GENERALIZED SET-VALUED VARIATIONAL-LIKE INCLUSIONS AND WIENER-HOPF EQUATIONS IN BANACH SPACES

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By using the notion of J^η-proximal mapping for a nonconvex, lower semicontinuous, η-subdifferentiable proper functional in reflexive Banach spaces, we introduce and study a class of generalized set-valued variational-like inclusions in Banach spaces and show their equivalences with a class of Wiener-Hopf equations. We propose two new iterative algorithms for the class of generalized set-valued variational-like inclusions. Furthermore, we prove the existence of solutions of the generalized set-valued variational-like inclusions and the convergence criteria of the two iterative algorithms for the generalized set-valued variational-like inclusions in reflexive Banach spaces. The results presented in this paper are new and are an extension of the corresponding results in this direction.

1. Introduction

Variational inequality theory has emerged as a powerful tool for a wide class of unrelated problems arising in various branches of physical, engineering, pure, and applied sciences in a unified and general framework, see, for example, [8, 9, 10]. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas both for their own sake and for their applications, see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. An important and useful generalization of variational(-like) and quasivariational(-like) inequalities is a variational(-like) inclusion.

Iterative algorithms have played a central role in the approximation solvability, especially in nonlinear variational inequalities as well as nonlinear equations. In general, we cannot use resolvent operator of proximal mapping to construct algorithms for finding the approximate solutions of variational-like inequalities (inclusions).

In the early 1990s, Shi [27] and Robinson [26] introduced and studied Wiener-Hopf equations. Very recently, Ahmad et al. [2] have defined a new notion of J^η-proximal mapping for a noncovex, lower semicontinuous, η-subdifferentiable proper functional in Banach spaces which is an extension of J-proximal mapping given by Ding and Xia [7].

Motivated and inspired by the recent work in this direction, we introduce a class of generalized set-valued variational-like inclusions and show their equivalences with a class

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of Wiener-Hopf equations by using the notion of J^η-proximal mapping for a nonconvex, lower semicontinuous, η-subdifferentiable proper functional in reflexive Banach spaces. Based on these equivalences, we propose two new iterative algorithms for the class of generalized set-valued variational-like inclusions. Furthermore, we prove the existence of solutions and discuss the convergence criteria for these generalized set-valued variational-like inclusions. The results obtained in this paper are new and are an extension of the corresponding results in [2, 3, 4, 5, 7, 13].

2. Preliminaries

Let E be a real Banach space equipped with the norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ denote the dual pair between E and its dual E^* and let $\Delta : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$\Delta(x) = \{f \in E^* \mid \langle f,x\rangle = \|x\|^2,\ \|x\| = \|f\|_{E^*}\}, \quad \forall x \in E. \tag{2.1}$$

In the sequel, we will denote a selection of normalized duality mapping Δ by j. It is well known that if E is smooth, then Δ is single-valued and if $E = H$, a Hilbert space, then Δ is an identity map.

We denote by $CB(E)$ the family of all nonempty closed and bounded subsets of E; $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A,B) = \max\left\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\right\}, \quad \forall A,B \in CB(E). \tag{2.2}$$

We recall the following definitions and results which are needed in the sequel.

Definition 2.1. Let $A : E \to CB(E^*)$ be a set-valued mapping, let $J : E \to E^*$ and $g : E \to E$ be two single-valued mappings, and let $\eta : E \times E \to E$ be a bifunction.

(i) A is said to be $(\lambda - H)$-Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$H(Ax,Ay) \le \lambda\|x-y\|, \quad \forall x,y \in E; \tag{2.3}$$

(ii) J is said to be α-strongly η-monotone if there exists a constant $\alpha > 0$ such that

$$\langle J(x) - J(y), \eta(x,y)\rangle \ge \alpha\|x-y\|^2, \quad \forall x,y \in E; \tag{2.4}$$

(iii) g is said to be k-strongly accretive if there exists a constant $k > 0$ such that

$$\langle j(x-y), g(x) - g(y)\rangle \ge k\|x-y\|^2, \quad \forall x,y \in E,\ j(x-y) \in \triangle(x-y); \tag{2.5}$$

(iv) η is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x-y, \eta(x,y)\rangle \ge \alpha\|x-y\|^2, \quad \forall x,y \in E. \tag{2.6}$$

Definition 2.2 [4]. Let $\eta : E \times E \to E$ be a single-valued mapping. A proper functional $\phi : E \to R \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in E$ if there exists a point $f^* \in E^*$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \quad \forall y \in E, \tag{2.7}$$

where f^* is called η-subgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial_\eta \phi(x)$. The mapping $\partial_\eta \phi : E \to 2^{E^*}$ defined by

$$\partial_\eta \phi(x) = \{f^* \in E^* : \phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \ \forall y \in E\} \tag{2.8}$$

is said to be η-subdifferential of ϕ at $x \in E$.

Definition 2.3 [5]. A functional $f : E \times E \to R \cup \{+\infty\}$ is said to be 0-diagonally quasiconcave (in short, 0-DQCV) in x if for any finite set $\{x_1,\dots,x_n\} \subset E$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, $\sum_{i=1}^n f(x_i, y) \leq 0$.

