ON THE CONCEPTS OF BALLS IN A D-METRIC SPACE

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The various concepts of open balls in D-metric spaces are studied in the case of certain D-metric spaces and many results in the literature on such balls are shown to be false.

1. Introduction

Dhage [1, 2, 3] introduced the concept of open balls in a D-metric space in two different ways and discussed at length the properties of the topologies generated by the family of all open balls of each kind. Here we observe that many of his results are either false or of doubtful validity. In some cases we give examples to show that either the results are false or that the proofs given by him are not valid. With regard to one type of open balls we observe that some of them may be empty and that the ball with a given center may not increase as the radius increases. The latter is contrary to a remark made by Dhage based on which he proves that the family of all open balls forms a base for a topology.

Definition 1.1 [1]. Let X be a nonempty set. A function $\rho : X \times X \times X \to [0,\infty)$ is called a D-metric on X if

(i) $\rho(x,y,z) = 0$ if and only if $x = y = z$ (coincidence),
(ii) $\rho(x,y,z) = \rho(p(x,y,z))$ for all $x,y,z \in X$ and for any permutation $p(x,y,z)$ of x, y, z (symmetry),
(iii) $\rho(x,y,z) \leq \rho(x,y,a) + \rho(x,a,z) + \rho(a,y,z)$ for all $x,y,z,a \in X$ (tetrahedral inequality).

If X is a nonempty set and ρ is a D-metric on X, then the ordered pair (X,ρ) is called a D-metric space. When the D-metric ρ is understood, X itself is called a D-metric space.

Definition 1.2 [1]. A sequence $\{x_n\}$ in a D-metric space (X,ρ) is said to be convergent (or ρ-convergent) if there exists an element x of X with the following property: given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\rho(x_m,x_n,x) < \varepsilon$ for all $m,n \geq N$.

In such a case, it is said that $\{x_n\}$ converges to x and x is a limit of $\{x_n\}$ and write $x_n \to x$.

Definition 1.3 [1]. A sequence $\{x_n\}$ in a D-metric space (X,ρ) is said to be Cauchy (or ρ-Cauchy) if given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\rho(x_m,x_n,x_p) < \varepsilon$ for all $m,n,p \geq N$.

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Notation. Let (X,ρ) be a D-metric space, $x_0 \in X$, and $r \in (0,\infty)$. Let

$$\begin{aligned} B^*(x_0,r) &= \{x \in X : \rho(x_0,x,x) < r\}, \\ B(x_0,r) &= \{x \in B^*(x_0,r) : \rho(x_0,x,y) < r \ \forall y \in B^*(x_0,r)\}, \\ \hat{B}(x_0,r) &= \{x_0\} \cup \Big\{x \in X : \sup_{y\in X} \rho(x_0,x,y) < r\Big\}. \end{aligned} \tag{1.1}$$

For a nonempty subset A of X, $\rho(x_0,x_0,A) = \inf\{\rho(x_0,x_0,x) : x \in A\}$.

Remark 1.4. Dhage referred to $B^*(x_0,r)$ as well as $B(x_0,r)$ as the open ball centered at x_0, and radius r. $B^*(x_0,r)$ defined above is denoted as $B(x_0,r)$ in Dhage [2] and as $B^*(x_0,r)$ in Dhage [1, 3]. Dhage defined $B(x_0,r)$ in [3] as $B(x_0,r) = \{y \in B^*(x_0,r) :$ if $y,z \in B^*(x_0,r)$ are any two points, then $\rho(x_0,y,z) < r\}$. This definition is meaningless. The definition as given in the notation is the natural refinement of it. Dhage defined $B(x_0,r)$ in [1] as

$$B(x_0,r) = \bigcap_{y\in X} \{x,y \in X : \rho(x_0,x,y) < r\}. \tag{1.2}$$

Probably due to the fact that the above definition is not meaningful, Ume and Kim [5] refined it and attributed it to Dhage. Their refined version reduces to the following:

$$B(x_0,r) = \{x_0\} \cup \Big\{x \in X : \sup_{y\in X} \rho(x_0,x,y) < r\Big\}. \tag{1.3}$$

Here we have denoted it as $\hat{B}(x_0,r)$.

