THE STABILITY OF COLLOCATION METHODS FOR VIDES OF SECOND ORDER

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Simplest results presented here are the stability criteria of collocation methods for the second-order Volterra integrodifferential equation (VIDE) by polynomial spline functions. The polynomial spline collocation method is stable if all eigenvalues of a matrix are in the unit disk and all eigenvalues with $|\lambda| = 1$ belong to a 1×1 Jordan block. Also many other conditions are derived depending upon the choice of collocation parameters used in the solution procedure.

1. Introduction

In order to discuss the numerical stability, we consider the linear second-order Volterra integrodifferential equation of the form

$$y^{(2)}(t) = q(t) + \sum_{i=0}^{1} p_i(t) y^{(i)}(t) + \sum_{i=0}^{1} \int_0^t k_i(t,s) y^{(i)}(s)\, ds, \quad t \in I := [0,T], \tag{1.1}$$

with

$$y(0) = y_0, \qquad y^{(1)}(0) = y_1, \tag{1.2}$$

where $q : I \to R$, $p_i : I \to R$, and $k_i : D \to R(i = 0,1)$ (with $D := \{(t,s) : 0 \le s \le t \le T\}$) are given functions and are assumed to be (at least) continuous in the respective domains. For more details of these equations, many other interesting methods for the approximated solution and stability procedure are available in earlier literature [1, 2, 3, 4, 5, 6, 9, 10, 12, 13]. The above equation is usually known as basis test equation and is suggested by Brunner and Lambert in [4]. Since then, it has been widely used for analyzing the stability properties of various methods.

Second-order VIDEs of the above form (1.1) will be solved numerically using polynomial spline spaces. In order to describe these approximating polynomial spline spaces,

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let $\prod_N : 0 = t_0 < t_1 < \cdots < t_N = T$ be the mesh for the interval I, and set

$$\begin{aligned} \sigma_n &:= [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad n = 0,1,\ldots,N-1, \\ h &= \max\{h_n : 0 \le n \le N-1\}, \quad \text{(mesh diameter)}, \\ Z_N &:= \{t_n : n = 1,2,\ldots,N-1\}, \qquad \overline{Z}_N = Z_N \cup \{T\}. \end{aligned} \tag{1.3}$$

Let π_{m+d} be the set of (real) polynomials of degree not exceeding $m+d$, where $m \ge 1$ and $d \ge -1$ are given integers. The solution to initial-value problem (1.1) will be approximated by an element u in the polynomial spline space,

$$\begin{aligned} S_{m+d}^{(d)}(Z_N) := \Big\{ u := u(t)|_{t\in\sigma_n} := u_n(t) \in \pi_{m+d},\ n = 0,1,\ldots,N-1, \\ u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } j = 0,1,\ldots,d,\ t_n \in Z_N \Big\}; \end{aligned} \tag{1.4}$$

that is, by a polynomial spline function of degree $m+d$ which possesses the knots Z_N and is d times continuously differentiable on I. If $d = -1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuities at the knots Z_N. There are many other papers which had treated such problem using $S_m^{(0)}(Z_N)$ and $S_m^{(1)}(Z_N)$ [3, 4, 6] polynomial spline spaces.

According to Miculá et al. [11], an element $u \in S_{m+d}^{(d)}(Z_N)$ has the following form (for all $n = 0,1,\ldots,N-1$ and $t \in \sigma_n$):

$$u(t) = u_n(t) = \sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!}(t-t_n)^r + \sum_{r=1}^{m} a_{n,r}(t-t_n)^{d+r}, \tag{1.5}$$

where

$$u_{-1}^{r}(0) := \left[\frac{d^r}{dt^r}u(t)\right]_{t=0} = y^{(r)}(0), \quad r = 0,1,\ldots,d. \tag{1.6}$$

From (1.5), we see that the element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined provided that the coefficients $\{a_{n,r}\}_{r=1,\ldots,m}$ for all $n = 0,1,\ldots,N-1$ are known. In order to determine these coefficients, we consider a set of collocation parameters $\{c_j\}_{j=1,\ldots,m}$, where $0 < c_1 < \cdots < c_m \le 1$, and define the set of collocation points as

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \quad \text{with } X_n := \{t_{n,j} := t_n + c_j h_n,\ j = 1,2,\ldots,m\}. \tag{1.7}$$

The approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined by imposing the condition that u satisfies the initial-value problem (1.1) on $X(N)$ and the initial conditions, that is,

$$u^{(2)}(t) = q(t) + \sum_{i=0}^{1} p_i(t)u^{(i)}(t) + \sum_{i=0}^{1}\int_0^t k_i(t,s)u^{(i)}(s)\,ds, \quad \forall t \in X(N), \tag{1.8}$$

with

$$u(0) = y_0, \qquad u^{(1)}(0) = y_1 \tag{1.9}$$

with a uniform mesh sequence $\{\prod_N\}$, $h_n = h$, for all $n = 0,1,\ldots,N-1$.

2. Numerical stability

In [7], Danciu studied the numerical stability of the collocation method for first-order integrodifferential equations. He studied the behavior of the method as applied to the initial-value problem integrodifferential test equation

$$y'(t) = q(t) + \alpha_0 y(t) + \nu \int_0^t y(s)\,ds, \quad \nu \neq 0. \tag{2.1}$$

Equation (2.1) has been suggested by Brunner and Lambert in 1974 (see [4]), since then it has been extensively used as a basis for investigating the stability properties of several other methods.

