A SEQUENTIAL RIESZ-LIKE CRITERION FOR THE RIEMANN HYPOTHESIS

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Received 7 June 2005 and in revised form 18 August 2005

Let $c_k := \sum_{j=0}^{k}(-1)^j\binom{k}{j}(1/\zeta(2j+2))$. We prove that the Riemann hypothesis is equivalent to $c_k \ll k^{-3/4+\epsilon}$ for all $\epsilon > 0$; furthermore, we prove that $c_k \ll k^{-3/4}$ implies that the zeros of $\zeta(s)$ are simple. This is closely related to M. Riesz's criterion which states that the Riemann hypothesis is equivalent to $\sum_{k=1}^{\infty}((-1)^{k+1}x^k/(k-1)!\zeta(2k)) \ll x^{1/4+\epsilon}$ as $x \to +\infty$, for all $\epsilon > 0$.

1. Introduction and preliminaries

The main theorem of this note is the following theorem.

Theorem 1.1. *Let*

$$c_k := \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{\zeta(2j+2)}; \tag{1.1}$$

then the Riemann hypothesis is true if and only if

$$c_k \ll k^{-3/4+\epsilon}, \quad \forall\epsilon > 0. \tag{1.2}$$

Furthermore, if $c_k \ll k^{-3/4}$, then the zeros of $\zeta(s)$ are simple.

The proof of this theorem is given in Section 3.

Remark 1.2. It will be seen below that unconditionally

$$c_k \ll k^{-1/2}, \tag{1.3}$$

and, on the other hand, that

$$c_k \not\ll k^{-3/4-\delta} \quad (\forall\delta > 0). \tag{1.4}$$

International Journal of Mathematics and Mathematical Sciences 2005:21 (2005) 3527–3537
DOI: 10.1155/IJMMS.2005.3527

Remark 1.3. It is quite obvious how one can trivially modify the proof of the theorem to obtain a more general result.

THEOREM 1.4. *A necessary and sufficient condition for* $\zeta(s) \neq 0$ *in the half-plane* $\Re(s) > 2(1-\alpha)$ *is*

$$c_k \ll k^{-\alpha+\epsilon} \quad (\forall \epsilon > 0). \tag{1.5}$$

However, we will eschew such gratuitous generalizing at this stage.

Necessary and sufficient conditions for the Riemann hypothesis depending only on values of $\zeta(s)$ at positive integers have been known for a long time, for example those of Riesz [7] and Hardy and Littlewood [3]. Riesz's criterion, for example, states that the Riemann hypothesis is true if and only if

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} = O(x^{1/4+\epsilon}) \quad (x \longrightarrow +\infty). \tag{1.6}$$

We believe our condition is new and it is definitely simpler, as it only involves finite rational combinations of the values $\zeta(2h)$, and seems well posed for numerical calculations. This work however did not originate as an attempt to simplify Riesz's criterion. It arose rather as a consequence of our note [1] on Maslanka's expression of the Riemann zeta function [4, 5] in the form

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} A_k P_k\left(\frac{s}{2}\right), \tag{1.7}$$

where

$$A_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j+1)\zeta(2j+2), \tag{1.8}$$

and the $P_k(s)$ are the *Pochhammer polynomials*

$$P_k(s) := \prod_{r=1}^{k} \left(1 - \frac{s}{r}\right), \tag{1.9}$$

which will appear prominently in the proof of Theorem 1.1. The necessary elementary facts about these polynomials are proved in Section 2.

In Section 4, we prove an *unconditional exact formula* for the coefficients c_k, stated in the following theorem, where we denote

$$R_k(\omega) := \operatorname{Res}\left(\frac{1}{\zeta(s)P_k(s/2)}; s = \omega\right). \tag{1.10}$$

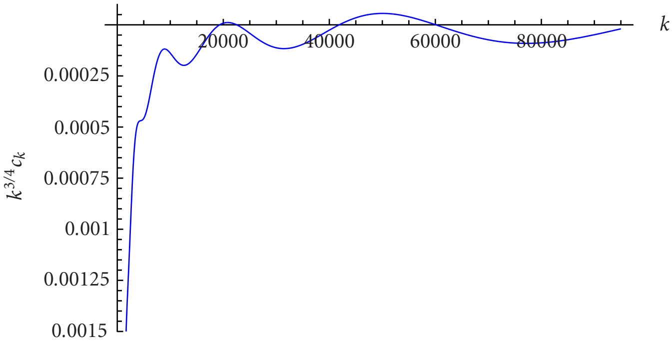

Figure 1.1

Theorem 1.5 (explicit formula). *For sufficiently large k,*

$$-2kc_{k-1} = \lim_{\nu\to\infty} \sum_{|\Im\rho|<T_\nu} R_k(\rho) + o(1), \tag{1.11}$$

where $\{T_\nu\}_{\nu=0}^{\infty}$ *is a certain sequence satisfying* $\nu < T_\nu < \nu+1$, *and the* ρ *denote complex zeta zeros. If simple zeros are assumed, then the above series becomes*

$$-2kc_{k-1} = \lim_{\nu\to\infty} \sum_{|\Im\rho|<T_\nu} \frac{1}{\zeta'(\rho)P_k(\rho/2)} + o(1). \tag{1.12}$$