Definition 2.4. Let $\eta : E \times E \to E$ be a single-valued mapping. Let $\phi : E \to R \cup \{+\infty\}$ be lower semicontinuous, η-subdifferentiable (may not be convex) proper functional and let $J : E \to E^*$ be a nonlinear mapping. If for any given point $x^* \in E^*$ and $\rho > 0$, there exists a unique point $x \in E$ satisfying

$$\langle Jx - x^*, \eta(y,x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E, \tag{2.9}$$

then the mapping $x^* \mapsto x$, denoted by $J_\rho^{\partial_\eta \phi}(x^*)$, is called $(J-\eta)$-proximal mapping of ϕ.

Clearly, there exists $x^* - Jx \in \rho \partial_\eta \phi(x)$ and then it follows that

$$J_\rho^{\partial_\eta \phi}(x^*) = (J + \rho \partial_\eta \phi)^{-1}(x^*). \tag{2.10}$$

Lemma 2.5 (Nadler [25]). *Let E be a complete metric space and let $T : E \to CB(E)$ be a set-valued mapping. Then for any $\epsilon > 0$ and for any $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \leq (1+\epsilon) H(T(x), T(y)). \tag{2.11}$$

Lemma 2.6 [7]. *Let D be a nonempty convex subset of a topology vector space and let $f : D \times D \to R \cup \{\pm\infty\}$ be such that*

(i) *for each $x \in D$, $y \mapsto f(x,y)$ is lower semicontinuous on each compact subset of D,*

(ii) *for each finite set $\{x_1,\dots,x_n\} \subset D$ and for each $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, $\min_{1 \leq i \leq n} f(x_i, y) \leq 0$,*

(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$, there is an $x \in co(D_0 \cup \{y\})$ satisfying $f(x,y) > 0$.*

Then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Very recently, exploiting Lemma 2.6 and using the similar technique of Ding and Xia [7], Ahmad et al. [2] have proved the existence and Lipschitz continuity of the $(J-\eta)$-proximal mapping of a proper functional ϕ on reflexive Banach spaces.

Lemma 2.7 [2]. *Let E be a reflexive Banach space with the dual space E^* and let $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable proper functional (which may not be convex). Let $J : E \to E^*$ be continuous and η-strongly monotone with constant $\alpha > 0$. Let $\eta : E \times E \to E$ be a continuous mapping such that $\eta(y', y) = -\eta(y, y')$ for all $y', y \in E$, and for any $x \in E$, the function $h(y,x) = \langle x^* - Jx, \eta(y,x)\rangle$ is 0-DQCV in y. Then for any $\rho > 0$, and any $x^* \in E^*$, there exists a unique $x \in E$ such that*

$$\langle Jx - x^*, \eta(y,x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E. \tag{2.12}$$

That is, $x = J_\rho^{\partial_\eta \phi}(x^*)$ and so the J^η-proximal mapping of ϕ is well defined. Furthermore, if $\eta : E \times E \to E$ is Lipschitz continuous with constant $\tau > 0$, then $J_\rho^{\partial_\eta \phi}$ is (τ/α)-Lipschitz continuous.

In order to obtain our results, we also use the following lemma.

Lemma 2.8 [2]. *Let E be a real Banach space and let $\Delta : E \to 2^{E^*}$ be the normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E, \ j(x+y) \in \Delta(x+y). \tag{2.13}$$

Let $A,B,C,D : E \to CB(E^*)$ and $G : E \to CB(E)$ be set-valued mappings; let $M,N : E^* \times E^* \to E^*$, $\eta : E \times E \to E$, and $g : E \to E$ be single-valued mappings; and let $\phi : E \times E \to R \cup \{+\infty\}$ be such that for each fixed $s \in E$, $\phi(\cdot,s) : E \to R \cup \{+\infty\}$ is a proper lower semicontinuous and η-subdifferentiable on E and $g(E) \cap \mathrm{dom}\partial_\eta \phi(\cdot,s) \neq \emptyset$. Consider the following generalized nonlinear set-valued variational-like inclusion problem: find $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, and $s \in G(x)$ such that $g(x) \in \mathrm{dom}(\partial_\eta \phi(\cdot,s))$ and

$$\langle M(u,v) - N(w,z), \eta(y, g(x))\rangle \geq \phi(g(x),s) - \phi(y,s), \quad \forall y \in E. \tag{2.14}$$

Special cases. (I) *If $G = I$, the identity mapping, then problem (2.14) reduces to the following generalized nonlinear variational-like inclusion: find $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$ such that $g(x) \in \mathrm{dom}(\partial_\eta \phi(\cdot,x))$ and*

$$\langle M(u,v) - N(w,z), \eta(y, g(x))\rangle \geq \phi(g(x),x) - \phi(y,x), \quad \forall y \in E, \tag{2.15}$$

which is to be a new one.