In order to discuss the numerical stability for second-order integrodifferential equations, we study the numerical stability of the collocation spline method when applied to the initial-value problem integrodifferential test equation of the following form:

$$y''(t) = q(t) + \alpha_0 y(t) + \alpha_1 y'(t) + \nu \int_0^t y(s)\,ds, \quad \nu \neq 0, \tag{2.2}$$

$$y(0) = y_0, \qquad y'(0) = y_1, \tag{2.3}$$

where α_1, α_2, and ν are constants, and the given function $g : I \to R$ is sufficiently smooth.

For simplicity, we use a polynomial spline collocation method in the space $S_{m+d}^{(d)}(Z_N)$, as an (m,d)-method (see [8]).

Definition 2.1. An (m,d)-method is said to be stable if all solutions $\{u(t_n)\}$ remain bounded, as $n \to \infty$, $h \to 0$ while h_n remains fixed.

From (1.5), we observe that the first $d+1$ coefficients of the $u \in S_{m+d}^{(d)}(Z_N)$ are determine by the smooth conditions, and then the collocation conditions can be used to determine the last m coefficients. Thus, for convenience, we introduce the following notations:

$$\begin{aligned} \eta_n &:= (\eta_{n,r})_{r=0,\dots,d}, \quad \text{with } \eta_{n,r} := \frac{u_{n-1}^{(r)}(t_n)}{r!}h^r; \\ \beta_n &:= (\beta_{n,r})_{r=1,\dots,m}, \quad \text{with } \beta_{n,r} := a_{n,r}h^{d+r},\ n = 0,1,\dots,N. \end{aligned} \tag{2.4}$$

Using (2.4) in (1.5), for all $t := t_n + \tau h \in \sigma_n$, we have the following equation:

$$u(t) = u_n(t_n + \tau h) = \sum_{r=0}^{d} \eta_{n,r}\tau^r + \sum_{r=1}^{m} \beta_{n,r}\tau^{d+r}, \quad \forall \tau \in (0,1],\ n = 0,1,\dots,N. \tag{2.5}$$

By applying the collocation method to test (2.2) for the case $d \geq 2$ and using (2.5) we have the following collocation equation:

$$V\beta_n = W\eta_n + h^2 R_n, \quad \forall n = 0,1,\dots,N-1, \tag{2.6}$$

where V is the $m\times m$ matrix, W is the $m\times(d+1)$ matrix, and R_n is the m-vector, whose elements are given by

$$V_{j,r} := \left[(d+r)(d+r-1)-\alpha_0 h^2 c_j^2-\alpha_1 h(d+r)c_j-\frac{\nu h^3}{(d+r+1)}c_j^3\right]c_j^{d+r-2},$$

$$W_{j,r} := \begin{cases} \nu h^3 c_j & \text{if } r=0, \\ \alpha_0 h^2 c_j+\nu\dfrac{h^3}{2}c_j^2 & \text{if } r=1, \\ \alpha_0 h^2 c_j^2+2\alpha_1 h c_j+\nu\dfrac{h^3}{3}c_j^3 & \text{if } r=2, \\ \left[\alpha_0 h^2 c_j^2+\alpha_1 h r c_j+\nu\dfrac{h^3}{r+1}c_j^3-r(r-1)\right]c_j^{r-2} & \text{if } 3\le r\le d, \end{cases} \tag{2.7}$$

$$R_{n,j} := \begin{cases} q(t_{0,j})-q(t_0) & \text{if } n=0, \\ q(t_{n,j})-q(t_{n-1,m})+u''_{n-1}(t_{n-1,m})-u''_{n-1}(t_n) & \\ +\alpha_0\left[u_{n-1}(t_n)-u_{n-1}(t_{n-1,m})\right]+\alpha_1\left[u'_{n-1}(t_n)-u'_{n-1}(t_{n-1,m})\right] & \\ +\nu h\displaystyle\int_{c_m}^{1} u_{n-1}(t_{n-1}+\tau h)\,d\tau & \text{if } n>0. \end{cases}$$

By direct differentiation of (2.5) and using the smooth conditions of the approximation $u\in S_{m+d}^{(d)}(Z_N)$, we get a relationship between vector η_{n+1} and vectors η_n and β_n, as follows:

$$\eta_{n+1}=A\eta_n+B\beta_n,\quad \forall n=0,1,\dots,N-1, \tag{2.8}$$

where A is the $(d+1)\times(d+1)$ upper triangular matrix and B is the $(d+1)\times m$ matrix, whose elements are given by

$$a_{j,r} := \begin{cases} 0 & \text{if } r<j, \\ \dbinom{r}{j} & \text{if } r\ge j, \end{cases} \qquad b_{j,r} := \binom{d+r}{j}. \tag{2.9}$$

For h small enough, the matrix V is invertible since the determinant of V is a Vandermonde-type determinant for $h=0$. Hence from (2.6) and (2.8), we have

$$\begin{aligned} \eta_{n+1} &= A\eta_n+BV^{-1}\left[W\eta_n+h^2R_n\right] \\ &= \left(A+BV^{-1}W\right)\eta_n+h^2BV^{-1}R_n. \end{aligned} \tag{2.10}$$

Thus we have the following recurrence relation:

$$\eta_{n+1}=M\eta_n+h^2BV^{-1}R_n, \tag{2.11}$$

where

$$M:=A+BV^{-1}W. \tag{2.12}$$

Therefore, we have the following theorem which represents a stability criterion for the present method. The proof of this theorem is quite similar to the proof given by Danciu [7] for first-order VIDEs.