Theorem 1.5. *Let X be a normed linear space and $p \in [1,\infty]$. Let ρ_p be defined on $X \times X \times X$ as*

$$\rho_p(x,y,z) = \begin{cases} \max\{\|x-y\|, \|y-z\|, \|z-x\|\} & \text{if } p = +\infty, \\ \big[\|x-y\|^p + \|y-z\|^p + \|z-x\|^p\big]^{1/p} & \text{if } 1 \le p < +\infty \end{cases} \tag{1.4}$$

for all $x,y,z \in X$. Then ρ_p is a D-metric on X.

Let $x_0 \in X$ and $r \in (0,\infty)$. Then

$$B_p^*(x_0,r) = \begin{cases} \{x \in X : \|x_0 - x\| < r\} & \text{if } p = +\infty, \\ \Big\{x \in X : \|x_0 - x\| < \dfrac{r}{2^{1/p}}\Big\} & \text{if } 1 \le p < +\infty, \end{cases} \tag{1.5}$$

and $B_p(x_0,r) = \{x_0\}$, where

$$\begin{aligned} B_p^*(x_0,r) &= \{x \in X : \rho_p(x_0,x,x) < r\}, \\ B_p(x_0,r) &= \{x \in B_p^*(x_0,r) : \rho_p(x_0,x,y) < r \ \forall y \in B_p^*(x_0,r)\}. \end{aligned} \tag{1.6}$$

Proof. From Naidu et al. [4, Corollaries 1, 3, and 4], it follows that ρ_p is a D-metric on X. We have

$$B_\infty^*(x_0,r) = \{x \in X : \rho_\infty(x_0,x,x) < r\} = \{x \in X : ||x_0 - x|| < r\}. \tag{1.7}$$

Clearly $x_0 \in B_\infty^*(x_0,r)$ and $\rho_\infty(x_0,x_0,y) = \|x_0 - y\| < r$ for all $y \in B_\infty^*(x_0,r)$. Hence $x_0 \in B_\infty(x_0,r)$. Let $y_0 \in B_\infty^*(x_0,r) \setminus \{x_0\}$. Then $0 < \|x_0 - y_0\| < r$. Hence $(r/\|x_0 - y_0\|) - 1 > 0$. Let $t_0 \in \{r/\|x_0 - y_0\| - 1, r/\|x_0 - y_0\|\}$. Let $z_0 = x_0 + t_0(x_0 - y_0)$. Then $\|z_0 - x_0\| = t_0 \|x_0 - y_0\| < r$. Hence $z_0 \in B_\infty^*(x_0,r)$. We have $\|y_0 - z_0\| = \|(1+t_0)(x_0 - y_0)\| = (1+t_0)\|x_0 - y_0\| > r$. Hence $y_0 \notin B_\infty(x_0,r)$. Hence $B_\infty(x_0,r) = \{x_0\}$.

Let $p \in [1,\infty)$. We have

$$\begin{aligned}
B_p^*(x_0,r) &= \{x \in X : \rho_p(x_0,x,x) < r\} \\
&= \left\{x \in X : \left[||x_0 - x||^p + \|x - x\|^p + ||x - x_0||^p\right]^{1/p} < r\right\} \\
&= \{x \in X : 2^{1/p}||x_0 - x|| < r\} \\
&= \left\{x \in X : ||x_0 - x|| < \frac{r}{2^{1/p}}\right\}, \\
B_p(x_0,r) &= \{x \in B_p^*(x_0,r) : \rho_p(x_0,x,y) < r \ \forall y \in B_p^*(x_0,r)\} \\
&= \Big\{x \in B_p^*(x_0,r) : ||x_0 - x||^p + \|x - y\|^p + ||y - x_0||^p < r^p \\
&\qquad \text{whenever } y \in X \text{ and } ||x_0 - y|| < \frac{r}{2^{1/p}}\Big\}.
\end{aligned} \tag{1.8}$$

Clearly $x_0 \in B_p(x_0,r)$. Let $y_0 \in B_p^*(x_0,r) \setminus \{x_0\}$. Then $0 < \|x_0 - y_0\| < r/2^{1/p}$. Hence $r/2^{1/p}\|x_0 - y_0\| > 1$. Hence $(r/2^{1/p}\|x_0 - y_0\|)^p > 1$. Hence $[(r/2^{1/p}\|x_0 - y_0\|)^p - 1] > 0$.