(II) *If $M(u,v) = f(x)$ for all $x \in E$, $u \in A(x)$, $v \in B(x)$, then problem (2.15) is equivalent to the following generalized quasivariational-like inclusion problem: find $x \in E$, $u \in A(x)$, $v \in B(x)$ such that $g(x) \in \mathrm{dom}(\partial_\eta \phi(\cdot,x))$ and*

$$\langle f(x) - N(u,v), \eta(y, g(x))\rangle \geq \phi(g(x),x) - \phi(y,x), \quad \forall y \in E. \tag{2.16}$$

This problem has been introduced and studied by Ahmad et al. [2] *very recently.*

(III) *If* $E = H$, *a Hilbert space,* $N(w,z) = 0$ *for all* $w \in C(x)$ *and* $z \in D(x)$, *and* $G = I$, *an identity mapping on* H, *then problem (2.14) reduces to the following generalized quasivariational-like inclusion problem: find* $x \in H$, $u \in A(x)$, $v \in B(x)$ *such that* $g(x) \in \mathrm{dom}(\partial_\eta \phi(\cdot,x))$ *and*

$$\langle M(u,v),\eta(y,g(x))\rangle \geq \phi(g(x),x) - \phi(y,x), \quad \forall y \in H, \tag{2.17}$$

which is introduced and studied by Ding [4].

(IV) *If* $E = H$, *a Hilbert space;* $M(u,v) = u - v$, *for all* $u,v \in H$; A, B *are both single-valued mappings;* $N(w,z) = 0$ *for all* $w \in C(x)$ *and* $z \in D(x)$;*and* $G = I$, *an identity mapping on* H, *then problem (2.14) reduces to the following variational inclusion considered by Ding and Luo* [5]: *find* $x \in H$ *such that* $g(x) \cap \mathrm{dom}\,\partial_\eta \phi(\cdot,x) \neq \emptyset$ *and*

$$\langle A(x) - B(x),\eta(y,g(x))\rangle \geq \phi(g(x),x) - \phi(y,x), \quad \forall y \in H. \tag{2.18}$$

We remark that for the appropriate and suitable choices of the mappings η, M, N, A, B, C, D, G, g, ϕ and the space E, one can obtain from problem (2.14) many known and new classes of generalized variational and quasivariational inequalities (inclusions) and complementarity problems, studied previously by many authors as special cases, see [2, 4, 5, 7, 12] and the references therein.

3. Wiener-Hopf equations and iterative algorithms

Assume that $g(E) \cap \mathrm{dom}\partial_\eta \phi(\cdot,s) \neq \emptyset$ for any $s \in E$.

Related to problem (2.14), we consider the following Wiener-Hopf equation (in short, WHE): find $t \in E^*$, $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $s \in G(x)$ such that

$$M(u,v) - N(w,z) + \rho^{-1} R_\rho^{\partial_\eta \phi(\cdot,s)}(t) = 0, \tag{3.1}$$

where $\rho > 0$ is a constant, $R_\rho^{\partial_\eta \phi(\cdot,s)} = I - J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}$, and $I : E^* \to E^*$ is the identity mapping.

Lemma 3.1. *The following statements are equivalent:*

(i) (x,u,v,w,z,s), *where* $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $s \in G(x)$, *is a solution of problem (2.14);*

(ii) (x,u,v,w,z,s), *where* $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $s \in G(x)$, *is a solution of the following equation:*

$$g(x) = J_\rho^{\partial_\eta \phi(\cdot,s)}(J \circ g(x) - \rho M(u,v) + \rho N(w,z)), \tag{3.2}$$

where $J_\rho^{\partial_\eta \phi(\cdot,s)}$ *denotes* J^η*-proximal mapping of* $\phi(\cdot,s)$ *for each fixed* $s \in E$, $J \circ g$ *denotes* J *composition* g, *and* $\rho > 0$ *is a constant;*

(iii) (t,x,u,v,w,z,s), *where* $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $s \in G(x)$, *is a solution of* WHE *(3.1), where*

$$g(x) = J_\rho^{\partial_\eta \phi(\cdot,s)}(t), \tag{3.3}$$

$$t = J \circ g(x) - \rho M(u,v) + \rho N(w,z). \tag{3.4}$$

Proof. It is obvious that (i) is equivalent to (ii) in view of the definition of $J_\rho^{\partial_\eta \phi}$. Hence, we only need to prove that (i) is equivalent to (ii). In fact, let (x,u,v,w,z,s) be the solution of problem (2.14), then (3.2) holds. Using the fact that $R_\rho^{\partial_\eta \phi(\cdot,s)} = I - J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}$ and (3.2), we get that

$$\begin{aligned} &R_\rho^{\partial_\eta \phi(\cdot,s)}[J \circ g(x) - \rho M(u,v) + \rho N(w,z)] \\ &\quad = J \circ g(x) - \rho M(u,v) + \rho N(w,z) - J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}[J \circ g(x) - \rho M(u,v) + \rho N(w,z)] \\ &\quad = J \circ g(x) - \rho M(u,v) + \rho N(w,z) - J \circ g(x) \\ &\quad = -\rho M(u,v) + \rho N(w,z), \end{aligned} \tag{3.5}$$

which implies that

$$M(u,v) - N(w,z) + \rho^{-1} R_\rho^{\partial_\eta \phi(\cdot,s)}(t) = 0, \tag{3.6}$$

with $t = J \circ g(x) - \rho M(u,v) + \rho N(w,z)$.