Theorem 2.2. *An (m,d)-method is stable if and only if all eigenvalues of matrix $M = A + BV^{-1}W$ are in the unit disk and all eigenvalues with $|\lambda| = 1$ belong to a 1×1 Jordan block.*

Remark 2.3. Note that the dimension of the matrix M is $d+1$. Moreover, if we denote by M_0 the matrix M with $h = 0$, and by $\lambda^{(0)}$ and λ the eigenvalues of M_0 and M, respectively, then it follows that the matrix M_0 has $\lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = 1$, for $m \geq 1$ and $d \geq 2$.

3. Applications

In this section, we will investigate the following special cases.

(I) For the case $d = 2$, the approximation space is $S_{m+2}^{(2)}(Z_N)$. From Theorem 2.2 and Remark 2.3, we have the following theorem.

Theorem 3.1. *For every choice of the collocation parameters $\{c_j\}_{j=1,m}$, an $(m,2)$-method is stable for all $m \geq 1$.*

(II) For the case $m = 1$, this choice of m corresponds to a classical spline function, that is, the approximate solution $u \in S_{1+d}^{(d)}(Z_N)$. Using notations from Remark 2.3 (i.e., M_0 is the matrix M with $h = 0$, and $\lambda^{(0)}$ and λ are the respective eigenvalues of M_0 and M), we have

$$\lambda = \lambda^{(0)} + O(h). \tag{3.1}$$

If $c_1 \in (0,1]$ is the collocation parameter, then for all $d \geq 1$, using the binomial expansion, we find that for matrix M_0 the trace is,

$$\operatorname{Tr}(M_0) = d + 2 + \frac{1}{c_1^{d-1}} - \left(1 + \frac{1}{c_1}\right)^{d-1}. \tag{3.2}$$

As regard the stability of the spline collocation method, we have the following result.

Theorem 3.2. *A $(1,d)$-method is stable if and only if one of the following conditions is true:*
(i) $d = 2$ *and* $c_1 \in (0,1]$,
(ii) $d = 3$ *and* $c_1 = 1$.

Proof. For the case $d = 2$, this theorem follows from Theorem 3.1. If $d = 3$, then the fourth eigenvalue of M_0 is $\lambda_4^{(0)} = 1 - (2/c_1) \leq -1$ for $c_1 \in (0,1]$, and its absolute value is 1, if and only if $c_1 = 1$. If $d \geq 4$, then setting $p = d - 1$ in (3.2), we have

$$\operatorname{Tr}(M_0) = p + 3 + \frac{1}{c_1^p} - \left(1 + \frac{1}{c_1}\right)^p, \tag{3.3}$$

so if $d > 4$ and $c_1 \in (0,1]$, then

$$\infty < \operatorname{Tr}(M_0) < -p = -(d-1). \tag{3.4}$$

Since $\mathrm{Tr}(M_0) = \lambda_1^{(0)} + \lambda_2^{(0)} + \cdots + \lambda_{d+1}^{(0)} < -d+1$, and $\lambda_1^{(0)} = 1$, it results that there exists an eigenvalue $\lambda^{(0)}$ whose value is smaller than -1. If $d = 4$ then from (3.2), $\lambda_4^{(0)} + \lambda_5^{(0)} \le -4$, and therefore $\lambda_4^{(0)} < -1$ or $\lambda_5^{(0)} < -1$. Thus from Theorem 2.2 we have that, for $d \ge 4$, a $(1,d)$-method is unstable for any choice of the collocation parameter $c_1 \in (0,1]$. □

(III) For $m = 2$, we can prove the following result but the proof is the same as in [7].

Theorem 3.3. *Let $0 < c_1 < c_2 \le 1$ be the collocation parameters, then*
(i) *a $(2,2)$-method is stable for every choice of the collocation parameters,*
(ii) *a $(2,3)$-method is stable if and only if $c_1 + c_2 \ge 3/2$,*
(iii) *if $c_2 = 1$, then a $(2,d)$-method is unstable for all $d \ge 4$.*

(IV) For the case $d = 3$, the approximation $u \in S_{m+3}^{(3)}(Z_N)$ and the dimension of the matrix M_0 is 4 and its first $p+1$ eigenvalues are $\lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = 1$. To compute the fourth eigenvalue, we need the following results. But, first we introduce the following notations:

$$\begin{gathered}
S_k := S_k(c_1,\dots,c_m) = \sum_{1\le i_1<\cdots<i_k\le m}^{m} c_{i_1}c_{i_2}\cdots c_{i_k}, \quad \text{for } 1 \le k \le m, \\
S_0 := S_0(c_1,\dots,c_m) = 1, \\
S_{k,j} := S_k(c_1,\dots,c_{j-1},c_{j+1},\dots,c_m), \quad \text{for } 1 \le k \le m-1,\ 1 \le j \le m.
\end{gathered} \tag{3.5}$$

Lemma 3.4. *Let $0 < c_1 < c_2 < \cdots < c_m \le 1$ be the collocation parameters, then*

$$\begin{vmatrix}
1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\
1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & c_m & c_m^2 & \cdots & c_m^{i-1} & c_m^{i+1} & \cdots & c_m^m
\end{vmatrix} = S_{m-i} \prod_{1\le k<j\le m}^{m} (c_j - c_k). \tag{3.6}$$

Proof. We will prove the lemma by induction on the dimension of the matrix, starting with 2×2 matrices. For the 2×2 matrices, the result is clearly true. For $m\times m$ matrices $(m > 2)$, we define

$$f(x) := \begin{vmatrix}
1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\
1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & c_{m-1} & c_{m-1}^2 & \cdots & c_{m-1}^{i-1} & c_{m-1}^{i+1} & \cdots & c_{m-1}^m \\
1 & x & x^2 & \cdots & x^{i-1} & x^{i+1} & \cdots & x^m
\end{vmatrix}. \tag{3.7}$$

