SOME INTERESTING SERIES ARISING FROM THE POWER SERIES EXPANSION OF $(\sin^{-1} x)^q$

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Received 3 February 2005 and in revised form 23 June 2005

Starting from the power series expansions of $(\sin^{-1} x)^q$, for $1 \le q \le 4$, formulae are obtained for the sum of several infinite series. Some of these evaluations involve $\zeta(3)$.

1. Introduction

In [10], Choe deduced the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.1}$$

from the power series expansion of $\sin^{-1}(x)$ (see also [1, 16]). By applying a generalization of the procedure used by Choe to the power series expansions of $(\sin^{-1} x)^q$ for $1 \le q \le 4$, we obtain explicit formulae for the sum of several infinite series, see (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6). For other applications based on the procedure used by Choe, see [11, 12, 17].

2. Main results

Let m denote an integer. For $m \ge 0$, we have the following theorems.

Theorem 2.1.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1)\binom{2k+2m}{k+m}} = 2^{-4m}\left(\sum_{\substack{r=1\\ r\equiv 1(\bmod 2)}}^{m} \frac{\binom{2m}{m-r}}{r^2} + \binom{2m}{m}\frac{\pi^2}{8}\right). \tag{2.1}$$

Theorem 2.2.

$$\sum_{k=1}^{\infty} \frac{\binom{2k+2m}{k+m}}{k^2\binom{2k}{k}} = \sum_{r=1}^{m} \frac{2\binom{2m}{m-r}}{r^2} + \binom{2m}{m}\frac{\pi^2}{6}. \tag{2.2}$$

International Journal of Mathematics and Mathematical Sciences 2005:14 (2005) 2329–2336
DOI: 10.1155/IJMMS.2005.2329

Theorem 2.3.

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1)\binom{2k+2m}{k+m}} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = 2^{-4m-1}\left(-\sum_{\substack{r=1 \\ r\equiv 1 (\bmod 2)}}^{m} \frac{\binom{2m}{m-r}}{2r^4} + \pi^2 \sum_{r=1}^{m} \frac{\binom{2m}{m-r}}{8r^2} + \binom{2m}{m}\frac{\pi^4}{192} \right). \tag{2.3}$$

Theorem 2.4.

$$\sum_{k=1}^{\infty} \frac{\binom{2k+2m+2}{k+m+1}}{(k+1)(2k+1)\binom{2k}{k}} \sum_{j=1}^{k} \frac{1}{j^2} = -4\sum_{r=1}^{m} \frac{\binom{2m}{m-r}}{r^4} + \frac{2\pi^2}{3} \sum_{r=1}^{m} \frac{\binom{2m}{m-r}}{r^2} + \binom{2m}{m}\frac{\pi^4}{60}. \tag{2.4}$$

In addition, we have the following theorems.

Theorem 2.5.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} &= \frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3), \\ \sum_{k=1}^{\infty} \frac{1}{(k+1)(2k+1)} \sum_{j=1}^{k} \frac{1}{j^2} &= \frac{\pi^2}{3}\log 2 - \frac{3}{2}\zeta(3). \end{aligned} \tag{2.5}$$

Theorem 2.6.

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)(2k+1)(2k-1)} \sum_{j=1}^{k} \frac{1}{j^2} = -\frac{\pi^2}{36} + \frac{2}{3}\log 2 + \frac{\pi^2}{9}\log 2 - \frac{1}{2}\zeta(3). \tag{2.6}$$

In (2.5) and (2.6), ζ represents the Riemann zeta function.

The following result in [14] ($m \geq 0$) should be compared with (2.1) :

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+2m+1)(2k+4m+1)\binom{2k+4m}{k+2m}} = \frac{\pi^2}{2^{8m+3}} \binom{2m}{m}^2. \tag{2.7}$$

Also, the series appearing above in (2.3), (2.4), (2.5), and (2.6) bear some resemblance to Euler sums (see, e.g., [3, 4, 5, 9]). A very broad generalization which generalizes both Euler sums and polylogarithms is studied in [6]. For other interesting evaluations of series involving binomial coefficients, see, for example, [7, 8, 15, 18].