Conversely, let (t,x,u,v,w,z,s) be a solution of WHE (3.1). It follows that

$$\rho M(u,v) - \rho N(w,z) = -R_\rho^{\partial_\eta \phi(\cdot,s)}(t) = J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}(t) - t. \tag{3.7}$$

From (3.2) and (3.7), we have

$$\begin{aligned} &\rho M(u,v) - \rho N(w,z) \\ &\quad = J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}[J \circ g(x) - \rho M(u,v) + \rho N(w,z)] - J \circ g(x) + \rho M(u,v) - \rho N(w,z) \\ &\quad \Longrightarrow J \circ g(x) = J \circ J_\rho^{\partial_\eta \phi(\cdot,s)}[J \circ g(x) - \rho M(u,v) + \rho N(w,z)] \\ &\quad \Longrightarrow g(x) = J_\rho^{\partial_\eta \phi(\cdot,s)}[J \circ g(x) - \rho M(u,v) + \rho N(w,z)], \end{aligned} \tag{3.8}$$

which means that (x,u,v,w,z,s) is the solution of problem (2.14). This completes the proof. □

Based on Lemma 3.1, we can suggest the following iterative algorithms for problems (2.14) and (2.15), respectively.

Algorithm 3.2. Let $A,B,C,D : E \to CB(E^*)$ and $G : E \to CB(E)$ be set-valued mappings, let $M,N : E^* \times E^* \to E^*$, $\eta : E \times E \to E$, $J : E \to E^*$ be single-valued mappings, and let $g : E \to E$ be the single-valued mapping with $g(E) = E$. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a lower semi-continuous, η-subdifferentiable proper functional on E satisfying $g(E) \cap \operatorname{dom} \partial_\eta \phi(\cdot, s) \neq \emptyset$. For given $t_0 \in E^*$, $x_0 \in E$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$, $z_0 \in D(x_0)$, $s_0 \in G(x_0)$, by (3.4), we have

$$t_1 = J \circ g(x_0) - \rho M(u_0, v_0) + \rho N(w_0, z_0). \tag{3.9}$$

By $g(E) = E$, there exists a point $x_1 \in E$ such that

$$g(x_1) = J_\rho^{\partial_\eta \phi(\cdot, s_0)}(t_1). \tag{3.10}$$

Since $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$, $z_0 \in D(x_0)$, $s_0 \in G(x_0)$, by Nadler's lemma (Lemma 2.5), there exist $u_1 \in A(x_1)$, $v_1 \in B(x_1)$, $w_1 \in C(x_1)$, $z_1 \in D(x_1)$, $s_1 \in G(x_1)$ such that

$$\begin{aligned}
||u_1 - u_0|| &\leq (1+1) H(A(x_1), A(x_0)), \\
||v_1 - v_0|| &\leq (1+1) H(B(x_1), B(x_0)), \\
||w_1 - w_0|| &\leq (1+1) H(C(x_1), C(x_0)), \\
||z_1 - z_0|| &\leq (1+1) H(D(x_1), D(x_0)), \\
||s_1 - s_0|| &\leq (1+1) H(G(x_1), G(x_0)).
\end{aligned} \tag{3.11}$$

Let $t_2 = J \circ g(x_1) - \rho M(u_1, v_1) + \rho N(w_1, z_1)$, by $g(E) = E$, there exists a point $x_2 \in E$ such that

$$g(x_2) = J_\rho^{\partial_\eta \phi(\cdot, s_1)}(t_2), \tag{3.12}$$

continuing the above process inductively, we can define the following iterative sequences $\{t_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, $\{s_n\}$ for solving problem (2.14) as follows:

(i)

$$\begin{aligned}
u_n \in A(x_n) &: ||u_{n+1} - u_n|| \leq (1 + (n+1)^{-1}) H(A(x_{n+1}), A(x_n)), \\
v_n \in B(x_n) &: ||v_{n+1} - v_n|| \leq (1 + (n+1)^{-1}) H(B(x_{n+1}), B(x_n)), \\
w_n \in C(x_n) &: ||w_{n+1} - w_n|| \leq (1 + (n+1)^{-1}) H(C(x_{n+1}), C(x_n)), \\
z_n \in D(x_n) &: ||z_{n+1} - z_n|| \leq (1 + (n+1)^{-1}) H(D(x_{n+1}), D(x_n)), \\
s_n \in G(x_n) &: ||s_{n+1} - s_n|| \leq (1 + (n+1)^{-1}) H(G(x_{n+1}), G(x_n));
\end{aligned} \tag{3.13}$$

(ii)

$$t_{n+1} = J \circ g(x_n) - \rho M(u_n, v_n) + \rho N(w_n, z_n); \tag{3.14}$$

(iii)

$$g(x_{n+1}) = J_{\rho}^{\partial_\eta \phi(\cdot,s_n)}(t_{n+1}), \tag{3.15}$$

for $n \geq 0$, where $\rho > 0$ is a constant.