We have

$$\left[\left(\frac{r}{2^{1/p}||x_0 - y_0||}\right)^p - 1\right]^{1/p} < \frac{r}{2^{1/p}||x_0 - y_0||}. \tag{1.9}$$

Let

$$t_0 \in \left\{\left[\left(\frac{r}{2^{1/p}||x_0 - y_0||}\right)^p - 1\right]^{1/p}, \frac{r}{2^{1/p}||x_0 - y_0||}\right\}. \tag{1.10}$$

Let $z_0 = x_0 + t_0(x_0 - y_0)$. Then $\|z_0 - x_0\| = t_0\|x_0 - y_0\| < r/2^{1/p}$. Hence $z_0 \in B_p^*(x_0,r)$.

We have

$$\begin{aligned}
&||x_0 - y_0||^p + ||y_0 - z_0||^p + ||z_0 - x_0||^p \\
&\quad = (1 + t_0^p)||x_0 - y_0||^p + ||y_0 - z_0||^p \\
&\quad = [1 + t_0^p + (1 + t_0)^p]||x_0 - y_0||^p \\
&\quad \geq 2(1 + t_0^p)||x_0 - y_0||^p > r^p.
\end{aligned} \tag{1.11}$$

Hence $y_0 \notin B_p(x_0,r)$. Hence $B_p(x_0,r) = \{x_0\}$. □

Remark 1.6. Theorem 1.5 shows that the conclusions about $B(x_0,r)$ in [3, Theorems 3.1, 3.2, 3.3, and 3.4] are false.

We now give an example of a D-metric space (X,ρ) in which

(i) the family $\{B(x,r) : x \in X \text{ and } r \in (0,\infty)\}$ does not form a base for any topology on X,
(ii) for each $x \in X$, there exists an $r_x \in (0,\infty)$ such that $B(x,r_x) = \phi$,
(iii) there exist $z_0 \in X$ and $r_1, r_2 \in (0,\infty)$ such that $r_1 < r_2$ and $B(z_0,r_1) \nsubseteq B(z_0,r_2)$.

Example 1.7 (Naidu et al. [4, Example 1]). Let $X = A \cup B \cup \{0\}$, where

$$A = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}, \qquad B = \{2^n : n \in \mathbb{N}\}. \tag{1.12}$$

Define $\rho : X \times X \times X \to [0,\infty)$ as follows:

$$\rho(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{x,y\},\max\{y,z\}, \\ \qquad \max\{z,x\}\} & \begin{array}{l}\text{if } x,y,z \in A \cup \{0\},\ 0 \\ \text{does not occur more} \\ \text{than once among } x,\ y,\ z \\ \text{and at least two among} \\ x,\ y,\ z \text{ are distinct,}\end{array} \\ 1 & \begin{array}{l}\text{if 0 and at least one} \\ \text{element of } B \text{ occur among} \\ x,\ y,\ z \text{ or 0 occurs exactly} \\ \text{twice among } x,\ y,\ z,\end{array} \\ \min\{x,y,z\} & \begin{array}{l}\text{if } x,y,z \in A \cup B \text{ and exactly} \\ \text{one element of } B \text{ occurs} \\ \text{exactly once among } x,y,z,\end{array} \\ \min\left\{\max\left\{\frac{1}{x},\frac{1}{y}\right\},\max\left\{\frac{1}{y},\frac{1}{z}\right\}, \right. \\ \qquad \left. \max\left\{\frac{1}{z},\frac{1}{x}\right\}\right\} & \begin{array}{l}\text{if } x,y,z \in A \cup B \text{ and exactly} \\ \text{one element of } A \text{ occurs} \\ \text{exactly once among } x,\ y,\ z,\end{array} \\ \left|\frac{1}{x}-\frac{1}{y}\right| + \left|\frac{1}{y}-\frac{1}{z}\right| + \left|\frac{1}{z}-\frac{1}{x}\right| & \text{if } x,y,z \in B. \end{cases} \tag{1.13}$$

Then (X,ρ) is a D-metric space and $\rho(x,y,z) \leq 1$ for all $x,y,z \in X$.

