THE STABILITY OF COLLOCATION METHODS FOR HIGHER-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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The numerical stability of the polynomial spline collocation method for general Volterra integro-differential equation is being considered. The convergence and stability of the new method are given and the efficiency of the new method is illustrated by examples. We also proved the conjecture suggested by Danciu in 1997 on the stability of the polynomial spline collocation method for the higher-order integro-differential equations.

1. Introduction

In this paper, we analyze the stability properties of the polynomial spline collocation method for the approximate solution of general Volterra integro-differential equation. Consider the linear pth-order Volterra integro-differential equation of the form

$$y^{(p)}(t) = q(t) + \sum_{j=0}^{p-1} p_j(t) y^{(j)}(t) + \sum_{j=0}^{p-1} \int_0^t k_j(t,s) y^{(j)}(s) ds, \quad t \in I := [0,T],$$
$$y^{(i)}(0) = y_0^{(i)}, \quad i = 0,1,\dots,p-1. \tag{1.1}$$

Here, the functions $q, p_j : I \to R$ and $k_j : D \to R$ $(j = 0,1,\dots,p-1)$ (with $D := \{(t,s) : 0 \le s \le t \le T\}$) are assumed to be (at least) continuous on their respective domains. For more detail of these equations and many other interesting methods for the approximated solution, stability procedures and applications are available in earlier literatures [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 18]. The above equation is usually known as basis test equation and is suggested by Brunner and Lambert [4]. Since then it, has been widely used for analyzing the stability properties [3, 4, 5, 6, 7, 8, 9, 18] of various methods.

Volterra integro-differential equation (1.1) will be solved numerically using polynomial spline spaces. To describe the polynomial spline spaces, let $\prod_N : 0 = t_0 < t_1 < \cdots < t_N = T$ be the mesh for the interval I, and set

$$\begin{aligned} \sigma_n &:= [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad n = 0,1,\dots,N-1, \\ h &= \max\{h_n : 0 \le n \le N-1\} \quad \text{(mesh diameter)}, \\ Z_N &:= \{t_n : n = 1,2,\dots,N-1\}, \qquad \overline{Z}_N = Z_N \cup \{T\}. \end{aligned} \tag{1.2}$$

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Let π_{m+d} be the set of (real) polynomials of degree not exceeding $m+d$, where $m \geq 1$ and $d \geq -1$ are given integers. The solution (y) to the initial-value problem (1.1) is approximated by an element u in the polynomial spline space

$$S_{m+d}^{(d)}(Z_N) := \{u := u(t)|_{t\in\sigma_n} := u_n(t) \in \pi_{m+d},\ n = 0,1,\dots,N-1,\\ u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n),\ \text{for } j = 0,1,\dots,d,\ t_n \in Z_N\}. \tag{1.3}$$

It is a polynomial spline function of degree $m+d$, which possesses the knots Z_N, and is d times continuously differentiable on I. If $d = -1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuities at the knots Z_N.

According to Micula [16] and Miculá and Micula [17], an element $u \in S_{m+d}^{(d)}(Z_N)$ for all $n = 0,1,\dots,N-1$ and $t \in \sigma_n$ has the following form:

$$u(t) = u_n(t) = \sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!}(t-t_n)^r + \sum_{r=1}^{m} a_{n,r}(t-t_n)^{d+r}, \tag{1.4}$$

where

$$u_{-1}^{r}(0) := \left[\frac{d^r}{dt^r}u(t)\right]_{t=0} = y^{(r)}(0), \quad r = 0,1,\dots,d. \tag{1.5}$$

From (1.4), we see that the element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined provided the coefficients $\{a_{n,r}\}_{r=1,\dots,m}$ for all $n = 0,1,\dots,N-1$ are known. In order to determine these coefficients, we consider a set of collocation parameters $\{c_j\}_{j=1,\dots,m}$, where $0 < c_1 < \cdots < c_m \leq 1$, and define the set of collocation points as

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \quad \text{with } X_n := \{t_{n,j} := t_n + c_j h_n,\ j = 1,2,\dots,m\}. \tag{1.6}$$

The approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ is determined by imposing the condition that u satisfies the initial-value problem (1.1) on $X(N)$ and the initial conditions, that is,

$$u^{(p)}(t) = q(t) + \sum_{j=0}^{p-1} p_j(t)u^{(j)}(t) + \sum_{j=0}^{p-1}\int_0^t k_j(t,s)u^{(j)}(s)ds, \quad t \in I := [0,T],\ \forall t \in X(N), \tag{1.7}$$

with

$$u^{(i)}(0) = u_0^{(i)}, \quad i = 0,1,\dots,p-1. \tag{1.8}$$

Here, we assume that the mesh sequence $\{\prod_N\}$ is uniform, that is, $h_n = h$, for all $n = 0,1,\dots,N-1$.