Remark 1.6. The $o(1)$ can be written explicitly in terms of the trivial zeros $-2,-4,\dots$ of $\zeta(s)$. This representation is the initial step to prove this *conjecture* that the condition $c_k \ll k^{-3/4}$ is both *a necessary and sufficient condition for the Riemann hypothesis with simple zeros*. The sufficiency is indeed true as will be seen below.

Some extended numerical computations of the c_k were kindly carried out for the author by K. Maslanka (personal communication). They seem to indicate good agreement with even $c_k \ll k^{-3/4}$. In fact, one sees $c_k k^{3/4}$ increase monotonically at first and then begin to oscillate around zero, where the wavelength fits well with the imaginary part of the first critical zero of $\zeta(s)$, see Figure 1.1.

2. Elementary properties of the Pochhammer polynomials

We begin with

$$(-1)^k \binom{\frac{s}{2}-1}{k} = P_k\left(\frac{s}{2}\right), \tag{2.1}$$

which is essentially a matter of notation.

The proofs of the following lemmas are rather standard, but we give them here for the sake of completeness.

The kth-degree polynomial $P_k(s)$ grows like s^k for large $|s|$; more precisely, we can state the following lemma.

Lemma 2.1.

$$|P_k(s)| > \frac{|s|^k}{k!2^k} \quad (|s| > 2k). \tag{2.2}$$

Proof. The condition $|s| > 2k$ implies that $|1 - r/s| > 1/2$ for $r = 1,2,\ldots,k$, thus

$$|P_k(s)| = \prod_{r=1}^{k} \left| \frac{r-s}{r} \right| = \frac{|s|^k}{k!} \prod_{r=1}^{k} \left| 1 - \frac{r}{s} \right| > \frac{|s|^k}{k!2^k}. \tag{2.3}$$

□

The following is the fundamental limit relation connecting the Pochhammer polynomials to the gamma function.

Lemma 2.2. *Uniformly on compact sets,*

$$\lim_{k\to\infty} P_k(s)k^s = \frac{1}{\Gamma(1-s)}. \tag{2.4}$$

Proof. From $\sum_{r=1}^{k}(1/r) = \gamma + \log k + O(1/k)$ and the infinite product for the gamma function, we obtain

$$P_k(s)k^s = e^{sO(1/k)}e^{-\gamma s}\prod_{r=1}^{k}\left(1 - \frac{s}{r}\right)e^{s/r} \longrightarrow \frac{1}{-s\Gamma(-s)} = \frac{1}{\Gamma(1-s)}, \tag{2.5}$$

the convergence being *uniform on compact sets* of the plane. □

An immediate corollary of Lemma 2.2 is the following lemma.

Lemma 2.3. *For every compact set $H \subset \mathbb{C}$, there is a positive constant C_H, not depending on k, such that*

$$|P_k(s)| \le C_H k^{-\Re(s)} \quad (s \in A,\ k = 1,2,\ldots). \tag{2.6}$$

Proof. Write the uniform limit (2.4) as

$$\left| P_k(s)k^s - \frac{1}{\Gamma(1-s)} \right| \le \epsilon_H(k) \longrightarrow 0 \quad (s \in H,\ k = 1,2,\ldots); \tag{2.7}$$

therefore

$$|P_k(s)| \le \left(\frac{1}{|\Gamma(1-s)|} + \epsilon_H(k) \right) k^{-\Re(s)}. \tag{2.8}$$

□

Lemma 2.4. *For* $\Re(s) < 0$,

$$\frac{1}{|P_k(s)|} > \frac{1}{|P_{k+1}(s)|} \quad (k \geq 1), \qquad \frac{1}{|P_k(s)|} \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{2.9}$$

Proof. Lemma 2.2 clearly implies that

$$\frac{1}{|P_k(s)|} \longrightarrow 0 \quad (\Re(s) < 0), \tag{2.10}$$

and for $\Re s < 0$, the trivial inequality $|w| \geq |\Re w|$ yields

$$\left| \frac{P_{k+1}(s)}{P_k(s)} \right| = \left| 1 - \frac{s}{k+1} \right| \geq 1 - \frac{\Re(s)}{k+1} > 1, \tag{2.11}$$

so the sequence $1/P_k(s)$ is strictly decreasing. □

The next lemma establishes an interesting connection between the partial fraction decomposition of $1/P_k(s)$ and the iterated forward difference operator involved in definition (1.1).