3. Proofs of Theorems 2.1, 2.2, 2.3, and 2.4

The power series expansions of $(\sin^{-1} x)^q$ for $1 \le q \le 4$ (valid for $|x| \le 1$) are given by (see [10], [2, pages 262-263])

$$\begin{aligned}
\sin^{-1} x &= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{x^{2k+1}}{2k+1}, \\
\left(\sin^{-1} x\right)^2 &= \sum_{k=1}^{\infty} \frac{2^{2k-1}}{\binom{2k}{k}} \frac{x^{2k}}{k^2}, \\
\left(\sin^{-1} x\right)^3 &= 6 \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \left(\sum_{j=1}^{k} \frac{1}{(2j-1)^2} \right) \frac{x^{2k+1}}{2k+1}, \\
\left(\sin^{-1} x\right)^4 &= 3 \sum_{k=1}^{\infty} \frac{2^{2k}}{\binom{2k}{k}} \left(\sum_{j=1}^{k} \frac{1}{j^2} \right) \frac{x^{2k+2}}{(k+1)(2k+1)}.
\end{aligned} \tag{3.1}$$

Multiplying each of (3.1) by x^{2m}, where m is an integer, putting $x = \sin\theta$ and integrating with respect to θ from $\theta = 0$ to $\theta = \pi/2$, and using the well-known results (valid for nonnegative integers p)

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2p+1}\theta \, d\theta &= \frac{2^{2p}}{(2p+1)\binom{2p}{p}}, \\
\int_0^{\pi/2} \sin^{2p}\theta \, d\theta &= \frac{\binom{2p}{p}}{2^{2p}} \frac{\pi}{2},
\end{aligned} \tag{3.2}$$

we obtain

$$\int_0^{\pi/2} \theta \sin^{2m}\theta \, d\theta = 2^{2m} \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1)\binom{2k+2m}{k+m}}, \quad m \ge 0, \tag{3.3}$$

$$\int_0^{\pi/2} \theta^2 \sin^{2m}\theta \, d\theta = \frac{\pi}{2^{2m+2}} \sum_{k=1}^{\infty} \frac{\binom{2k+2m}{k+m}}{k^2 \binom{2k}{k}}, \quad m \ge -1, \tag{3.4}$$

$$\int_0^{\pi/2} \theta^3 \sin^{2m}\theta \, d\theta = 3\left(2^{2m+1}\right) \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{(2k+1)(2k+2m+1)\binom{2k+2m}{k+m}} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}, \quad m \ge -1, \tag{3.5}$$

$$\int_0^{\pi/2} \theta^4 \sin^{2m}\theta \, d\theta = \frac{3\pi}{2^{2m+3}} \sum_{k=1}^{\infty} \frac{\binom{2k+2m+2}{k+m+1}}{(k+1)(2k+1)\binom{2k}{k}} \sum_{j=1}^{k} \frac{1}{j^2}, \quad m \ge -2. \tag{3.6}$$

For $m \geq 0$, we evaluate the integrals on the left of (3.3), (3.4), (3.5), and (3.6) using the following formula valid for a nonnegative integer m (see [13, page 31]):

$$\sin^{2m}\theta = 2^{-2m}\left\{\sum_{j=0}^{m-1}(-1)^{m+j}2\binom{2m}{j}\cos\left(2(m-j)\theta\right)+\binom{2m}{m}\right\}, \tag{3.7}$$

and the following easily checked formulae (valid for positive integers l):

$$\begin{gathered}\int_0^{\pi/2}\theta\cos(2l\theta)\,d\theta=\frac{(-1)^l-1}{4l^2},\\ \int_0^{\pi/2}\theta^2\cos(2l\theta)\,d\theta=\frac{(-1)^l\pi}{4l^2},\\ \int_0^{\pi/2}\theta^3\cos(2l\theta)\,d\theta=3\left(\frac{(-1)^l\pi^2}{16l^2}+\frac{1-(-1)^l}{8l^4}\right),\\ \int_0^{\pi/2}\theta^4\cos(2l\theta)\,d\theta=(-1)^l\pi\left(\frac{\pi^2}{8l^2}-\frac{3}{4l^4}\right).\end{gathered} \tag{3.8}$$

After some simplification, we obtain (2.1), (2.2), (2.3), and (2.4).

4. Special cases of Theorems 2.1, 2.2, 2.3, and 2.4

We record the special cases corresponding to $0 \leq m \leq 2$.