2. Numerical stability

In order to discuss numerical stability, we study the behavior of the method as applied to the pth-order test Volterra integro-differential equation

$$\begin{gathered}
y^{(p)}(t) = q(t) + \sum_{j=0}^{p-1} \alpha_j y^{(j)}(t) + \nu \int_0^t y(s)ds, \quad \nu \neq 0,\ t \in I = [0,T], \\
y^{(i)}(0) = y_0^{(i)}, \quad i = 0,1,\dots,p-1,
\end{gathered} \tag{2.1}$$

where α_j, ν are constants and $q : I \to R$ is sufficiently smooth.

We refer to a polynomial spline collocation method in the space $S_{m+d}^{(d)}(Z_N)$, as an (m,d,p)-method, where p is the order of the integro-differential equation.

Definition 2.1. An (m,d,p)-method is said to be stable if all solutions $\{u(t_n)\}$ remain bounded as $n \to \infty$, $h \to 0$, while hN remains fixed.

From (1.4), we observe that the first $d+1$ coefficients of $u \in S_{m+d}^{(d)}(Z_N)$ are determined by the smooth conditions, and the exact collocation equation (1.7) can be used to determine the last m coefficients. For the convenience, we introduce the following notations:

$$\begin{aligned}
\eta_n &:= (\eta_{n,r})_{r=0,\dots,d}, \quad \text{with } \eta_{n,r} := \frac{u_{n-1}^{(r)}(t_n)}{r!} h^r, \\
\beta_n &:= (\beta_{n,r})_{r=1,\dots,m}, \quad \text{with } \beta_{n,r} := a_{n,r} h^{d+r} \ (n = 0,1,\dots,N-1).
\end{aligned} \tag{2.2}$$

Using (2.2) and $t := t_n + \tau h \in \sigma_n$ in (1.4), we obtain the following:

$$u(t) = u_n(t_n + \tau h) = \sum_{r=0}^{d} \eta_{n,r} \tau^r + \sum_{r=1}^{m} \beta_{n,r} \tau^{d+r}, \quad \forall \tau \in (0,1],\ n = 0,1,\dots,N-1. \tag{2.3}$$

By direct differentiation of (2.3) and using the smooth conditions of the approximation $u \in S_{m+d}^{(d)}(Z_N)$, we get a relationship between vector η_{n+1} and vectors η_n and β_n as follows:

$$\eta_{n+1} = A\eta_n + B\beta_n, \quad \forall n = 0,1,\dots,N-1, \tag{2.4}$$

where A is the $(d+1) \times (d+1)$ upper triangular matrix, and B is the $(d+1) \times m$ matrix, whose elements are given by

$$a_{j,r} := \begin{cases} 0 & \text{if } r < j, \\ \dbinom{r}{j} & \text{if } r \geq j, \end{cases} \qquad b_{j,r} := \binom{d+r}{j}. \tag{2.5}$$

For $d \geq p$, apply the collocation method to test (2.1) and use the representation (2.3) to obtain the following collocation equation:

$$V\beta_n = W\eta_n + h^p R_n, \quad \text{for } n = 0,1,\dots,N-1, \tag{2.6}$$

where V is the $m \times m$-matrix, W is the $m \times (d+1)$-matrix, and R_n is the m-vector, whose elements are given by

$$V_{j,r} := \binom{d+r}{p} p! c_j^{d+r-p} - \nu h^{p+1} \frac{c_j^{d+r+1}}{d+r+1} - \sum_{i=0}^{p-1} \alpha_i h^{p-i} \binom{d+r}{i} i! c_j^{d+r-i}, \tag{2.7}$$

$$W_{j,r} := \begin{cases} \nu h^{p+1} c_j & \text{if } r = 0, \\ \nu h^{p+1} \dfrac{c_j^{r+1}}{r+1} + \displaystyle\sum_{i=0}^{r-1} \alpha_i h^{p-i} \binom{r}{i} i! c_j^{r-i} & \text{if } r = 1, 2, \dots, p, \\ -\dbinom{r}{p} p! c_j^{r-p} + \nu h^{p+1} \dfrac{c_j^{r+1}}{r+1} + \displaystyle\sum_{i=0}^{p-1} \alpha_i h^{p-i} \binom{r}{i} i! c_j^{r-i} & \text{if } p+1 \le r \le d, \end{cases} \tag{2.8}$$

$$R_{n,j} := \begin{cases} q(t_{0,j}) - q(t_0) & \text{if } n = 0, \\ q(t_{n,j}) - q(t_{n-1,m}) + u_{n-1}^{(p)}(t_{n-1,m}) - u_{n-1}^{(p)}(t_n) & \\ \qquad + \displaystyle\sum_{i=0}^{p-1} \alpha_i \Big[u_{n-1}^{(i)}(t_n) - u_{n-1}^{(i)}(t_{n-1,m}) \Big] & \\ \qquad + \lambda h \displaystyle\int_{c_m}^{1} u_{n-1}(t_{n-1} + \tau h)\, d\tau & \text{if } n > 0. \end{cases} \tag{2.9}$$