Let $r \in (0, \infty)$. We have

$$B^*(0,r) = \begin{cases} \{0\} \cup \{x \in A : x < r\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases} \tag{1.14}$$

For $x_0 \in A$, we have

$$B^*(x_0,r) = \begin{cases} \{x_0\} \cup \{x \in A : x < r\} \cup \left\{x \in B : x > \dfrac{1}{r}\right\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases} \tag{1.15}$$

For $x_0 \in B$, we have

$$B^*(x_0,r) = \begin{cases} \{x \in A : x < r\} \cup \left\{x \in B : \left|\dfrac{1}{x_0} - \dfrac{1}{x}\right| < \dfrac{r}{2}\right\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases} \tag{1.16}$$

We note that for $x_0 \in B$,

$$B^*(x_0,r) = \begin{cases} \{x_0\} \cup \{x \in A : x < r\} & \text{if } r \leq \dfrac{1}{x_0}, \\ \{x \in A : x < r\} \cup \{x \in B : x \geq x_0\} & \text{if } \dfrac{1}{x_0} < r \leq \min\left\{1, \dfrac{2}{x_0}\right\}, \\ \{x \in A : x < r\} \cup \left\{x \in B : x \geq \dfrac{x_0}{2}\right\} & \text{if } \dfrac{2}{x_0} < r \leq \min\left\{1, \dfrac{6}{x_0}\right\}. \end{cases} \tag{1.17}$$

We have

$$B(0,r) = \begin{cases} \phi & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases} \tag{1.18}$$

For $x_0 \in A$, we have

$$B(x_0,r) = \begin{cases} \phi & \text{if } r \leq x_0, \\ B^*(x_0,r) & \text{if } x_0 < r \leq 1, \\ X & \text{if } r > 1. \end{cases} \tag{1.19}$$

For $x_0 \in B$, we have

$$B(x_0,r) = \begin{cases} X & \text{if } r > 1, \\ \phi & \text{if } r \leq \dfrac{1}{x_0}, \\ B^*(x_0,r) & \text{if } \dfrac{1}{x_0} < r \leq \min\left\{1, \dfrac{2}{x_0}\right\}, \\ \{x_0\} \cup \{x \in A : x < r\} & \text{if } \dfrac{2}{x_0} < r \leq \min\left\{1, \dfrac{3}{x_0}\right\}. \end{cases} \tag{1.20}$$

We have

$$B^*(4,r)=\begin{cases}\{x\in A: x<r\}\\ \quad\cup\{x\in B: 4\le x\le 2^{n+1}\} & \text{if } n\in\mathbb{N} \text{ and } \dfrac{1}{2}-\dfrac{1}{2^n}<r\le\dfrac{1}{2}-\dfrac{1}{2^{n+1}},\\ \{x\in A: x<r\}\cup\{x\in B: x\ge 4\} & \text{if } r=\dfrac{1}{2},\\ A\cup B & \text{if } \dfrac{1}{2}<r\le 1,\\ X & \text{if } r>1,\end{cases}$$

$$B(4,r)=\begin{cases}\phi & \text{if } r\le\dfrac{1}{4},\\ B^*(4,r) & \text{if } \dfrac{1}{4}<r\le\dfrac{1}{2},\\ \{x\in A: x<r\}\\ \quad\cup\{x\in B: 4\le x\le 2^{n+1}\} & \text{if } n\in\mathbb{N} \text{ and } 1-\dfrac{1}{2^n}<r\le 1-\dfrac{1}{2^{n+1}},\\ A\cup B & \text{if } r=1,\\ X & \text{if } r>1.\end{cases} \tag{1.21}$$

We have

$$\begin{gathered}B\left(4,\frac{3}{4}\right)=A\cup\{4\},\\ B\left(\frac{1}{4},\frac{1}{2}\right)=B^*\left(\frac{1}{4},\frac{1}{2}\right)=\left\{x\in A: x<\frac{1}{2}\right\}\cup\{x\in B: x>2\}.\end{gathered} \tag{1.22}$$

Hence

$$B\left(4,\frac{3}{4}\right)\cap B\left(\frac{1}{4},\frac{1}{2}\right)=\left\{x\in A: x<\frac{1}{2}\right\}\cup\{4\}. \tag{1.23}$$

Suppose that there exist $z_0\in X$ and $r_0\in(0,\infty)$ such that $4\in B(z_0,r_0)$ and

$$B(z_0,r_0)\subseteq B\left(4,\frac{3}{4}\right)\cap B\left(\frac{1}{4},\frac{1}{2}\right). \tag{1.24}$$

Then $B(z_0,r_0)\cap B=\{4\}$.