Algorithm 3.3. Let $A,B,C,D : E \to CB(E^*)$ be four set-valued mappings, let $M,N : E^* \times E^* \to E^*$, $\eta : E \times E \to E$, $J : E \to E^*$, $g : E \to E$ be single-valued mappings. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable proper functional on E satisfying $g(E) \cap \operatorname{dom}\partial_\eta \phi(\cdot,x) \neq \emptyset$. For given $t_0 \in E^*$, $x_0 \in E$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$, $z_0 \in D(x_0)$, compute the sequences $\{t_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ for solving problem (2.15) as follows:

(i)

$$\begin{aligned} u_n \in A(x_n) &: \|u_{n+1} - u_n\| \leq (1+(n+1)^{-1})H(A(x_{n+1}),A(x_n)), \\ v_n \in B(x_n) &: \|v_{n+1} - v_n\| \leq (1+(n+1)^{-1})H(B(x_{n+1}),B(x_n)), \\ w_n \in C(x_n) &: \|w_{n+1} - w_n\| \leq (1+(n+1)^{-1})H(C(x_{n+1}),C(x_n)), \\ z_n \in D(x_n) &: \|z_{n+1} - z_n\| \leq (1+(n+1)^{-1})H(D(x_{n+1}),D(x_n)); \end{aligned} \tag{3.16}$$

(ii)

$$t_{n+1} = (1-\lambda)t_n + \lambda[J \circ g(x_n) - \rho M(u_n,v_n) + \rho N(w_n,z_n)], \tag{3.17}$$

for $n \geq 0$;

(iii)

$$g(x_n) = J_{\rho}^{\partial_\eta \phi(\cdot,x_n)}(t_n), \quad \forall n \geq 1, \tag{3.18}$$

where $\rho > 0$ is a constant and $0 < \lambda < 1$ is a relaxation parameter.

4. Convergence

In this section, we prove the existence of solutions to problems (2.14) and (2.15) and convergence of Algorithms 3.2 and 3.3 in different methods.

Theorem 4.1. *Let E be a real reflexive Banach space. Let $\eta : E \times E \to E$ be continuous and τ-Lipschitz continuous such that $\eta(x,y) = -\eta(y,x)$ for all $x,y \in E$; let for any given $x^* \in E^*$, the function $h(y,u) = \langle x^* - Ju, \eta(y,u)\rangle$ be* 0-DQCV *in y; let $J : E \to E^*$ be α-strongly η-monotone and λ_j Lipschitz continuous; let $\phi : E \times E \to R \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable proper functional on E; let $A,B,C,D : E \to CB(E^*)$ and $G : E \to CB(E)$ be λ_A, λ_B, λ_C, λ_D and $(\lambda_G - H)$-Lipschitz continuous, respectively; let g be k-strongly accretive and λ_g-Lipschitz continuous; let $M : E^* \times E^* \to E^*$ be λ_{M_1} and λ_{M_2}-Lipschitz continuous in the first and second arguments, respectively; and let $N : E^* \times E^* \to E^*$ be λ_{N_1} and λ_{N_2}-Lipschitz continuous in the first and second arguments, respectively. Suppose that there exist* *constants* $\sigma > 0$, $\rho > 0$ *such that for each* $s_1, s_2 \in E$, $v^* \in E^*$,

$$\left\| J_\rho^{\partial_\eta \phi(\cdot, s_1)}(v^*) - J_\rho^{\partial_\eta \phi(\cdot, s_2)}(v^*) \right\| \leq \sigma \| s_1 - s_2 \|, \tag{4.1}$$

$$\rho < \sqrt{\frac{2k - 1 - 2\sigma^2 \lambda_G^2 - 4b^2 \lambda_j^2 \lambda_g^2}{16b^2 (\lambda_{M_1}^2 \lambda_A^2 + \lambda_{M_2}^2 \lambda_B^2 + \lambda_{N_1}^2 \lambda_C^2 + \lambda_{N_2}^2 \lambda_D^2)}}, \tag{4.2}$$

$$b = \frac{\tau}{a}, \quad k > 0.5 + \sigma^2 \lambda_G^2 + 2b^2 \lambda_j^2 \lambda_g^2, \tag{4.3}$$

then the iterative sequences $\{t_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, $\{s_n\}$ *generated by Algorithm 3.2 strongly converge to* t, x, u, v, w, z, s, *respectively, and* (x, u, v, w, z, s) *is a solution of problem (2.14).*

Proof. By the Lipschitz continuity and k-strong accretivity of g, and by Lemma 2.7, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|g(x_{n+1}) - g(x_n) - g(x_{n+1}) + g(x_n) + x_{n+1} - x_n\|^2 \\ &\leq \|g(x_{n+1}) - g(x_n)\|^2 - 2\langle g(x_{n+1}) - g(x_n) - x_{n+1} + x_n, j(x_{n+1} - x_n)\rangle \\ &\leq \|g(x_{n+1}) - g(x_n)\|^2 - 2(k-1)\|x_{n+1} - x_n\|^2. \end{aligned} \tag{4.4}$$

