class $\Sigma^*(\alpha, \beta, k)$.

This completes the proof of the theorem. $\square$

COROLLARY 2.3. *Let the function $f(z)$ be defined by (1.1) and let $f(z) \in \Sigma$. If $f \in \Sigma^*(\alpha, \beta, k)$, then*

$$a_n \leq \frac{\beta(k+1-2\alpha) + 1-k}{(n+k) + \beta|2\alpha+n-k|}, \quad n \geq 0. \quad (2.8)$$

The result is sharp for functions $f_n(z)$ given by (2.7).

REMARK 2.4. If $f \in \Sigma^*(\alpha, \beta, k)$ with $a_0 = 0$, then Corollary 2.3 is true for some β ($0 < \beta \leq (k-1)/(k+1-2\alpha) \leq 1$) and α ($0 \leq \alpha \leq (k+1)/2 < 1$).

If $\beta = 1$, we get the following corollary.

COROLLARY 2.5 (see [10]). *Let the function $f(z)$ be defined by (1.1) and let $f(z) \in \Sigma$. If $f \in \Sigma^*(\alpha, 1, k)$, then*

$$a_n \leq \frac{1-\alpha}{n+\alpha}, \quad n \geq 0, \quad (2.9)$$

for some α ($1/2 \leq \alpha < 1$).

3. Distortion theorem. A distortion property for functions in the class $\Sigma^*(\alpha, \beta, k)$ is given as follows.

THEOREM 3.1. *If the function $f(z)$ defined by (1.1) is in the class $\Sigma^*(\alpha, \beta, k)$, then for $0 < |z| = r < 1$,*

$$\begin{aligned} \frac{1}{r} - a_0 - \frac{\{\beta(k+1-2\alpha) + 1-k\} r}{(1+k) + \beta|2\alpha+1-k|} \\ \leq |f(z)| \leq \frac{1}{r} + a_0 + \frac{\{\beta(k+1-2\alpha) + 1-k\} r}{(1+k) + \beta|2\alpha+1-k|}, \end{aligned} \quad (3.1)$$

with equality for

$$\begin{aligned} f_1(z) &= \frac{1}{z} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}z}{(1+k)+\beta|2\alpha+1-k|} \quad (z = ir, r), \\ \frac{1}{r^2} - a_0 - \frac{\beta(k+1-2\alpha)+1-k}{(1+k)+\beta|2\alpha+1-k|} \\ &\leq |f'(z)| \leq \frac{1}{r^2} + a_0 + \frac{\beta(k+1-2\alpha)+1-k}{(1+k)+\beta|2\alpha+1-k|}, \end{aligned} \quad (3.2)$$

with equality for

$$f_1(z) = \frac{1}{z} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}z}{(1+k)+\beta|2\alpha+1-k|} \quad (z = \pm ir, \pm r). \quad (3.3)$$

PROOF. Since $f \in \Sigma^*(\alpha, \beta, k)$, [Theorem 2.2](#) yields the inequality

$$\sum_{n=0}^{\infty} a_n \leq \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}, \quad n \geq 0. \quad (3.4)$$

Thus, for $0 < |z| = r < 1$, and making use of (3.4), we have

$$\begin{aligned} |f(z)| &\leq \left| \frac{1}{z} \right| + a_0 + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{r} + a_0 + r \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{r} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|}, \\ |f(z)| &\geq \left| \frac{1}{z} \right| - a_0 - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{r} - a_0 - r \sum_{n=1}^{\infty} a_n \\ &\geq \frac{1}{r} - a_0 - \frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|}. \end{aligned} \quad (3.5)$$

Also from [Theorem 2.1](#) it follows that

$$\sum_{n=1}^{\infty} na_n \leq a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}. \quad (3.6)$$

Thus

$$\begin{aligned} |f'(z)| &\leq \frac{1}{|z|^2} + a_0 + \sum_{n=1}^{\infty} na_n |z|^{n-1} \leq \frac{1}{r^2} + a_0 + \sum_{n=1}^{\infty} na_n \\ &\leq \frac{1}{r^2} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}, \\ |f'(z)| &\geq \frac{1}{|z|^2} - a_0 - \sum_{n=1}^{\infty} na_n |z|^{n-1} \geq \frac{1}{r^2} - a_0 - \sum_{n=1}^{\infty} na_n \\ &\geq \frac{1}{r^2} - a_0 - \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}. \end{aligned} \quad (3.7)$$

Hence we complete the proof of [Theorem 3.1](#). □

4. Radii of starlikeness and convexity. The radii of starlikeness and convexity for the class $\Sigma^*(\alpha, \beta, k)$ are given by the following theorem.