If $x_0\in A\cup\{0\}$, $s\in(0,\infty)$ and $B(x_0,s)\ne\phi$, then $B(x_0,s)\cap B$ is infinite. Hence $z_0\notin A\cup\{0\}$. Hence $z_0\in B$. We note that for any $x_0\in B$, $x_0\in B(x_0,s)$ if $s\in(1/x_0,\infty)$ and that $B(x_0,s)=\phi$ if $s\in(0,1/x_0]$. Since $B(z_0,r_0)$ is nonempty, it follows that $r_0>1/z_0$ and $z_0\in B(z_0,r_0)$. Thus $z_0\in B(z_0,r_0)\cap B=\{4\}$. Hence $z_0=4$ and $r_0>1/4$. Since $B(z_0,r_0)\cap B=\{4\}$ and $z_0=\{4\}$, we have $B(4,r_0)\cap B=\{4\}$. Hence $1/2<r_0\le 3/4$.

We have

$$A = \{x \in A : x < r_0\} \subseteq B(4, r_0) = B(z_0, r_0). \tag{1.25}$$

Since $B(z_0, r_0) \subseteq B(4, 3/4) \cap B(1/4, 1/2)$, it follows that $A \subseteq \{x \in A : x < 1/2\}$. This is a contradiction since $1/2 \in A$. Hence there is no ball of the form $B(x, r)$ containing 4 and contained in $B(4, 3/4) \cap B(1/4, 1/2)$. Hence $\{B(x, r) : x \in X \text{ and } r \in (0, \infty)\}$ does not form a base for any topology on X.

Let $x_0 \in B \setminus \{2\}$. Then we have

$$\begin{gathered} B\left(x_0, \frac{2}{x_0}\right) = B^*\left(x_0, \frac{2}{x_0}\right) = \left\{x \in A : x < \frac{2}{x_0}\right\} \cup \{x \in B : x \geq x_0\}, \\ B\left(x_0, \frac{3}{x_0}\right) = \left\{x \in A : x < \frac{3}{x_0}\right\} \cup \{x_0\}. \end{gathered} \tag{1.26}$$

Hence $2x_0 \in B(x_0, 2/x_0)$ but $2x_0 \notin B(x_0, 3/x_0)$. Hence $B(x_0, 2/x_0) \nsubseteq B(x_0, 3/x_0)$. Let $y_0 \in A \cap B(x_0, 3/x_0)$. Then $B^*(y_0, s) \cap B$ is infinite for any $s \in (0, \infty)$. But $B(x_0, 3/x_0) \cap B = \{x_0\}$. Hence $B^*(y_0, s) \nsubseteq B(x_0, 3/x_0)$ for any $s \in (0, \infty)$.

Remark 1.8. Dhage asserted in [3, Theorem 4.1] that if (X, ρ) is a D-metric space, then $\{B(x, r) : x \in X \text{ and } r \in (0, \infty)\}$ is a base for a topology on X and called it the D-metric topology on X. Example 1.7 shows that Dhage [3, Theorem 4.1] is false. So we may interpret the D-metric topology on a D-metric space (X, ρ) as the topology generated by $\{B(x, r) : x \in X \text{ and } r \in (0, \infty)\}$. In [3, Theorem 4.2] it is stated that the topology of D-metric convergence and the D-metric topology on a D-metric space are equivalent. Naidu et al. [4] proved that in the D-metric space of Example 1.7, D-metric convergence does not define a topology. In [3, Theorems 4.3 and 4.4] it is stated that the D-metric function $\rho(x, y, z)$ is continuous in one variable and also in all the three variables. Naidu et al. [4] gave examples to show that the D-metric need not be sequentially continuous even in a single variable even when D-metric convergence defines a metrizable topology.

Remark 1.9. Dhage stated that if (X, ρ) is a D-metric space, $x_0 \in X$, and $0 < r_1 < r_2 < +\infty$, then $B(x_0, r_1) \subseteq B(x_0, r_2)$ (see [3, Remark 3.2(ii)]). Example 1.7 shows that Dhage [3, Remark 3.2(ii)] is false.