Lemma 2.5.

$$\frac{1}{P_k(s)} = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{j}{s-j}, \quad k \geq 1. \tag{2.12}$$

The proof of this lemma is an elementary exercise in computing

$$\operatorname{Res}\left(\frac{1}{P_k(s)}; s = j \right) = (-1)^j \binom{k}{j} j, \quad j = 1, 2, \ldots, k. \tag{2.13}$$

3. Proof of the main theorem (Theorem 1.1)

3.1. Proof of sufficiency. The sufficiency of the condition (1.5) follows from writing $1/\zeta(s)$ as a series of Pochhammer polynomials. We state it as a separate proposition as we believe it deserves special attention.

Proposition 3.1. *If* $c_k \ll k^{-3/4+1/2\epsilon}$ *for any* $\epsilon > 0$, *then*

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k\left(\frac{s}{2}\right), \quad \Re(s) > \frac{1}{2}, \tag{3.1}$$

where the series converges uniformly on compact subsets of the half-plane. A fortiori, $\zeta(s)$ *does not vanish for* $\Re(s) > 1/2$.

Remark 3.2. Since it will be shown that actually $c_k \ll k^{-1/2}$, it follows rather trivially that the representation (3.1) for $1/\zeta(s)$ is *unconditionally* valid at least in the half-plane

$\Re(s) > 1$. On the other hand, as announced in (1.4), Proposition 3.1 shows that

$$c_k \not\ll k^{-3/4-\delta} \quad (\forall \delta > 0), \tag{3.2}$$

since the contrary statement would imply by (3.1) that $\zeta(s)$ has no zeros on the critical line.

We need a lemma before proving Proposition 3.1.

Lemma 3.3. *Define*

$$q_k := \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 - \frac{1}{n^2}\right)^k, \tag{3.3}$$

then

$$q_k \ll k^{-1/2}. \tag{3.4}$$

Proof. The contribution of the terms with $n > \sqrt{k}$ is trivially $\ll k^{-1/2}$, whereas the contribution of the remaining terms is

$$\sum_{n \le \sqrt{k}} \frac{1}{n^2}\left(1 - \frac{1}{n^2}\right)^k \ll \int_1^{\infty} e^{-k/x^2} \frac{dx}{x^2} = \frac{k^{-1/2}}{2} \int_0^{\infty} e^{-u} u^{-1/2}\, du \ll k^{-1/2}. \tag{3.5}$$

□

Proof of Proposition 3.1. First, note that

$$\begin{aligned} c_k &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2j+2}} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{n^{2j}} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(1 - \frac{1}{n^2}\right)^k. \end{aligned} \tag{3.6}$$

For $\Re(s) > 1$, we have

$$\begin{aligned} \frac{1}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(\frac{1}{n^2}\right)^{s/2-1} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(1 - \left(1 - \frac{1}{n^2}\right)\right)^{s/2-1} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} (-1)^k \binom{\frac{s}{2}-1}{k} \left(1 - \frac{1}{n^2}\right)^k \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right)\left(1 - \frac{1}{n^2}\right)^k. \end{aligned} \tag{3.7}$$

These summations can be interchanged because letting

$$S=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n^2}\left|P_k\left(\frac{s}{2}\right)\right|\left(1-\frac{1}{n^2}\right)^k, \tag{3.8}$$

we see from Lemmas 2.3 and 3.3 that

$$\begin{aligned} S&=\sum_{k=0}^{\infty}\left|P_k\left(\frac{s}{2}\right)\right|\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{n^2}\right)^k \\ &=\sum_{k=0}^{\infty}\left|P_k\left(\frac{s}{2}\right)\right|q_k \ll \sum_{k=1}^{\infty}k^{-\Re(s)/2-1/2}<\infty. \end{aligned} \tag{3.9}$$

Thus, we proceed to interchange summations in (3.7), taking into account (3.6), to obtain, unconditionally for $\Re(s)>1$,

$$\frac{1}{\zeta(s)}=\sum_{k=0}^{\infty}P_k\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\left(1-\frac{1}{n^2}\right)^k=\sum_{k=0}^{\infty}c_kP_k\left(\frac{s}{2}\right). \tag{3.10}$$