Then,

$$\|x_{n+1} - x_n\|^2 \leq \frac{1}{2k-1} \|g(x_{n+1}) - g(x_n)\|^2. \tag{4.5}$$

By Algorithm 3.2(ii) and (iii), (4.1), $(\lambda_G - H)$-Lipschitz continuity of G, and Lemma 2.5, we have

$$\begin{aligned} &\frac{1}{2}\|g(x_{n+1}) - g(x_n)\|^2 \\ &\quad\leq \left\| J_\rho^{\partial_\eta \phi(\cdot, s_n)}(t_{n+1}) - J_\rho^{\partial_\eta \phi(\cdot, s_{n-1})}(t_{n+1}) \right\|^2 + \left\| J_\rho^{\partial_\eta \phi(\cdot, s_{n-1})}(t_{n+1}) - J_\rho^{\partial_\eta \phi(\cdot, s_{n-1})}(t_n) \right\|^2 \\ &\quad\leq \sigma^2 \|s_n - s_{n-1}\|^2 + b^2 \|t_{n+1} - t_n\|^2 \\ &\quad\leq \sigma^2 (1 + n^{-1})^2 \lambda_G^2 \|x_n - x_{n-1}\|^2 + b^2 \|t_{n+1} - t_n\|^2. \end{aligned} \tag{4.6}$$

Now we estimate $\|t_{n+1} - t_n\|^2$. Since M is $\lambda_{M_1}, \lambda_{M_2}$-Lipschitz continuous in the first and second arguments, respectively, and N is $\lambda_{N_1}, \lambda_{N_2}$-Lipschitz continuous in the first and second arguments, respectively, and using the Lipschitz continuity of A, B, C, D, G, J, g, and (4.2) and (4.3), we obtain that

$$\begin{aligned}
\|t_{n+1}-t_n\|^2 &= \|J\circ g(x_n)-\rho M(u_n,v_n)+\rho N(w_n,z_n)-J\circ g(x_{n-1})\\
&\qquad +\rho M(u_{n-1},v_{n-1})-\rho N(w_{n-1},z_{n-1})\|^2\\
&= \|J\circ g(x_n)-J\circ g(x_{n-1})\\
&\qquad -\rho[M(u_n,v_n)-M(u_{n-1},v_{n-1})-(N(w_n,z_n)-N(w_{n-1},z_{n-1}))]\|^2\\
&\le 2\lambda_j^2\lambda_g^2\|x_n-x_{n-1}\|^2\\
&\qquad +2\rho^2\|M(u_n,v_n)-M(u_{n-1},v_{n-1})-(N(w_n,z_n)-N(w_{n-1},z_{n-1}))\|^2\\
&\le 2\lambda_j^2\lambda_g^2\|x_n-x_{n-1}\|^2+4\rho^2\|M(u_n,v_n)-M(u_{n-1},v_{n-1})\|^2\\
&\qquad +4\rho^2\|N(w_n,z_n)-N(w_{n-1},z_{n-1})\|^2\\
&\le 2\lambda_j^2\lambda_g^2\|x_n-x_{n-1}\|^2\\
&\qquad +8\rho^2\left(1+\frac{1}{n}\right)^2(\lambda_{M_1}^2\lambda_A^2+\lambda_{M_2}^2\lambda_B^2+\lambda_{N_1}^2\lambda_C^2+\lambda_{N_2}^2\lambda_D^2)\|x_n-x_{n-1}\|^2.
\end{aligned}\tag{4.7}$$

It follows from (4.5)–(4.7) that

$$\|x_{n+1}-x_n\|\le\alpha_n\|x_n-x_{n-1}\|,\tag{4.8}$$

where

$$\alpha_n=\sqrt{\frac{(1+n^{-1})^2[2\sigma^2\lambda_G^2+16b^2\rho^2(\lambda_{M_1}^2\lambda_A^2+\lambda_{M_2}^2\lambda_B^2+\lambda_{N_1}^2\lambda_C^2+\lambda_{N_2}^2\lambda_D^2)]+4b^2\lambda_j^2\lambda_g^2}{2k-1}}.\tag{4.9}$$

Let

$$\alpha=\sqrt{\frac{2\sigma^2\lambda_G^2+16b^2\rho^2(\lambda_{M_1}^2\lambda_A^2+\lambda_{M_2}^2\lambda_B^2+\lambda_{N_1}^2\lambda_C^2+\lambda_{N_2}^2\lambda_D^2)+4b^2\lambda_j^2\lambda_g^2}{2k-1}}.\tag{4.10}$$

It is easy to see that $\alpha_n\to\alpha$ as $n\to\infty$. Since $0<\alpha<1$ by conditions (4.2) and (4.3), $\alpha_n<(1+\alpha)/2<1$ for sufficiently large n. It follows from (4.8) that $\{x_n\}$ is a Cauchy sequence and hence there is an $x\in E$ such that $x_n\to x\in E$ as $n\to\infty$. By the Lipschitz continuity of A, B, C, D, G, it follows from Algorithm 3.2(i) that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, $\{s_n\}$ are also Cauchy sequences. Let $u_n\to u$, $v_n\to v$, $w_n\to w$, $z_n\to z$, $s_n\to s$ as $n\to\infty$. By Algorithm 3.2 (ii) and (iii), we get that

$$\begin{gathered}
t_{n+1}=J\circ g(x_n)-\rho M(u_n,v_n)+\rho N(w_n,z_n),\\
g(x_{n+1})=J_\rho^{\partial_\eta\phi(\cdot,s_n)}(t_{n+1}).
\end{gathered}\tag{4.11}$$