We state the following result for pth-order VIDEs which describes a stability criterion for the collocation spline method. The proof of this theorem is similar to the proof given by Danciu [9] for first-order VIDEs.

Theorem 2.2. *An (m,d,p)-method is stable if and only if all eigenvalues of matrix $M := A + BV^{-1}W$ are in the unit disk, and all eigenvalues with $|\lambda| = 1$ belong to a 1×1 Jordan block, where the matrices A and B are defined in (2.5).*

Remark 2.3. The dimension of the matrix M is $\dim(d+1)$. Moreover, let M_0 be the matrix M with $h = 0$, and let $\lambda^{(0)}$ and λ be the eigenvalues of M_0 and M, respectively, then it follows that the matrix M_0 has

$$\lambda_1^{(0)} = \lambda_2^{(0)} = \cdots = \lambda_{p+1}^{(0)} = 1, \quad \forall m \ge 1,\ d \ge p. \tag{2.10}$$

3. Applications

In this section, we will investigate some special cases.

(I) For the case $d = p$, the approximation space is $S_{m+p}^{(p)}(Z_N)$. From the above theorem and Remark 2.3, we have the following theorem.

Theorem 3.1. *For every choice of the collocation parameters $\{c_j\}_{j=1,\dots,m}$, an (m,p,p)-method is stable for all $m \ge 1$.*

(II) For the case $m = 1$, this choice of m corresponds to a classical spline function, that is, the approximate solution $u \in S_{1+d}^{(d)}(Z_N)$. By Remark 2.3, M_0 is the matrix M with $h = 0$, and $\lambda^{(0)}$ and λ are the respective eigenvalues of M_0 and M, and we have

$$\lambda = \lambda^{(0)} + O(h). \tag{3.1}$$

LEMMA 3.2. *If $c_1 \in (0,1]$ is the collocation parameter, then, for $m = 1$ and $d \geq p$, the trace of the matrix M_0 can be computed by the following formula:*

$$\operatorname{Tr}(M_0) = d + 2 + \frac{1}{c_1^{d-p+1}} - \left(1 + \frac{1}{c_1}\right)^{d-p+1}. \tag{3.2}$$

Proof. Let V_0 and W_0 be the matrices V and W with $h = 0$, respectively. Then, for $m = 1$, we have from (2.7) and (2.8) that V_0 is a 1×1-matrix and W_0 is a $1 \times (d+1)$-matrix, whose elements are given by

$$\begin{aligned} V_0 &:= \binom{d+1}{p} p! c_1^{d-p+1}, \\ \left(W_0\right)_{1,r} &:= \begin{cases} 0 & \text{if } r = 0,1,\dots,p, \\ -\binom{r}{p} p! c_1^{r-p} & \text{if } p+1 \leq r \leq d. \end{cases} \end{aligned} \tag{3.3}$$

Now, from the definition of the matrices A and B as in (2.5) (note that the diagonal entry of the matrix A is one), we have

$$\begin{aligned} \operatorname{Tr}(M_0) &= \operatorname{Tr}\left(A + B V_0^{-1} W_0\right) \\ &= \operatorname{Tr}(A) + \frac{1}{\binom{d+1}{p} p! c_1^{d-p+1}} \operatorname{Tr}(B W_0) \\ &= d + 1 - \frac{1}{\binom{d+1}{p} p! c_1^{d-p+1}} \sum_{i=p+1}^{d} \binom{d+1}{i} \binom{i}{p} p! c_1^{i-p}. \end{aligned} \tag{3.4}$$

However, by the binomial expansion, we have the following identity:

$$\sum_{i=p+1}^{d} \binom{d+1}{i} \binom{i}{p} p! c_1^{i-p} = \binom{d+1}{p} p! \left(\left(1 + c_1\right)^{d-p+1} - 1 - c_1^{d-p+1}\right). \tag{3.5}$$

Hence,

$$\operatorname{Tr}(M_0) = d + 2 + \frac{1}{c_1^{d-p+1}} - \left(1 + \frac{1}{c_1}\right)^{d-p+1}. \tag{3.6}$$

□

For the stability of the spline collocation method $S_{1+d}^{(d)}(Z_N)$ $(m = 1)$, we have the following theorem.