But Lemma 2.3, together with the hypothesis $c_k \ll k^{-3/4+(1/2)\epsilon}$, implies that the above series converges uniformly on compacts of the half-plane $\Re(s)>1/2+\epsilon$. Thus, the series extends $1/\zeta(s)$ analytically to the half-plane $\Re(s)>1/2$. We have thus proved the validity of (3.1). □

Finally, we prove the assertion on simple zeros in the main theorem (Theorem 1.1). Assume that $c_k \ll k^{-3/4}$. Take any *fixed* $s=1/2+i\beta$ on the critical line and $0<h\le\delta$ for a fixed, finite $\delta>0$. By (3.1),

$$\left|\frac{1}{\zeta(s+h)}\right|\le|c_0|+\sum_{k=1}^{\infty}O(k^{-3/4})\left|P_k\left(\frac{s+h}{2}\right)\right|. \tag{3.11}$$

Now it is clear that

$$\alpha_1=\sup_{0\le h\le\delta}\frac{1}{|\Gamma(3/4+h/2-i(\beta/2))|}<\infty. \tag{3.12}$$

But by Lemma 2.2, there is a constant $\alpha_2>0$ such that

$$\sum_{k=1}^{\infty}k^{-3/4}\left|P_k\left(\frac{s+h}{2}\right)\right|\le\alpha_2\sum_{k=1}^{\infty}\frac{k^{-1-h/2}}{|\Gamma(3/4+h/2-i(\beta/2))|}\le\alpha_1\alpha_2\zeta\left(1+\frac{h}{2}\right)\ll\frac{1}{h}. \tag{3.13}$$

Applying this in (3.11), we obtain

$$\left|\frac{1}{\zeta(s+h)}\right|\ll\frac{1}{h}. \tag{3.14}$$

This shows that if $\zeta(s)=0$, then s can only be a simple zero.

3.2. Necessity of the condition

Proof of the necessity of the condition. Assume now that the Riemann hypothesis is true. If, as usual, we write

$$M(x) := \sum_{n \le x} \mu(n), \tag{3.15}$$

we then have

$$M(x) \ll x^{1/2+2\epsilon} \quad (\forall \epsilon > 0), \tag{3.16}$$

which, actually, is well known to be equivalent to the Riemann hypothesis (see, e.g., [8, Theorem 14.25(C)]).

We can transform the second expression for c_k in (3.6) summing it by parts to obtain

$$\begin{aligned} c_k &= -\int_1^{\infty} M(x) \frac{d}{dx}\left(\frac{1}{x^2}\left(1-\frac{1}{x^2}\right)^k\right) dx \\ &= 2\int_0^1 M\left(\frac{1}{x}\right)\left(1-x^2\right)^{k-1}\left(x-(k+1)x^3\right) dx. \end{aligned} \tag{3.17}$$

Therefore,

$$|c_k| \le 2(k+1)\int_0^1 \left|M\left(\frac{1}{x}\right)\right| x^3\left(1-x^2\right)^{k-1} dx + 2\int_0^1 \left|M\left(\frac{1}{x}\right)\right| x\left(1-x^2\right)^{k-1} k\, dx, \tag{3.18}$$

but (on the Riemann hypothesis)

$$M\left(\frac{1}{x}\right) \ll x^{-1/2-2\epsilon} \quad (x \downarrow 0), \tag{3.19}$$

so that

$$c_k \ll k\int_0^1 x^{5/2-2\epsilon}(1-x^2)^{k-1} dx + \int_0^1 x^{1/2-2\epsilon}(1-x^2)^{k-1} dx. \tag{3.20}$$

On the other hand, for $\Re(\lambda) > -1$, a classical beta integral result (see, e.g., [2, Section 9.3]) gives

$$\int_0^1 x^{\lambda}\left(1-x^2\right)^{k-1} dx = \frac{1}{2}\Gamma\left(\frac{1}{2}(\lambda+1)\right)\frac{\Gamma(k)}{\Gamma(k+(1/2)(\lambda+1))} \ll k^{-1/2-\lambda/2}, \tag{3.21}$$

where the last estimate follows from Stirling's formula for the logarithm of the gamma function; hence (3.20) becomes

$$c_k \ll k^{-3/4+\epsilon}. \tag{3.22}$$

□

4. An exact formula for c_k

The c_k have a nice exact expression as an integral in the complex plane, as shown by the following proposition.