Note that

$$\begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ 1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{i-1} & c_m^{i+1} & \cdots & c_m^m \end{vmatrix} = f(c_m). \tag{3.8}$$

Now, since $f(c_1) = f(c_2) = \cdots = f(c_{m-1}) = 0$, we have

$$f(x) = a(x-b)\prod_{i=1}^{m-1}(x-c_i), \tag{3.9}$$

where a, b are constants to be determined. By the induction hypothesis, we obtain

$$a = S_{m-1-i}(c_1,\dots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k). \tag{3.10}$$

Moreover, from (3.9),

$$f(0) = a(-1)^m c_1c_2\cdots c_{m-1}b. \tag{3.11}$$

On the other hand, from the definition of f and by the induction hypothesis, we have

$$\begin{aligned} f(0) &= (-1)^{m+1}\begin{vmatrix} c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m-1} & c_{m-1}^2 & \cdots & c_{m-1}^{i-1} & c_{m-1}^{i+1} & \cdots & c_{m-1}^m \end{vmatrix} \\ &= (-1)^{m+1}c_1c_2\cdots c_{m-1}S_{m-i}(c_1,\dots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k). \end{aligned} \tag{3.12}$$

Thus, from (3.11) and (3.12), we have

$$-ab = S_{m-i}(c_1,\dots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k), \tag{3.13}$$

and so

$$\begin{aligned} f(c_m) &= a(c_m-b)\prod_{i=1}^{m-1}(c_m-c_i) \\ &= \Bigg[c_mS_{m-1-i}(c_1,\dots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k) \\ &\quad + S_{m-i}(c_1,\dots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k)\Bigg]\prod_{i=1}^{m-1}(c_m-c_i). \end{aligned} \tag{3.14}$$

However, since

$$c_m S_{m-1-i}(c_1,\dots,c_{m-1}) + S_{m-i}(c_1,\dots,c_{m-1}) = S_{m-i}(c_1,\dots,c_m) = S_{m-i},$$
$$\prod_{k<j}^{m-1}(c_j-c_k)\prod_{i=1}^{m-1}(c_m-c_i) = \prod_{k<j}^{m}(c_j-c_k), \tag{3.15}$$

we have

$$f(c_m) = S_{m-i}\prod_{k<j}^{m}(c_j-c_k), \tag{3.16}$$

which proves the lemma. □

Remark 3.5. Note that in Lemma 3.4, if $i=m$, then we have the Vandermonde determinant.

Corollary 3.6. *Let V_0 be the matrix V with $h=0$, $d=3$, that is, V_0 is the $m\times m$ matrix whose elements are*

$$(V_0)_{j,r} := ((r+3)(r+2))C_j^{r+1}. \tag{3.17}$$

Then, V_0^{-1} is the matrix whose elements are

$$(V_0^{-1})_{r,j} = \frac{1}{\det(V_0)}(-1)^{r+j}S^2_{m-1,j}S_{m-r,j}\prod_{l<k,(l,k\neq j)}(c_k-c_l)\prod_{k=1,(k\neq r)}^{m}(k+2)(k+3), \tag{3.18}$$

where

$$\det(V_0) = \left[\prod_{k=1}^{m}(k+2)(k+3)\prod_{l<k}(c_k-c_l)\right]S_m^2. \tag{3.19}$$

Proof. From Lemma 3.4, we have

$$\det(V_0) = \left[\prod_{k=1}^{m}(k+2)(k+3)\prod_{l<k}(c_k-c_l)\right]S_m^2. \tag{3.20}$$

Now

$$V_0^{-1} = \frac{\operatorname{Adj}(V_0)}{\det(V_0)}, \tag{3.21}$$

where $\mathrm{Adj}(V_0)$ is the adjoint matrix of V_0, however,

$$\mathrm{Adj}(V_0)_{r,j} = (-1)^{r+j} S_{m-1,j}^2 \prod_{k=1,(k\neq r)}^{m} (k+2)(k+3) \times \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{r-2} & c_1^r & \cdots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-2} & c_2^r & \cdots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{j-1} & c_{j-1}^2 & \cdots & c_{j-1}^{r-2} & c_{j-1}^r & \cdots & c_{j-1}^{m-1} \\ 1 & c_{j+1} & c_{j+1}^2 & \cdots & c_{j+1}^{r-2} & c_{j+1}^r & \cdots & c_{j+1}^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{r-2} & c_m^r & \cdots & c_m^{m-1} \end{vmatrix}. \tag{3.22}$$

Again, by Lemma 3.4 and using relations

$$S_{m-1-(r-1)}(c_1,\dots,c_{j-1},c_{j+1},\dots,c_m) = S_{m-r}(c_1,\dots,c_{j-1},c_{j+1},\dots,c_m) = S_{m-r,j}, \tag{3.23}$$

we have

$$\begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{r-2} & c_1^r & \cdots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-2} & c_2^r & \cdots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{j-1} & c_{j-1}^2 & \cdots & c_{j-1}^{r-2} & c_{j-1}^r & \cdots & c_{j-1}^{m-1} \\ 1 & c_{j+1} & c_{j+1}^2 & \cdots & c_{j+1}^{r-2} & c_{j+1}^r & \cdots & c_{j+1}^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{r-2} & c_m^r & \cdots & c_m^{m-1} \end{vmatrix} = S_{m-r,j} \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l). \tag{3.24}$$

Thus,

$$(V_0^{-1})_{r,j} = \frac{1}{\det(V_0)} (-1)^{r+j} S_{m-1,j}^2 S_{m-r,j} \prod_{l<k,(l,k\neq j)} (c_k - c_l) \prod_{k=1,(k\neq r)}^{m} (k+2)(k+3), \tag{3.25}$$

which completes the proof of the corollary. □

Now, we can derive a formula for computing the $p+2$-eigenvalue of the matrix M_0.