Theorem 3.3. *A $(1,d,p)$-method $(d \geq p)$ is stable if and only if one of the following conditions is true:*

(i) $d = p$ *and* $c_1 \in (0,1]$,

(ii) $d = p+1$ *and* $c_1 = 1$.

Proof. For the case $d = p$, the conclusion follows from Theorem 3.1.

If $d = p+1$, then, using (2.10) and (3.2), the $p+2$-eigenvalue of M_0 can be computed as follows:

$$\lambda_{p+2}^{(0)} = \operatorname{Tr}(M_0) - p - 1 = p + 3 + \frac{1}{c_1^2} - \left(1 + \frac{1}{c_1}\right)^2 - p - 1 = 1 - \frac{2}{c_1}. \tag{3.7}$$

Therefore, if $c_1 \in (0,1]$, then $\lambda_{p+2}^{(0)} \leq -1$, and its absolute value is 1 if and only if $c_1 = 1$.

If $d \geq p+2$, then, setting $\theta = d - p + 1$ in (3.2), we have

$$\operatorname{Tr}(M_0) = p - 1 + \theta + 2 + \frac{1}{c_1^\theta} - \left(1 + \frac{1}{c_1}\right)^\theta = p + \theta - \sum_{i=1}^{\theta-1} \binom{\theta}{i} \frac{1}{c_1^i}. \tag{3.8}$$

If $\theta > 3$ (i.e., $d > p+2$), then, by induction, we can prove $\binom{\theta}{i} > \theta$ $(i = 1,2,\ldots,\theta-1)$ and $\theta(\theta-1) > 2(\theta+1)$.

Thus, if $c_1 \in (0,1]$, then

$$\operatorname{Tr}(M_0) < p + \theta - \theta(\theta-1) < p + \theta - 2(\theta+1), \tag{3.9}$$

and therefore

$$-\infty < \operatorname{Tr}(M_0) < p - \theta - 2 = 2p - 3 - d. \tag{3.10}$$

If $p = 1$, then from (3.10),

$$\operatorname{Tr}(M_0) = \lambda_1^{(0)} + \lambda_2^{(0)} + \cdots + \lambda_{d+1}^{(0)} < -(d+1), \quad \text{for } d > 3. \tag{3.11}$$

Therefore, there exists an eigenvalue $\lambda^{(0)}$ whose value is smaller than -1.

If $p > 1$, then, from (2.10) and (3.10), we have

$$\lambda_p^{(0)} + \lambda_{p+1}^{(0)} + \cdots + \lambda_{d+1}^{(0)} < -(d+2-p), \quad \text{for } d > p+2 > 3. \tag{3.12}$$

Thus, there exists an eigenvalue $\lambda^{(0)}$ whose value is less than -1.

If $d = p+2$ and $c_1 \in (0,1]$, then, from (3.2), we have

$$\begin{aligned} \lambda_{p+2}^{(0)} + \lambda_{p+3}^{(0)} &= \operatorname{Tr}(M_0) - p - 1 \\ &= p + 4 + \frac{1}{c_1^3} - \left(1 + \frac{1}{c_1}\right)^3 - p - 1 \\ &= 2 - \frac{3}{c_1} - \frac{3}{c_1^2} \leq -4. \end{aligned} \tag{3.13}$$

Hence,

$$\lambda_{p+2}^{(0)} < -1 \quad \text{or} \quad \lambda_{p+3}^{(0)} < -1. \tag{3.14}$$

Thus, from Theorem 2.2, a $(1,d,p)$-method is unstable for any choice of the collocation parameter $c_1 \in (0,1]$ when $d \geq p+2$. □

(III) For the case $m = 2$, we can prove the following theorem. The proof is similar to the proof given in [9] for first-order integro-differential equation ($p = 1$).