In view of Lipschitz continuity of g, J, M, N and (4.1), letting $n \to \infty$, we have that

$$\begin{gathered} t = \left(J \circ g(x) - \rho M(u,v) + \rho N(w,z)\right), \\ g(x) = J_\rho^{\partial_\eta \phi(\cdot,s)}(t). \end{gathered} \tag{4.12}$$

Next, we claim that $u \in A(x)$. In fact,

$$\begin{aligned} d(u,A(x)) &\le \|u - u_n\| + d(u_n, A(x_n)) + H(A(x_n), A(x)) \\ &\le \|u - u_n\| + \lambda_A \|x_n - x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.13}$$

Hence, $d(u,A(x)) = 0$ and so $u \in A(x)$ since $A(x) \in CB(E^*)$. In a similar way, we can also prove that $v \in B(x)$, $w \in C(x)$, $z \in D(x)$, $s \in G(x)$. From the above argument, we know that (x,t,u,v,w,z,s) satisfies WHE (3.1). It follows from Lemma 3.1 that (x,u,v,w,z,s) is a solution of problem (2.14). This completes the proof. □

Corollary 4.2. *Let E be a real reflexive Banach space. Let $\eta : E \times E \to E$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\tau > 0$ and continuous such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$; let, for any given $x^* \in E^*$, the function $h(y,u) = \langle x^* - u, \eta(y,u)\rangle$ be 0-DQCV in y; let $J = I$, the identity mapping; and let ϕ, A, B, C, D, G, g, M, N be as in Theorem 4.1. Suppose that there exist constants $\sigma > 0$, $\rho > 0$ satisfying (4.1) and the following conditions:*

$$\rho < \sqrt{\frac{2k - 1 - 2\sigma^2\lambda_G^2 - 4b^2\lambda_g^2}{16b^2\left(\lambda_{M_1}^2\lambda_A^2 + \lambda_{M_2}^2\lambda_B^2 + \lambda_{N_1}^2\lambda_C^2 + \lambda_{N_2}^2\lambda_D^2\right)}}, \tag{4.14}$$

$$b = \frac{\tau}{a}, \quad k > 0.5 + \sigma^2\lambda_G^2 + 2b^2\lambda_g^2. \tag{4.15}$$

Then the iterative sequences generated by Algorithm 3.2 strongly converge to t, x, u, v, w, z, s, respectively, and (x,u,v,w,z,s) is a solution of problem (2.14).

Remark 4.3. Theorem 4.1 and Corollary 4.2 extend and improve the corresponding results in [2].

Theorem 4.4. *Let E, M, N, A, B, C, D, g, h, J, η be as in Theorem 4.1. Suppose that there exist constants $\sigma > 0$, $\rho > 0$ satisfying (4.1) and the following conditions:*

$$\rho < \frac{\sqrt{k - 0.5 - \sigma^2} - b\lambda_j\lambda_g}{b\left(\lambda_{M_1}\lambda_A + \lambda_{M_2}\lambda_B + \lambda_{N_1}\lambda_C + \lambda_{N_2}\lambda_D\right)}, \tag{4.16}$$

$$b = \frac{\tau}{\alpha}, \quad b^2\lambda_j^2\lambda_g^2 + \sigma^2 + 0.5 < k, \tag{4.17}$$

then the iterative sequences $\{t_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ generated by Algorithm 3.3 strongly converge to x, u, v, w, z, respectively, and (x,u,v,w,z) is a solution of problem (2.15).

Proof. From Algorithm 3.3, the Lipschitz continuity of J, g and M, N in the first and second arguments, and H-Lipschitz continuity of A, B, C, D, we have

$$\begin{aligned}
||t_{n+2}-t_{n+1}|| &= ||(1-\lambda)t_{n+1}+\lambda[J\circ g(x_{n+1})-\rho M(u_{n+1},v_{n+1})+\rho N(w_{n+1},z_{n+1})] \\
&\quad -(1-\lambda)t_n-\lambda[J\circ g(x_n)-\rho M(u_n,v_n)+\rho N(w_n,z_n)]|| \\
&\le (1-\lambda)||t_{n+1}-t_n||+\lambda||J\circ g(x_{n+1})-J\circ g(x_n)|| \\
&\quad +\lambda\rho[||M(u_{n+1},v_{n+1})-M(u_n,v_n)||+||N(w_{n+1},z_{n+1})-N(w_n,z_n)||] \\
&\le (1-\lambda)||t_{n+1}-t_n||+\lambda\lambda_j\lambda_g||x_{n+1}-x_n||+\lambda\rho(1+(n+1)^{-1}) \\
&\quad \times(\lambda_{M_1}\lambda_A+\lambda_{M_2}\lambda_B+\lambda_{N_1}\lambda_C+\lambda_{N_2}\lambda_D)||x_{n+1}-x_n|| \\
&\le (1-\lambda)||t_{n+1}-t_n||+\lambda[\lambda_j\lambda_g+\rho(1+(n+1)^{-1}) \\
&\quad \times(\lambda_{M_1}\lambda_A+\lambda_{M_2}\lambda_B+\lambda_{N_1}\lambda_C+\lambda_{N_2}\lambda_D)]||x_{n+1}-x_n||.
\end{aligned} \tag{4.18}$$