Theorem 3.7. *For the case $d = 3$ and $m \geq 1$, the $p+2$-eigenvalue of M_0 can be computed by using the following relation:*

$$\lambda_4^{(0)} = \frac{S_m - 2S_{m-1} + 3S_{m-2} + \cdots + (-1)^{m-1} m S_1 + (-1)^m (m+1)}{S_m}. \tag{3.26}$$

Proof. Let V_0 and W_0 be the matrices V and W, respectively, with $h=0$, then for $d=3$, W_0 is an $m\times 4$ matrix whose elements are given by

$$\left(W_0\right)_{j,r} := \begin{cases} 0 & \text{if } r=0,1,2, \\ -6c_j & \text{if } r=3. \end{cases} \tag{3.27}$$

Now, the fourth eigenvalue of $M_0 = A + BV_0^{-1}W_0$ is

$$\lambda_4^{(0)} = 1 + \sum_{r=1}^{m} (B)_{4,r}\left(V_0^{-1}W_0\right)_{r,4}. \tag{3.28}$$

From (2.8), the entries of the last row of matrix B are

$$(B)_{4,r} = \binom{3+r}{3}. \tag{3.29}$$

Moreover, from (3.27) and Corollary 3.6, we have

$$\begin{aligned}\left(V_0^{-1}W_0\right)_{r,4} = \frac{-6}{\det\left(V_0\right)} \sum_{j=1}^{m} \Bigg[& (-1)^{(r+j)} S_{m-1,j}^2 S_{m-r,j} c_j \\ & \times \prod_{l<k,(l,k\neq j)} \left(c_k - c_l\right) \prod_{k=1,(k\neq r)}^{m} (k+2)(k+3) \Bigg].\end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned}\lambda_4^{(0)} = 1 + \frac{6}{\det\left(V_0\right)} \sum_{r=1}^{m}\sum_{j=1}^{m} \Bigg[& \binom{3+r}{3} (-1)^{(r+j+1)} S_{m-1,j}^2 S_{m-r,j} c_j \\ & \times \prod_{l<k,(l,k\neq j)} \left(c_k - c_l\right) \prod_{k=1,(k\neq r)}^{m} (k+2)(k+3) \Bigg].\end{aligned} \tag{3.31}$$

By using relations

$$\begin{gathered} c_j S_{m-1,j}^2 = S_m S_{m-1,j}, \\ 6\binom{3+r}{3} \prod_{k=1,(k\neq r)}^{m} (k+2)(k+3) = (r+1)\prod_{k=1}^{m}(k+2)(k+3), \end{gathered} \tag{3.32}$$

and $\det(V_0)$, the above expression can be simplified as follows:

$$\lambda_4^{(0)} = 1 + \frac{\sum_{r=1}^{m}(-1)^r(r+1)\sum_{j=1}^{m}(-1)^{(j+1)} S_{m-1,j}\, S_{m-r,j} \prod_{l<k,(l,k\neq j)}\left(c_k - c_l\right)}{S_m \prod_{l<k}\left(c_k - c_l\right)}. \tag{3.33}$$

However, from Lemma 3.4, we have

$$\sum_{j=1}^{m}(-1)^{(j+1)}S_{m-1,j}S_{m-r,j}\prod_{l<k,(l,k\neq j)}(c_k-c_l)=\begin{vmatrix}1 & c_1 & c_1^2 & \cdots & c_1^{r-1} & c_1^{r+1} & \cdots & c_1^m\\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-1} & c_2^{r+1} & \cdots & c_2^m\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & c_m & c_m^2 & \cdots & c_m^{r-1} & c_m^{r+1} & \cdots & c_m^m\end{vmatrix}$$
$$=S_{m-r}\prod_{l<k}^{m}(c_k-c_l). \tag{3.34}$$

Hence,

$$\begin{aligned}\lambda_{p+2}^{(0)}&=1+\frac{\sum_{r=1}^{m}(-1)^r(r+1)S_{m-r}}{S_m}\\ &=\frac{\sum_{r=0}^{m}(-1)^r(r+1)S_{m-r}}{S_m}\\ &=\frac{S_m-2S_{m-1}+3S_{m-2}+\cdots+(-1)^{m-1}mS_1+(-1)^m(m+1)}{S_m},\end{aligned}\tag{3.35}$$

which concludes the proof of Theorem 3.7. □

Remark 3.8. Theorem 3.7 proves the conjecture asserted by Danciu [7] for first-order integrodifferential equations ($p=1,d=2$).

As an application to Theorem 3.7, we can prove the following results. The proofs are quite similar to [7] for the first-order Volterra integrodifferential equation.

Corollary 3.9. *An $(m,3)$-method is stable if and only if*

$$\left|\frac{[d/dt(t\cdot R_m(t))]_{t=1}}{R_m(0)}\right|\leq 1,\tag{3.36}$$

where $R_m(t)$ is the polynomial of degree m whose zeroes are the collocation parameters $\{c_j\}_{j=1,\dots,m}$.

Regarding the stability of local superconvergent solution $u\in S_{m+4}^{(3)}(Z_n)$, we have the following corollary.