Theorem 3.4. *Let $0 < c_1 < c_2 \leq 1$ be the collocation parameters, then*

(i) $(2,p,p)$*-method is stable for every choice of the collocation parameters,*

(ii) $(2,p+1,p)$*-method is stable if and only if $c_1 + c_2 \geq 3/2$,*

(iii) *if $c_2 = 1$, then $(2,d,p)$-method is unstable for all $d \geq p+2$.*

(IV) For the case $d = p+1$, the approximation $u \in S_{m+p+1}^{(p+1)}(Z_N)$ and the dimension of the matrix M_0 are $p+2$, whose $\lambda_1^{(0)} = \lambda_2^{(0)} = \cdots = \lambda_{p+1}^{(0)} = 1$ are its first $p+1$-eigenvalues. To compute the $p+2$-eigenvalue, we need the following results. But, first we introduce the following notations:

$$\begin{gathered} S_k := S_k(c_1,\ldots,c_m) = \sum_{1\leq i_1<\cdots<i_k\leq m}^{m} c_{i_1}c_{i_2}\cdots c_{i_k}, \quad \text{for } 1 \leq k \leq m, \\ S_0 := S_0(c_1,\ldots,c_m) = 1, \\ S_{k,j} := S_k(c_1,\ldots,c_{j-1},c_{j+1},\ldots,c_m), \quad \text{for } 1 \leq k \leq m-1,\ 1 \leq j \leq m. \end{gathered} \tag{3.15}$$

Lemma 3.5. *Let $0 < c_1 < c_2 < \cdots < c_m \leq 1$ be the collocation parameters, then*

$$\begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ 1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{i-1} & c_m^{i+1} & \cdots & c_m^m \end{vmatrix} = S_{m-i} \prod_{1\leq k<j\leq m}^{m} (c_j - c_k). \tag{3.16}$$

Proof. We will prove the lemma by induction on the dimension of the matrix, starting with 2×2-matrices. For the 2×2-matrices, the result is clearly true. For $m\times m$-matrices ($m > 2$), we define

$$f(x) := \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ 1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{m-1} & c_{m-1}^2 & \cdots & c_{m-1}^{i-1} & c_{m-1}^{i+1} & \cdots & c_{m-1}^m \\ 1 & x & x^2 & \cdots & x^{i-1} & x^{i+1} & \cdots & x^m \end{vmatrix}. \tag{3.17}$$

Note that

$$\begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ 1 & c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{i-1} & c_m^{i+1} & \cdots & c_m^m \end{vmatrix} = f(c_m). \tag{3.18}$$

Now, since $f(c_1) = f(c_2) = \cdots = f(c_{m-1}) = 0$, we have

$$f(x) = a(x-b)\prod_{i=1}^{m-1}(x-c_i), \tag{3.19}$$

where a, b are constants to be determined. By the induction hypothesis, we obtain

$$a = S_{m-1-i}(c_1,\ldots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k). \tag{3.20}$$

Moreover, from (3.19),

$$f(0) = a(-1)^m c_1 c_2 \cdots c_{m-1} b. \tag{3.21}$$

On the other hand, from the definition of f and by the induction hypothesis, we have

$$\begin{aligned} f(0) &= (-1)^{m+1}\begin{vmatrix} c_1 & c_1^2 & \cdots & c_1^{i-1} & c_1^{i+1} & \cdots & c_1^m \\ c_2 & c_2^2 & \cdots & c_2^{i-1} & c_2^{i+1} & \cdots & c_2^m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m-1} & c_{m-1}^2 & \cdots & c_{m-1}^{i-1} & c_{m-1}^{i+1} & \cdots & c_{m-1}^m \end{vmatrix} \\ &= (-1)^{m+1} c_1 c_2 \cdots c_{m-1} S_{m-i}(c_1,\ldots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k). \end{aligned} \tag{3.22}$$

Thus, from (3.21) and (3.22), we have

$$-ab = S_{m-i}(c_1,\ldots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k), \tag{3.23}$$

and so

$$\begin{aligned} f(c_m) &= a(c_m-b)\prod_{i=1}^{m-1}(c_m-c_i) \\ &= \left[c_m S_{m-1-i}(c_1,\ldots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k)\right. \\ &\qquad \left. + S_{m-i}(c_1,\ldots,c_{m-1})\prod_{k<j}^{m-1}(c_j-c_k)\right]\prod_{i=1}^{m-1}(c_m-c_i). \end{aligned} \tag{3.24}$$

However, since

$$\begin{gathered}
c_m S_{m-1-i}(c_1,\dots,c_{m-1}) + S_{m-i}(c_1,\dots,c_{m-1}) = S_{m-i}(c_1,\dots,c_m) = S_{m-i}, \\
\prod_{k<j}^{m-1} (c_j - c_k) \prod_{i=1}^{m-1} (c_m - c_i) = \prod_{k<j}^{m} (c_j - c_k),
\end{gathered} \tag{3.25}$$

we have

$$f(c_m) = S_{m-i} \prod_{k<j}^{m} (c_j - c_k), \tag{3.26}$$

which proves the lemma. □

Remark 3.6. Note that in Lemma 3.5 if $i = m$, then we have the Vandermonde determinant.