Now, we estimate $\|x_{n+1}-x_n\|$. Since g is k-strongly accretive, by using Algorithm 3.3(ii) and (iii), (4.1), and Lemmas 2.6 and 2.7, we have

$$\begin{aligned}
||x_{n+1}-x_n||^2 &= ||x_{n+1}-x_n-(g(x_{n+1})-g(x_n))+(g(x_{n+1})-g(x_n))||^2 \\
&\le ||g(x_{n+1})-g(x_n)||^2-2\langle g(x_{n+1})-g(x_n)-x_{n+1}+x_n, j(x_{n+1}-x_n)\rangle \\
&\le \left[\frac{\tau}{\alpha}||t_{n+1}-t_n||+\sigma||x_{n+1}-x_n||\right]^2-(2k-2)||x_{n+1}-x_n||^2 \\
&\le 2\left[b^2||t_{n+1}-t_n||^2+\sigma^2||x_{n+1}-x_n||^2\right]-(2k-2)||x_{n+1}-x_n||^2.
\end{aligned} \tag{4.19}$$

It follows that

$$||x_{n+1}-x_n|| \le \frac{b}{\sqrt{k-0.5-\sigma^2}}||t_{n+1}-t_n||. \tag{4.20}$$

Combining (4.18) and (4.19), we have

$$\begin{aligned}
||t_{n+2}-t_{n+1}|| &\le (1-\lambda)||t_{n+1}-t_n||+\lambda[\lambda_j\lambda_g+\rho(1+(n+1)^{-1}) \\
&\quad \times(\lambda_{M_1}\lambda_A+\lambda_{M_2}\lambda_B+\lambda_{N_1}\lambda_C+\lambda_{N_2}\lambda_D)]\frac{b}{\sqrt{k-0.5-\sigma^2}}||t_{n+1}-t_n|| \qquad (4.21) \\
&\le [1-\lambda(1-\alpha_n)]||t_{n+1}-t_n||,
\end{aligned}$$

where

$$\alpha_n = \frac{b[\lambda_j\lambda_g + \rho(1+(n+1)^{-1})(\lambda_{M_1}\lambda_A + \lambda_{M_2}\lambda_B + \lambda_{N_1}\lambda_C + \lambda_{N_2}\lambda_D)]}{\sqrt{k-0.5-\sigma^2}}. \tag{4.22}$$

Letting $n \to \infty$, we see that $\alpha_n \to \alpha$, where

$$\alpha = \frac{b[\lambda_j\lambda_g + \rho(\lambda_{M_1}\lambda_A + \lambda_{M_2}\lambda_B + \lambda_{N_1}\lambda_C + \lambda_{N_2}\lambda_D)]}{\sqrt{k-0.5-\sigma^2}}. \tag{4.23}$$

Since $\alpha < 1$ by conditions (4.16) and (4.17), $0 < (1-\lambda(1-\alpha_n)) < (1+\alpha)/2$ for sufficiently large n. It follows from (4.21) that $\{t_n\}$ is a Cauchy sequence and hence there is a $t \in E$ such that $t_n \to t$ in E as $n \to \infty$. Similarly, by (4.20), we observe that $x_n \to x$ in E as $n \to \infty$. Also from Algorithm 3.3(i) we have that $u_n \to u$, $v_n \to v$, $w_n \to w$, and $z_n \to z$ in E as $n \to \infty$. As the same argument in Theorem 4.1, we know that $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $z \in D(x)$. Finally, the continuity of J, A, B, C, D, g, $J_\rho^{\partial_\eta\phi(\cdot,x)}$ and Algorithm 3.3 ensure that (t,x,u,v,w,z) is a solution of WHE (3.1) with $G = I$. It follows from Lemma 3.1 that (x,u,v,w,z) is a solution of problem (2.15). This completes the proof. □

Corollary 4.5. *Let E be a real reflexive Banach space. Let $\eta : E \times E \to E$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\tau > 0$ and continuous such that $\eta(x,y) = -\eta(y,x)$ for all $x,y \in E$; let, for any given $x^* \in E^*$, the function $h(y,u) = \langle x^* - u, \eta(y,u)\rangle$ be 0-DQCV in y; let $J = I$, the identity mapping; let ϕ, A, B, C, D, g, M, N be as in Theorem 4.4. Suppose that there exist constants $\sigma > 0$, $\rho > 0$ satisfying (4.1) and the following conditions:*

$$\rho < \frac{\sqrt{k-0.5-\sigma^2} - b\lambda_g}{b(\lambda_{M_1}\lambda_A + \lambda_{M_2}\lambda_B + \lambda_{N_1}\lambda_C + \lambda_{N_2}\lambda_D)}, \tag{4.24}$$

$$b = \frac{\tau}{\alpha}, \quad b^2\lambda_g^2 + \sigma^2 + 0.5 < k, \tag{4.25}$$

then the iterative sequences $\{t_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ generated by Algorithm 3.3 strongly converge to t, x, u, v, w, z, respectively, and (x,u,v,w,z) is a solution of problem (2.15).

Remark 4.6. Theorem 4.4 and Corollary 4.5 extend and improve the corresponding results in [3, 4, 5, 7].

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