THEOREM 4.1. *If the function $f(z)$ defined by (1.1) is in the class $\Sigma^*(\alpha, \beta, k)$, then $f(z)$ is meromorphically starlike of order ρ ($0 \leq \rho < 1$) in the disk $|z| < r_1(\alpha, \beta, \rho)$, where $r_1(\alpha, \beta, \rho)$ is the largest value for which*

$$r_1 = r_1(\alpha, \beta, \rho) = \inf_{n \geq 0} \left(\frac{(1-\rho)\{(n+k) + \beta|2\alpha + n - k|\}}{(n+2-\rho)\{\beta(k+1-2\alpha) + 1 - k\}} \right)^{1/(n+1)}. \quad (4.1)$$

The result is sharp for functions $f_n(z)$ given by (2.7).

PROOF. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \rho \quad (4.2)$$

for $|z| \leq r_1$. We have

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq \frac{\sum_{n=0}^{\infty} (n+1) |a_n| |z|^{n+1}}{1 - \sum_{n=0}^{\infty} |a_n| |z|^{n+1}} \leq 1 - \rho. \quad (4.3)$$

Hence (4.3) holds true if

$$\sum_{n=0}^{\infty} (n+1) |a_n| |z|^{n+1} \leq (1-\rho) \left(1 - \sum_{n=0}^{\infty} |a_n| |z|^{n+1} \right) \quad (4.4)$$

or

$$\sum_{n=0}^{\infty} \frac{(n+2-\rho)}{1-\rho} a_n |z|^{n+1} \leq 1; \quad (4.5)$$

with the aid of (2.8), (4.5) is true if

$$\frac{(n+2-\rho)}{1-\rho} |z|^{n+1} \leq \frac{(n+k) + \beta|2\alpha + n - k|}{\beta(k+1-2\alpha) + 1 - k}. \quad (4.6)$$

Solving (4.6) for $|z|$, we obtain

$$|z| \leq \left(\frac{(1-\rho)\{(n+k) + \beta|2\alpha + n - k|\}}{(n+2-\rho)\{\beta(k+1-2\alpha) + 1 - k\}} \right)^{1/(n+1)} \quad (n \geq 0). \quad (4.7)$$

This completes the proof of Theorem 4.1. $\square$

THEOREM 4.2. *If the function $f(z)$ defined by (1.1) is in the class $\Sigma^*(\alpha, \beta, k)$, then $f(z)$ is meromorphically convex of order ρ ($0 \leq \rho < 1$) in the disk $|z| < r_2(\alpha, \beta, \rho)$, where $r_2(\alpha, \beta, \rho)$ is the largest value for which*

$$r_2 = r_2(\alpha, \beta, \rho) = \inf_{n \geq 0} \left(\frac{(1-\rho)\{(n+k) + \beta|2\alpha + n - k|\}}{n(n+2-\rho)\{\beta(k+1-2\alpha) + 1 - k\}} \right)^{1/(n+1)}. \quad (4.8)$$

The result is sharp for functions $f_n(z)$ given by (2.7).

PROOF. We omit the details of the proof as they are very tedious. It suffices to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \rho \quad (4.9)$$

for $|z| \leq r_2$, with the aid of Theorem 2.2. Thus we have the assertion of Theorem 4.2. $\square$

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REFERENCES

- [1] M. K. Aouf, *On a certain class of meromorphic univalent functions with positive coefficients*, Rend. Mat. Appl. (7) **11** (1991), no. 2, 209–219.
- [2] S. K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roumaine Math. Pures Appl. **22** (1977), no. 3, 295–297.
- [3] N. E. Cho, S. H. Lee, and S. Owa, *A class of meromorphic univalent functions with positive coefficients*, Kobe J. Math. **4** (1987), no. 1, 43–50.
- [4] J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.
- [5] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York, 1983.
- [6] R. M. Goel and N. S. Sohi, *On a class of meromorphic functions*, Glas. Mat. Ser. III **17**(37) (1982), no. 1, 19–28.
- [7] J. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc. **25** (1970), 220–228.
- [8] M. L. Mogra, T. R. Reddy, and O. P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. **32** (1985), no. 2, 161–176.
- [9] Z. Nehari and E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1957), 15–23.
- [10] S. Owa and N. N. Pascu, *Coefficient inequalities for certain classes of meromorphically starlike and meromorphically convex functions*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 17, 1–6.
- [11] Ch. Pommerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Math. Z. **78** (1962), 263–284 (German).
- [12] ———, *On meromorphic starlike functions*, Pacific J. Math. **13** (1963), 221–235.
- [13] W. C. Royster, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. **107** (1963), 300–308.
- [14] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing, New Jersey, 1992.
- [15] B. A. Uralegaddi and M. D. Ganigi, *A certain class of meromorphically starlike functions with positive coefficients*, Pure Appl. Math. Sci. **26** (1987), no. 1-2, 75–81.
- [16] B. A. Uralegaddi and C. Somanatha, *Certain differential operators for meromorphic functions*, Houston J. Math. **17** (1991), no. 2, 279–284.
- [17] ———, *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc. **43** (1991), no. 1, 137–140.

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