Proposition 4.1.

$$-2kc_{k-1} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} \quad (k \geq 2,\ 1 < a < 2), \tag{4.1}$$

where the integral is absolutely convergent. The path of integration is the line $\Re(s) = a$ traversed in the upward direction.

Proof. Note first that for any $\sigma > 1$,

$$\frac{1}{|\zeta(s)|} = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma). \tag{4.2}$$

By Lemma 2.1, we may move the path of integration to a vertical line with any abscissa $b > 2k$. Calculating the residues with the help of Lemma 2.5, we get

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} = \sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{2j}{\zeta(2j)} + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{ds}{\zeta(s)P_k(s/2)}, \tag{4.3}$$

but

$$-2k\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{\zeta(2j+2)} = -2kc_{k-1}. \tag{4.4}$$

For fixed k, let $b \to +\infty$. By Lemma 2.1, this yields

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} \longrightarrow 0 \quad (b \longrightarrow \infty), \tag{4.5}$$

so that (4.1) follows. □

Proof of the explicit formula (Theorem 1.5). We intend now to move the path of integration in (4.1) to the left of the critical strip. As this procedure is a little more delicate than that of the previous lemma, we will proceed in more detail. We begin with a fixed but arbitrary $k \geq \max(4, A)$, for some fixed $A > 0$ to be determined later in the proof, so that, of course, to begin with Proposition 4.1 can be applied. For any $T_\nu > 0$, consider the integral

$$I(k,\nu) := \frac{1}{2\pi i}\int_{L_\nu} \frac{ds}{\zeta(s)P_k(s/2)}, \tag{4.6}$$

where L_ν is the rectangle $\{3/2 - iT_\nu, 3/2 + iT_\nu, -1 + iT_\nu, -1 - iT_\nu\}$ traversed in the positive direction. By the residue theorem, we have

$$I(k,\nu) = \sum_{|\Im\rho|<T_\nu} \operatorname{Res}\left(\frac{1}{\zeta(s)P_k(s/2)};\ s = \rho\right), \tag{4.7}$$

where the finite sum runs over the zeros ρ of the zeta function in the interior of the rectangle L_ν. The choice of the T_ν is dictated by Theorem 9.7 in Titchmarsh's monograph [8], where it is attributed to Valiron et al., independently. As a consequence of this *unconditional* theorem for some constant $A > 0$, there is a sequence T_ν with $\nu < T_\nu < \nu + 1$ such that

$$\frac{1}{|\zeta(\sigma + iT_\nu)|} < T_\nu^A \quad (-1 \leq \sigma \leq 2). \tag{4.8}$$

This estimate, together with Lemma 2.1, implies that the contribution of the horizontal rungs in $I(k,\nu)$ tends to zero as $\nu \to \infty$.

On the other hand, it is clear that as $\nu \to \infty$, the integral on the right-hand vertical side of L_ν tends to the absolutely convergent integral on the right-hand side of (4.1), thus to $-2kc_{k-1}$.

Likewise, to see that the contribution of the left-hand side of the rectangle L_ν converges as $\nu \to \infty$ to

$$J_k := -\frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \frac{ds}{\zeta(s) P_k(s/2)}, \tag{4.9}$$

it suffices to show that this integral is absolutely convergent. To prove this, note that the functional align implies that

$$\begin{aligned} \frac{1}{|\zeta(-1+it)|} &= \frac{1}{|2^{1-it}\pi^{-2-it}\cos(i\pi t/2)\Gamma(2-it)\zeta(2-it)|} \\ &\ll \frac{1}{e^{\pi t/2} + e^{-\pi t/2}} \frac{1}{|\Gamma(2-it)|} \ll |t|^{3/2}, \end{aligned} \tag{4.10}$$

where we used again (4.2) in writing $|\zeta(2-it)^{-1}| \leq \zeta(2)$, and the well-known estimates for the gamma function on vertical strips (see, e.g., formula (21.52) in Rademacher's treatise [6]). Now (4.10) and the trivial bound (2.2) yield the absolute integrability of (4.9).

We have thus proved that the limit as $\nu \to \infty$ of $I(k,\nu)$ exists, arriving at

$$-2kc_{k-1} = \lim_{\nu \to \infty} \sum_{|\Im \rho| < T_\nu} \operatorname{Res}\left(\frac{1}{\zeta(s) P_k(s/2)}; s = \rho\right) + J_k, \tag{4.11}$$

where the limit of the summation has also been shown to exist. But $J_k \to 0$ as $k \to \infty$ by the monotone convergence theorem on account of Lemma 2.4. This completes the proof of (1.11) of Theorem 1.5, which immediately implies (1.12) under the assumption of simple zeros. □

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