Corollary 3.7. *Let V_0 be the matrix V with $h = 0$ and $d = p+1$, that is, V_0 is the $m \times m$-matrix, whose elements are*

$$(V_0)_{j,r} := \binom{p+r+1}{p} p! c_j^{r+1}. \tag{3.27}$$

Then, V_0^{-1} is the matrix, whose elements are given by

$$(V_0^{-1})_{r,j} = \frac{1}{\det(V_0)} (-1)^{r+j} S_{m-1,j}^2 S_{m-r,j} \prod_{l<k,\,(l,k\neq j)}^{m} (c_k - c_l) \prod_{k=1,\,(k\neq r)}^{m} \binom{p+k+1}{p} p!, \tag{3.28}$$

where

$$\det(V_0) = \left[\prod_{k=1}^{m} \binom{p+k+1}{p} p! \prod_{l<k}^{m} (c_k - c_l) \right] S_m^2. \tag{3.29}$$

Proof. From Lemma 3.5, we have $\det(V_0) = [\prod_{k=1}^{m} \binom{p+k+1}{p} p! \prod_{l<k}^{m} (c_k - c_l)] S_m^2$. Now,

$$V_0^{-1} = \frac{\mathrm{Adj}(V_0)}{\det(V_0)}, \tag{3.30}$$

where $\operatorname{Adj}(V_0)$ is the adjoint matrix of V_0, however,

$$\operatorname{Adj}(V_0)_{r,j} = (-1)^{r+j} S_{m-1,j}^2 \prod_{k=1,(k\neq r)}^{m} \binom{p+k+1}{p} p!$$

$$\times \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{r-2} & c_1^r & \cdots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-2} & c_2^r & \cdots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{j-1} & c_{j-1}^2 & \cdots & c_{j-1}^{r-2} & c_{j-1}^r & \cdots & c_{j-1}^{m-1} \\ 1 & c_{j+1} & c_{j+1}^2 & \cdots & c_{j+1}^{r-2} & c_{j+1}^r & \cdots & c_{j+1}^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{r-2} & c_m^r & \cdots & c_m^{m-1} \end{vmatrix}. \tag{3.31}$$

Again, by Lemma 3.5 and using the following relations:

$$S_{m-1-(r-1)}(c_1,\ldots,c_{j-1},c_{j+1},\ldots,c_m) = S_{m-r}(c_1,\ldots,c_{j-1},c_{j+1},\ldots,c_m) = S_{m-r,j}, \tag{3.32}$$

we have

$$\begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{r-2} & c_1^r & \cdots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-2} & c_2^r & \cdots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{j-1} & c_{j-1}^2 & \cdots & c_{j-1}^{r-2} & c_{j-1}^r & \cdots & c_{j-1}^{m-1} \\ 1 & c_{j+1} & c_{j+1}^2 & \cdots & c_{j+1}^{r-2} & c_{j+1}^r & \cdots & c_{j+1}^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \cdots & c_m^{r-2} & c_m^r & \cdots & c_m^{m-1} \end{vmatrix} = S_{m-r,j} \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l). \tag{3.33}$$

Thus,

$$(V_0^{-1})_{r,j} = \frac{1}{\det(V_0)} (-1)^{r+j} S_{m-1,j}^2 S_{m-r,j} \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l) \prod_{k=1,(k\neq r)}^{m} \binom{p+k+1}{p} p!, \tag{3.34}$$

which completes the proof of the corollary. □

Now, we can develop a formula for computing the $p+2$-eigenvalue of the matrix M_0.

Theorem 3.8. *For the case $d = p+1$ and $m \geq 1$, the $p+2$-eigenvalue of M_0 can be computed by using the following relation:*

$$\lambda_{p+2}^{(0)} = \frac{S_m - 2S_{m-1} + 3S_{m-2} + \cdots + (-1)^{m-1} m S_1 + (-1)^m (m+1)}{S_m}. \tag{3.35}$$

Proof. Let V_0 and W_0 be the matrices V and W, respectively, with $h=0$, then for $d=p+2$, we have from (2.8) that W_0 is a $m\times(p+2)$-matrix, whose elements are given by

$$(W_0)_{j,r} := \begin{cases} 0 & \text{if } r=0,1,\dots,p, \\ -(p+1)!c_j & \text{if } r=p+1. \end{cases} \tag{3.36}$$

Now, the $p+2$th-eigenvalue of $M_0 = A + BV_0^{-1}W_0$ is

$$\lambda_{p+2}^{(0)} = 1 + \sum_{r=1}^{m} (B)_{p+2,r}\left(V_0^{-1}W_0\right)_{r,p+2}. \tag{3.37}$$