Corollary 3.10. (i) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are uniformly distributed in $(0,1]$ (i.e., $c_j=j/m$, for all $j=1,2,\dots,m$), then $(m,3)$-method is stable for $m\geq 1$.*

(ii) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Radau II points in the interval $(0,1]$, then $(m,3)$-method is unstable for $m\geq 2$.*

(iii) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Gauss points in the interval $(0,1]$, then $(m,3)$-method is unstable for $m\geq 2$.*

(iv) *If the first $m-1$ collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Gauss points in the interval $(0,1)$ and the last collocation parameter is $c_m=1$, then $(m,3)$-method is stable for $m\geq 2$.*

4. Stability of $S_m^{(0)}(Z_n)$

In this section, we will investigate the stability when $d = 0$.

From (2.5), the restriction of $u \in S_m^{(0)}(Z_n)$ to the subinterval σ_n is given by

$$u(t) = u_n(t_n + \tau h) = u_{n-1}(t_n) + \sum_{r=1}^{m} \beta_{n,r}\tau^r, \quad \text{for } \tau \in (0,1],\ n = 0,1,\dots,N-1. \tag{4.1}$$

If we denote by u_{n+1} and by u''_{n+1} the vectors with m elements

$$u_{n+1} := \left(u_n(t_n + c_j h)\right)^T_{j=1,\dots m}, \qquad u''_{n+1} := \left(u''_n(t_n + c_j h)\right)^T_{j=1,\dots m}, \tag{4.2}$$

then from (4.1), we obtain

$$u_{n+1} = (1,1,\dots,1)^T u_{n-1}(t_n) + E\beta_n, \quad \text{for } n = 0,1,\dots,N-1, \tag{4.3}$$

$$u''_{n+1} = h^{-2}E''\beta_n, \quad \text{for } n = 0,1,\dots,N-1. \tag{4.4}$$

Here the matrices E and E'' are $m \times m$ matrices defined by $E := (c_j^r)_{j,r=1,\dots,m}$ and $E'' := (r(r-1)c_j^{r-2})_{j,r=1,\dots,m}$.

In this case, the collocation equation becomes

$$V\beta_n = h^2 W'\left(u_{n-1}(t_n), u'_{n-1}(t_n), u''_{n-1}(t_n)\right)^T + h^2 R_n, \tag{4.5}$$

for $n = 0,1\dots,N-1$, where W' is the $m \times 3$ matrix whose elements are

$$(W')_{j,r} := \begin{cases} \nu h c_j & \text{if } r = 1, \\ -\alpha_1 & \text{if } r = 2, \\ 1 & \text{if } r = 3, \end{cases} \tag{4.6}$$

and the matrix V and the vector R_n are defined in (2.6) when $d = 0$.

Since $V = E'' + O(h)$, the elimination of β_n between (4.4) and (4.5) yields

$$\begin{aligned} u''_n(t_{n,j}) &= (1 + O(h))u''_{n-1}(t_n) + (1 + O(h))u'_{n-1}(t_n) \\ &\quad + O(h)u_{n-1}(t_n) + (1 + O(h))R_{n,j}, \\ &\quad \text{for } j = 1,2,\dots,m \quad (n = 0,1,\dots,N-1). \end{aligned} \tag{4.7}$$

For $\tau \in [0,1]$, the second derivatives of the approximation $u \in S_m^{(0)}(Z_n)$ may be written in the form

$$u''_n(t + \tau h) = \sum_{j=1}^{m} L_j(\tau)u''_n(t_{n,j}), \quad \text{for } n = 0,1,\dots,N-1, \tag{4.8}$$

where

$$L_j(\tau) := \prod_{j=1, i\neq j}^{m} \frac{(\tau - c_i)}{(c_j - c_i)}, \quad \text{for } j = 0,1,\dots,m, \tag{4.9}$$

are the Lagrange fundamental polynomials associated with the collocation parameters $\{c_j\}_{j=1,m}$. Now, replacing $u''(t_{n,j})$ in (4.8) with its values given by (4.7), for $n = 0,1,\ldots, N-1$, we obtain

$$\begin{aligned} u_n''(t_{n+1}) &= (1+O(h))\left(u_{n-1}''(t_n) + u_{n-1}'(t_n) + \sum_{j=1}^{m} L_j(1)R_{n,j}\right) \\ &\quad + O(h)u_{n-1}(t_n), \quad \text{for } j = 1,2,\ldots,m \quad (n = 0,1,\ldots,N-1). \end{aligned} \tag{4.10}$$

By integrating relation (4.8), for $\tau \in [0,1]$, and using relation (4.7), we obtain

$$\begin{aligned} u_n'(t_{n+1}) &= h(1+O(h))u_{n-1}''(t_n) + (1+h(1+O(h)))u_{n-1}'(t_n) \\ &\quad + hO(h)u_{n-1}(t_n) + h(1+O(h))\int_0^1 \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau, \\ &\quad \text{for } j = 1,2,\ldots,m \quad (n = 0,1,\ldots,N-1). \end{aligned} \tag{4.11}$$

Integrating (4.8) one more time and using relation (4.7) yields

$$\begin{aligned} u_n(t_{n+1}) &= h^2(1+O(h))(u_{n-1}''(t_n) + u_{n-1}'(t_n)) + (1+h^2O(h))u_{n-1}(t_n) \\ &\quad + h^2(1+O(h))\int_0^1\int_0^s \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau\, ds, \\ &\quad \text{for } j = 1,2,\ldots,m \quad (n = 0,1,\ldots,N-1). \end{aligned} \tag{4.12}$$