Putting $m = 0, 1, 2$ in (2.1), we get

$$\begin{gathered}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8},\\ \sum_{k=0}^{\infty}\frac{k+1}{(2k+1)^2(2k+3)}=\frac{1}{8}+\frac{\pi^2}{32},\\ \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)(2k+5)\binom{2k+4}{k+2}}=\frac{1}{64}+\frac{3\pi^2}{1024}.\end{gathered} \tag{4.1}$$

Putting $m = 0,1,2$ in (2.2), we get

$$\begin{gathered}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6},\\ \sum_{k=1}^{\infty}\frac{\binom{2k+2}{k+1}}{k^2\binom{2k}{k}}=2+\frac{\pi^2}{3},\\ \sum_{k=1}^{\infty}\frac{\binom{2k+4}{k+2}}{k^2\binom{2k}{k}}=\frac{17}{2}+\pi^2.\end{gathered} \tag{4.2}$$

The first results of (4.1) and (4.2) are of course well-known classical results.

Putting $m = 0, 1, 2$ in (2.3), we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\sum_{j=1}^{k}\frac{1}{(2j-1)^2} &= \frac{\pi^4}{384},\\ \sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{(2k+1)(2k+3)\binom{2k+2}{k+1}}\sum_{j=1}^{k}\frac{1}{(2j-1)^2} &= \frac{-1}{64}+\frac{\pi^2}{256}+\frac{\pi^4}{3072},\\ \sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{(2k+1)(2k+5)\binom{2k+4}{k+2}}\sum_{j=1}^{k}\frac{1}{(2j-1)^2} &= \frac{-1}{256}+\frac{17\pi^2}{16384}+\frac{\pi^4}{16384}.\end{aligned} \tag{4.3}$$

Putting $m = 0, 1, 2$ in (2.4) gives

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\sum_{j=1}^{k}\frac{1}{j^2} &= \frac{\pi^4}{120},\\ \sum_{k=1}^{\infty}\frac{\binom{2k+4}{k+2}}{(k+1)(2k+1)\binom{2k}{k}}\sum_{j=1}^{k}\frac{1}{j^2} &= -4+\frac{2\pi^2}{3}+\frac{\pi^4}{30},\\ \sum_{k=1}^{\infty}\frac{\binom{2k+6}{k+3}}{(k+1)(2k+1)\binom{2k}{k}}\sum_{j=1}^{k}\frac{1}{j^2} &= -\frac{65}{4}+\frac{17\pi^2}{6}+\frac{\pi^4}{10}.\end{aligned} \tag{4.4}$$

We note that the first series evaluated in (4.4) is an Euler sum and the result is classical and was known to Euler (see, e.g., [5]).

5. Proof of Theorem 2.5

We consider the case $m = -1$ of (3.5), (3.6) (the case $m = -1$ of (3.4) gives a trivial result). We need the following result valid for a positive integer n and $|x| < 2\pi$ (see [2, page 260]):

$$\int_0^x \frac{u^n}{2}\cot\left(\frac{u}{2}\right)du = \cos\left(\frac{n\pi}{2}\right)n!\zeta(n+1) - \sum_{j=0}^{n}(-1)^{j(j+1)/2}\frac{\Gamma(n+1)}{\Gamma(n+1-j)}x^{n-j}\,\mathrm{Cl}_{j+1}(x), \tag{5.1}$$

where

$$\begin{aligned}\mathrm{Cl}_{2n}(x) &= \sum_{k=1}^{\infty}\frac{\sin(kx)}{k^{2n}},\\ \mathrm{Cl}_{2n+1}(x) &= \sum_{k=1}^{\infty}\frac{\cos(kx)}{k^{2n+1}},\end{aligned} \tag{5.2}$$

and Γ and ζ represent the Gamma function and the Riemann zeta function respectively. We note that

$$\begin{gathered} \mathrm{Cl}_{2n}(\pi) = 0, \\ \mathrm{Cl}_{2n+1}(\pi) = \left(\frac{1}{2^{2n}} - 1\right)\zeta(2n+1), \quad n \geq 1, \\ \mathrm{Cl}_1(\pi) = -\log 2. \end{gathered} \tag{5.3}$$

Putting $x = \pi$ in (5.1), we obtain

$$2^n \int_0^{\pi/2} \theta^n \cot\theta \, d\theta = n! \cos\left(\frac{n\pi}{2}\right)\zeta(n+1) - \sum_{j=0}^{n} (-1)^{j(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} \pi^{n-j} \mathrm{Cl}_{j+1}(\pi). \tag{5.4}$$