The entries of the last row of matrix B are

$$(B)_{p+2,r} = \binom{p+r+1}{p+1}. \tag{3.38}$$

Moreover, from (3.36) and Corollary 3.7, we have

$$\begin{aligned}\left(V_0^{-1}W_0\right)_{r,p+2} = \frac{-(p+1)!}{\det(V_0)} \sum_{j=1}^{m} \Bigg[&(-1)^{(r+j)} S_{m-1,j}^2 S_{m-r,j} c_j \\ &\times \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l) \prod_{k=1,(k\neq r)}^{m} \binom{p+k+1}{p} p! \Bigg].\end{aligned} \tag{3.39}$$

Therefore,

$$\begin{aligned}\lambda_{p+2}^{(0)} = 1 + \frac{(p+1)!}{\det(V_0)} \sum_{r=1}^{m}\sum_{j=1}^{m} \Bigg[&\binom{p+r+1}{p+1} (-1)^{(r+j+1)} S_{m-1,j}^2 S_{m-r,j} c_j \\ &\times \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l) \prod_{k=1,(k\neq r)}^{m} \binom{p+k+1}{p} p! \Bigg].\end{aligned} \tag{3.40}$$

Now, by using the relations

$$\begin{gathered} c_j S_{m-1,j}^2 = S_m S_{m-1,j}, \\ (p+1)!\binom{p+r+1}{p+1} \prod_{k=1,(k\neq r)}^{m} \binom{p+k+1}{p} p! = (r+1)\prod_{k=1}^{m}\binom{p+k+1}{p} p!, \end{gathered} \tag{3.41}$$

and by substituting (3.29) for $\det(V_0)$, (3.40) can be simplified as follows:

$$\lambda_{p+2}^{(0)} = 1 + \frac{\sum_{r=1}^{m}(-1)^r (r+1) \sum_{j=1}^{m} (-1)^{(j+1)} S_{m-1,j} S_{m-r,j} \prod_{l<k,(l,k\neq j)}^{m} (c_k - c_l)}{S_m \prod_{l<k}^{m} (c_k - c_l)}. \tag{3.42}$$

However, from Lemma 3.5, we have

$$\sum_{j=1}^{m}(-1)^{(j+1)}S_{m-1,j}S_{m-r,j}\prod_{l<k,(l,k\neq j)}(c_k-c_l)=\begin{vmatrix}1 & c_1 & c_1^2 & \cdots & c_1^{r-1} & c_1^{r+1} & \cdots & c_1^m\\ 1 & c_2 & c_2^2 & \cdots & c_2^{r-1} & c_2^{r+1} & \cdots & c_2^m\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & c_m & c_m^2 & \cdots & c_m^{r-1} & c_m^{r+1} & \cdots & c_m^m\end{vmatrix}$$
$$=S_{m-r}\prod_{l<k}^{m}(c_k-c_l). \tag{3.43}$$

Hence,

$$\begin{aligned}\lambda_{p+2}^{(0)}&=1+\frac{\sum_{r=1}^{m}(-1)^r(r+1)S_{m-r}}{S_m}\\ &=\frac{\sum_{r=0}^{m}(-1)^r(r+1)S_{m-r}}{S_m}\\ &=\frac{S_m-2S_{m-1}+3S_{m-2}+\cdots+(-1)^{m-1}mS_1+(-1)^m(m+1)}{S_m},\end{aligned} \tag{3.44}$$

which concludes the proof of Theorem 3.8. □

Remark 3.9. Theorem 3.8 proves the conjecture asserted by Danciu [9] for first-order integro-differential equations ($p=1, d=2$).

As an application to Theorem 3.8, we can prove the following results. The proofs are identical to the proof given in [9] for the first-order integro-differential equation.

Corollary 3.10. *An $(m,p+1,p)$-method is stable if and only if*

$$\left|\frac{\left[(d/dt)(t\cdot R_m(t))\right]_{t=1}}{R_m(0)}\right|\leq 1, \tag{3.45}$$

where $R_m(t)$ is the polynomial of degree m, whose zeroes are the collocation parameters $\{c_j\}_{j=1,\dots,m}$.

Regarding the stability of local superconvergent solution $u\in S_{m+p+1}^{(p+1)}(Z_n)$, we have the following corollary.