Equations (4.7), (4.11), and (4.12) together form a system which may be written in the form

$$\begin{pmatrix} u_n(t_{n+1}) \\ u_n'(t_{n+1}) \\ u_n''(t_{n+1}) \end{pmatrix} = M' \begin{pmatrix} u_{n-1}(t_n) \\ u_{n-1}'(t_n) \\ u_{n-1}''(t_n) \end{pmatrix} + (1+O(h))R'_n \quad \text{for } n = 0,1,\ldots,N-1, \tag{4.13}$$

where

$$\begin{aligned} M' &:= \begin{pmatrix} (1+h^2O(h)) & h^2(1+O(h)) & h^2(1+O(h)) \\ hO(h) & (1+h(1+O(h))) & h(1+O(h)) \\ O(h) & (1+O(h)) & (1+O(h)) \end{pmatrix}, \\ R_n' &:= \begin{pmatrix} h^2\displaystyle\int_0^1\int_0^s \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau\, ds \\ h\displaystyle\int_0^1 \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau \\ \displaystyle\sum_{j=1}^{m} L_j(1)R_{n,j} \end{pmatrix}. \end{aligned} \tag{4.14}$$

Equation (4.13) has the same form as (2.11). Since $h = 0$ implies that the matrix M' has eigenvalues $\lambda_1' = \lambda_2' = \lambda_3' = 1$, we can prove the following theorem.

THEOREM 4.1. *For every choice of the collocation parameters $\{c_j\}_{j=1,\dots,m}$, an $(m,0)$-method is stable for all $m \geq 1$.*

5. Stability of $S_{m+1}^{(1)}(Z_n)$

In this section, we will investigate the stability when $d = 1$.

From (2.5), the restriction of $u \in S_{m+1}^{(1)}(Z_n)$ to the subinterval σ_n is given by

$$u(t) = u_n(t+\tau h) = u_{n-1}(t_n) + u'_{n-1}(t_n)\tau + \sum_{r=1}^{m} \beta_{n,r}\tau^{r+1}, \quad \text{for } \tau \in (0,1],\ n = 0,1,\dots,N-1. \tag{5.1}$$

In this case, the collocation equation becomes

$$V\beta_n = h^2 W''\left(u_{n-1}(t_n), u'_{n-1}(t_n), u''_{n-1}(t_n)\right)^T + h^2 R_n, \tag{5.2}$$

for $n = 0,1\dots,N-1$, where W'' is the $m \times 3$ matrix whose elements are

$$(W'')_{j,r} := \begin{cases} \nu h c_j & \text{if } r = 1, \\ c_j h\left(\alpha_0 + \dfrac{\nu h c_j}{2}\right) & \text{if } r = 2, \\ 1 & \text{if } r = 3, \end{cases} \tag{5.3}$$

and the matrix V and the vector R_n are defined in (2.6) when $d = 1$.

Using the same procedure as in Section 4, we can derive the system

$$\begin{pmatrix} u_n(t_{n+1}) \\ u'_n(t_{n+1}) \\ u''_n(t_{n+1}) \end{pmatrix} = M'' \begin{pmatrix} u_{n-1}(t_n) \\ u'_{n-1}(t_n) \\ u''_{n-1}(t_n) \end{pmatrix} + (1+O(h))R''_n, \quad \text{for } n = 0,1,\dots,N-1, \tag{5.4}$$

where

$$M'' := \begin{pmatrix} 1+h^2O(h) & h^2O(h) & h^2(1+O(h)) \\ hO(h) & 1+hO(h) & h(1+O(h)) \\ O(h) & O(h) & 1+O(h) \end{pmatrix},$$
$$R''_n := \begin{pmatrix} h^2 \displaystyle\int_0^1 \int_0^s \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau\, ds \\ h \displaystyle\int_0^1 \left(\sum_{j=1}^{m} L_j(\tau)R_{n,j}\right) d\tau \\ \displaystyle\sum_{j=1}^{m} L_j(1)R_{n,j} \end{pmatrix}. \tag{5.5}$$

Equation (5.4) has the same form as (2.11). Since $h = 0$ implies that the matrix M'' has eigenvalues $\lambda'_1 = \lambda'_2 = \lambda'_3 = 1$, we can prove the following theorem.

THEOREM 5.1. *For every choice of the collocation parameters* $\{c_j\}_{j=1,\dots,m}$, *an* $(m,1)$*-method is stable for all* $m \geq 1$.

6. Numerical examples

The $(3,d)$-method is tested using the following three examples in the interval $[0,1]$ with step size $h = 0.05$. The following notations will be used in the presentation.

$$e_1 := |y(t_1) - u(t_1)|, \qquad e_{N/2} := |y(0.5) - u(0.5)|, \qquad e_N := |y(1) - u(1)|, \tag{6.1}$$

where $u \in S^d_{3+d}(Z_n)$ is the approximate solution.

Example 6.1. Consider the following integrodifferential equation of second order:

$$y''(t) = 1 + \frac{1}{2}y(t) + \frac{1}{2}\int_0^t y(s)\,ds, \qquad y(0) = 2, \qquad y'(0) = 2, \tag{6.2}$$

where $y(t) = 2e^t$ is the exact solution.

Example 6.2. Consider the following integrodifferential equation of second order:

$$y''(t) = q(t) - \frac{t^2}{16}y'(t) + \int_0^t t^2 s y'(s)\,ds, \qquad y(0) = 1, \qquad y'(0) = 4, \tag{6.3}$$

where $q(t)$ is chosen so that $y(t) = \sin 4t$ is the exact solution.

Example 6.3. Consider the following integrodifferential equation of second order:

$$\begin{aligned} y''(t) &= q(t) + p_1(t)y(t) + p_2(t)y'(t) + \int_0^t y(s)\,ds \\ &\quad + \int_0^t ts^2 y'(s)\,ds, \qquad y(0) = 2, \qquad y'(0) = 0, \end{aligned} \tag{6.4}$$

with

$$p_1(t) = -t^3 + 2t - 1, \qquad p_2(t) = 1 - 2t^2, \tag{6.5}$$

where $q(t)$ is chosen so that $y(t) = 1 + \cos t$ is the exact solution.