Using

$$\int_0^{\pi/2} \theta^n \cot\theta \, d\theta = \frac{1}{n+1} \int_0^{\pi/2} \theta^{n+1} \csc^2\theta \, d\theta, \quad n \geq 1, \tag{5.5}$$

in (5.4), we get

$$\begin{aligned} &\frac{2^n}{n+1} \int_0^{\pi/2} \theta^{n+1} \csc^2\theta \, d\theta \\ &\quad = n! \cos\left(\frac{n\pi}{2}\right)\zeta(n+1) - \sum_{j=0}^{n} (-1)^{j(j+1)/2} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} \pi^{n-j} \mathrm{Cl}_{j+1}(\pi). \end{aligned} \tag{5.6}$$

From (5.6) and (5.3) we obtain

$$\int_0^{\pi/2} \theta^2 \csc^2\theta \, d\theta = \pi \log 2, \tag{5.7}$$

$$\int_0^{\pi/2} \theta^3 \csc^2\theta \, d\theta = \frac{3}{4}\pi^2 \log 2 - \frac{21}{8}\zeta(3), \tag{5.8}$$

$$\int_0^{\pi/2} \theta^4 \csc^2\theta \, d\theta = \frac{\pi^3}{2} \log 2 - \frac{9}{4}\pi\zeta(3). \tag{5.9}$$

Putting $m = -1$ in (3.5) and (3.6) and using (5.8) and (5.9) give (2.5).

6. Proof of Theorem 2.6

We consider the case $m = -2$ of (3.6). We need to evaluate $\int_0^{\pi/2} \theta^4 \csc^4\theta \, d\theta$. We have

$$\begin{aligned} \int_0^{\pi/2} \theta^4 \csc^4\theta \, d\theta &= \theta^4 \csc^2\theta(-\cot\theta)\Big]_0^{\pi/2} + \int_0^{\pi/2} \cot\theta \frac{d}{d\theta}(\theta^4 \csc^2\theta) \, d\theta \\ &= 4\int_0^{\pi/2} \theta^3 \cot\theta \csc^2\theta \, d\theta - 2\int_0^{\pi/2} \theta^4 \csc^2\theta \cot^2\theta \, d\theta. \end{aligned} \tag{6.1}$$

Using $\cot^2\theta = \csc^2\theta - 1$ in the second integral on the right gives

$$\int_0^{\pi/2} \theta^4 \csc^4\theta\, d\theta = \frac{4}{3}\int_0^{\pi/2} \theta^3 \cot\theta \csc^2\theta\, d\theta + \frac{2}{3}\int_0^{\pi/2} \theta^4 \csc^2\theta\, d\theta. \tag{6.2}$$

Also,

$$\begin{aligned}\int_0^{\pi/2} \theta^3 \cot\theta \csc^2\theta\, d\theta &= \theta^3 \csc\theta(-\csc\theta)\big]_0^{\pi/2} + \int_0^{\pi/2} \csc\theta \frac{d}{d\theta}\left(\theta^3 \csc\theta\right) d\theta \\ &= -\frac{\pi^3}{8} + 3\int_0^{\pi/2} \theta^2 \csc^2\theta\, d\theta - \int_0^{\pi/2} \theta^3 \cot\theta \csc^2\theta\, d\theta,\end{aligned} \tag{6.3}$$

so that

$$\int_0^{\pi/2} \theta^3 \cot\theta \csc^2\theta\, d\theta = -\frac{\pi^3}{16} + \frac{3}{2}\int_0^{\pi/2} \theta^2 \csc^2\theta\, d\theta. \tag{6.4}$$

From (6.2), (6.4), (5.7), and (5.9), we obtain

$$\int_0^{\pi/2} \theta^4 \csc^4\theta\, d\theta = -\frac{\pi^3}{12} + 2\pi\log 2 + \frac{\pi^3}{3}\log 2 - \frac{3}{2}\pi\zeta(3). \tag{6.5}$$

Putting $m = -2$ in (3.6) and using (6.5), we obtain (2.6).

7. Final remarks

In a future paper, we plan to investigate what happens when we multiply (3.1) by x^{2m+1} and carry out the same steps as we did here.

Acknowledgments

The research for this paper was carried out while the author was a Postdoctoral Fellow at the University of Waterloo, Canada, under C. L. Stewart. Also, the author would like to thank J. M. Borwein for valuable comments on earlier versions of the paper.

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