Corollary 3.11. (i) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are uniformly distributed in $(0,1]$ (i.e., $c_j=j/m$, for all $j=1,2,\dots,m$), then $(m,p+1,p)$-method is stable for $m\geq 1$.*

(ii) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Radau II points in the interval $(0,1]$, then $(m,p+1,p)$-method is unstable for $m\geq 2$.*

(iii) *If the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Gauss points in the interval $(0,1]$, then $(m,p+1,p)$-method is unstable for $m\geq 2$.*

(iv) *If the first $m-1$ collocation parameters $\{c_j\}_{j=1,\dots,m}$ are the Gauss points in the interval $(0,1)$ and the last collocation parameter is $c_m=1$, then $(m,p+1,p)$-method is stable for $m\geq 2$.*

Table 4.1. Approximate error for Example 4.1 with $c_1 = (5-\sqrt{15})/10$, $c_2 = 1/2$, $c_3 = (5+\sqrt{15})/10$.

d	e_1	$e_{N/2}$	e_N
$d=3$	2.70×10^{-10}	3.31×10^{-4}	6.54×10^{-3}
$d=4$	2.70×10^{-10}	2.61×10^{23}	5.97×10^{58}
$d=5$	2.70×10^{-10}	4.58×10^{49}	4.25×10^{113}

Table 4.2. Approximate error for Example 4.2 with $c_1 = (5-\sqrt{15})/10$, $c_2 = 1/2$, $c_3 = (5+\sqrt{15})/10$.

d	e_1	$e_{N/2}$	e_N
$d=4$	0	2.91×10^{-5}	1.07×10^{-3}
$d=5$	0	1.45×10^{21}	3.32×10^{56}
$d=6$	0	1.20×10^{48}	1.02×10^{112}

4. Numerical examples

The method is tested using the following two examples in the interval $[0,1]$ with step size $h = 0.05$, errors are computed in Tables 4.1 and 4.2 for various $(3,d,p)$-methods with $p = 3,4$. The following notations will also be used in the presentation:

$$e_1 := |y(t_1) - u(t_1)|, \qquad e_{N/2} := |y(0.5) - u(0.5)|, \qquad e_N := |y(1) - u(1)|, \quad (4.1)$$

where $u \in S^d_{3+d}$ $(m = 3)$ is the approximated solution.

Example 4.1. Consider the following integro-differential equation of third order:

$$y^{(3)}(t) = \int_0^t y(s)ds, \quad y(0) = 1,\ y'(0) = 2,\ y''(0) = 1, \tag{4.2}$$

with exact solution $y(t) = e^t + \sin t$.

Example 4.2. Consider the following fourth-order integro-differential equation:

$$y^{(4)}(t) = 1 + \int_0^t y(s)ds, \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1, \tag{4.3}$$

with exact solution $y(t) = e^t$.

(a) Let us consider the Gauss points as the collocation parameters, that is, $c_1 = (5 - \sqrt{15})/10$, $c_2 = 1/2$, and $c_3 = (5+\sqrt{15})/10$, then we have Tables 4.1 and 4.2 corresponding to Examples 4.1 and 4.2, respectively.

(b) If the first two collocation parameters are the Gauss points, that is, $c_1 = (3-\sqrt{3})/6$, $c_2 = (3+\sqrt{3})/6$, and $c_3 = 1$, then we have Tables 4.3 and 4.4 for Examples 4.1 and 4.2.

From Tables 4.3 and 4.4, one can observe that $(3,d,p)$-method $(p = 3,4)$ is stable for $d = p$ and it is unstable for $d = p+2$. In the case $d = p+1$, this method is stable if the first two collocation parameters are the Gauss points, that is, $c_1 = (3-\sqrt{3})/6$, $c_2 = (3+\sqrt{3})/6$,

Table 4.3. Approximate error for Example 4.1 with $c_1 = (3-\sqrt{3})/6$, $c_2 = (3+\sqrt{3})/6$, $c_3 = 1$.

d	e_1	$e_{N/2}$	e_N
$d = 3$	2.70×10^{-10}	6.00×10^{-10}	4.20×10^{-9}
$d = 4$	2.70×10^{-10}	4.00×10^{-10}	5.20×10^{-9}
$d = 5$	2.70×10^{-10}	3.95×10^{10}	4.53×10^{40}

Table 4.4. Approximate error for Example 4.2 with $c_1 = (3-\sqrt{3})/6$, $c_2 = (3+\sqrt{3})/6$, $c_3 = 1$.

d	e_1	$e_{N/2}$	e_N
$d = 4$	0	5.00×10^{-9}	1.00×10^{-8}
$d = 5$	0	5.00×10^{-9}	1.40×10^{-8}
$d = 6$	0	4.99×10^{7}	5.73×10^{37}

and $c_3 = 1$ as in case (b), and unstable if the collocation parameters are the Gauss points, that is, $c_1 = (5-\sqrt{15})/10$, $c_2 = 1/2$, and $c_3 = (5+\sqrt{15})/10$ as in case (a).

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