(a) If the collocation parameters are uniformly distributed, that is, $c_1 = 1/3$, $c_2 = 2/3$, and $c_3 = 1$, then we have Tables 6.1, 6.2, and 6.3 corresponding to Examples 6.1, 6.2, and 6.3, respectively.

(b) If the collocation parameters are the Radau II points, that is, $c_1 = (4 - \sqrt{6})/10$, $c_2 = (4 + \sqrt{6})/10$, and $c_3 = 1$, then we have Tables 6.4, 6.5, and 6.6 corresponding to Examples 6.1, 6.2, and 6.3, respectively.

(c) If the collocation parameters are the Gauss points, that is, $c_1 = (5 - \sqrt{15})/10$, $c_2 = 1/2$, and $c_3 = (5 + \sqrt{15})/10$, then we have Tables 6.7, 6.8, and 6.9 corresponding to Examples 6.1, 6.2, and 6.3, respectively.

(d) If the first two collocation parameters are the Gauss points, that is, $c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$, and $c_3 = 1$, then we have Tables 6.10, 6.11, and 6.12 corresponding to Examples 6.1, 6.2, and 6.3, respectively.

Table 6.1. Approximate error for Example 6.1.

d	e_1	$e_{N/2}$	e_N
2	0	0	1.00×10^{-7}
3	0	1.00×10^{-9}	3.00×10^{-9}
4	0	2.40×10^{10}	1.46×10^{38}

Table 6.2. Approximate error for Example 6.2.

d	e_1	$e_{N/2}$	e_N
2	1.3×10^{-9}	3.32×10^{-4}	2.10×10^{-3}
3	3.00×10^{-10}	3.32×10^{-4}	2.11×10^{-3}
4	3.00×10^{-10}	4.20×10^{13}	2.56×10^{41}

Table 6.3. Approximate error for Example 6.3.

d	e_1	$e_{N/2}$	e_N
2	0	8.36×10^{-6}	1.04×10^{-3}
3	1.00×10^{-10}	8.35×10^{-6}	1.04×10^{-3}
4	0	2.02×10^{12}	1.24×10^{40}

Table 6.4. Approximate error for Example 6.1.

d	e_1	$e_{N/2}$	e_N
2	0	0	7.00×10^{-9}
3	0	7.35×10^{-4}	6.23×10^{8}
4	0	8.02×10^{23}	7.81×10^{65}

Table 6.5. Approximate error for Example 6.2.

d	e_1	$e_{N/2}$	e_N
2	1.40×10^{-9}	3.32×10^{-4}	2.1×10^{-3}
3	2.00×10^{-10}	4.52×10^{-2}	3.87×10^{10}
4	8.00×10^{-10}	5.61×10^{27}	5.48×10^{69}

Table 6.6. Approximate error for Example 6.3.

d	e_1	$e_{N/2}$	e_N
2	0	8.36×10^{-6}	1.04×10^{-3}
3	1.00×10^{-11}	8.55×10^{-3}	7.34×10^{9}
4	3.00×10^{-10}	2.34×10^{27}	2.28×10^{69}

From these numerical examples, we observe that a $(3,d)$-method is stable for $d=2$ and it is unstable for $d=4$. In the case $d=3$, this method is stable if the collocation parameters are uniformly distributed (i.e., $c_j := j/3$, for $j=1,2,3$) as in case a, or $c_1=(3-\sqrt{3})/6$,

Table 6.7. Approximate error for Example 6.1.

d	e_1	$e_{N/2}$	e_N
2	1.55×10^{-6}	2.55×10^{-3}	2.46×10^{-2}
3	9.30×10^{-6}	7.84×10^{20}	1.80×10^{56}
4	1.01×10^{-4}	5.42×10^{52}	5.03×10^{116}

Table 6.8. Approximate error for Example 6.2.

d	e_1	$e_{N/2}$	e_N
2	1.6×10^{-9}	3.59×10^{-4}	1.00×10^{-4}
3	1.10×10^{-9}	4.19×10^{17}	3.78×10^{53}
4	9.40×10^{-9}	4.80×10^{48}	4.47×10^{112}

Table 6.9. Approximate error for Example 6.3.

d	e_1	$e_{N/2}$	e_N
2	0	4.83×10^{-3}	3.65×10^{-2}
3	0	3.40×10^{23}	1.44×10^{59}
4	2.00×10^{-4}	9.96×10^{52}	9.40×10^{116}

Table 6.10. Approximate error for Example 6.1.

d	e_1	$e_{N/2}$	e_N
2	1.00×10^{-9}	4.00×10^{-9}	1.20×10^{-8}
3	0	0	7.00×10^{-9}
4	0	4.45×10^{13}	5.12×10^{43}

Table 6.11. Approximate error for Example 6.2.

d	e_1	$e_{N/2}$	e_N
2	1.10×10^{-9}	3.32×10^{-4}	2.11×10^{-3}
3	6.00×10^{-10}	3.32×10^{-4}	2.11×10^{-3}
4	3.00×10^{-10}	5.43×10^{16}	6.26×10^{46}

Table 6.12. Approximate error for Example 6.3.

d	e_1	$e_{N/2}$	e_N
2	0	8.36×10^{-6}	1.04×10^{-3}
3	1.00×10^{-10}	8.36×10^{-6}	1.04×10^{-3}
4	1.00×10^{-10}	3.06×10^{15}	3.54×10^{45}

$c_2 = (3+\sqrt{3})/6$, and $c_3 = 1$ as in case (d). These examples illustrate the conclusions of Theorem 5.1.

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