Remark 1.10. Dhage asserted in [3, Theorem 3.5] that if (X, ρ) is a D-metric space, $x_0 \in X$, $r \in (0, \infty)$, and $y_0 \in B(x_0, r)$, then there exists $s \in (0, \infty)$ such that $B(y_0, s) \subseteq B(x_0, r)$. In proving this statement he concluded that if $y_0 \in B(x_0, r)$, there exists $s \in (0, \infty)$ such that $B^*(y_0, s) \subseteq B(x_0, r)$. Example 1.7 shows that such a conclusion is false. Hence the validity of Dhage [3, Theorem 3.5] is doubtful.

We now give an example of a D-metric space (X, ρ) in which

(i) $\{B(x, r) : x \in X \text{ and } r \in (0, \infty)\}$ forms a base for a topology on X which is T_0 but not T_1,
(ii) $\{B^*(x, r) : x \in X \text{ and } r \in (0, \infty)\}$ forms a base for a topology τ on X which is T_1 but not Hausdorff.

Example 1.11. Let $X = \{1/2^n : n \in \mathbb{N}\}$. Define $\rho : X \times X \times X \to [0,\infty)$ as follows:

$$\rho(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \min\{\max\{x,y\},\max\{y,z\},\max\{z,x\}\} & \text{otherwise.} \end{cases} \tag{1.27}$$

Then (X,ρ) is a D-metric space, $\rho(x,y,y) = y$ for all $x,y \in X$, and ρ satisfies condition (v) of Dhage [2] on D-metric, that is, $\rho(x,y,y) \leq \rho(x,z,z) + \rho(z,y,y)$ for all $x,y,z \in X$.

Let $x_0 \in X$ and $r \in (0,\infty)$.

We have

$$\begin{aligned} B^*(x_0,r) &= \{x_0\} \cup \{x \in X : x < r\}, \\ B(x_0,r) &= \begin{cases} \{x \in X : x < r\} & \text{if } r > x_0, \\ \phi & \text{if } r \leq x_0, \end{cases} \\ \hat{B}(x_0,r) &= \begin{cases} \{x_0\} & \text{if } r \leq x_0, \\ \{x \in X : x < r\} & \text{if } r > x_0. \end{cases} \end{aligned} \tag{1.28}$$

Obviously $\{B(x,r) : x \in X \text{ and } r \in (0,\infty)\}$ forms a base for a topology, say, τ_1 on X. If $x_1,x_2 \in X$ and $x_1 < x_2$, then any neighbourhood of x_2 contains x_1. Hence τ_1 is not T_1. In particular, it is not Hausdorff. If $x_1,x_2 \in X$ and $x_1 < r < x_2$, then $x_1 \in B(x_1,r)$ but $x_2 \notin B(x_1,r)$. Hence τ_1 is T_0.

Clearly $\{\hat{B}(x,r) : x \in X \text{ and } r \in (0,\infty)\}$ forms a base for the discrete topology, say, τ_2 on X. A sequence $\{x_n\}$ in X converges to an element x_0 of X with respect to τ_2 if and only if $x_n = x_0$ for all sufficiently large n (since $\{x_0\}$ is a τ_2-open set). However, $\{1/2^n\}$ converges to zero with respect to the D-metric ρ. Thus for sequences in X τ_2-convergence and convergence with respect to the D-metric ρ are not equivalent.

Evidently, $\{B^*(x,r) : x \in X \text{ and } r \in (0,\infty)\}$ forms a base for a topology, say, τ on X. A nonempty subset U of X is τ-open if and only if $U = \{x \in X : x < r\} \cup S$ for some $r \in (0,\infty)$ and for some subset S of X. If $x_1,x_2 \in X$, $r_1,r_2 \in (0,\infty)$, and $r_3 = \min\{r_1,r_2\}$, then $\phi \neq \{y \in X : y < r_3\} \subseteq B^*(x_1,r_1) \cap B^*(x_2,r_2)$. Hence τ is not Hausdorff. A subset A of X is τ-closed if and only if there exists an $r \in (0,\infty)$ such that $A \subseteq \{x \in X : x \geq r\}$. Since each element of X is positive, it follows that $\{x\}$ is τ-closed for each x in X. Hence X is T_1. Since τ is T_1 and not Hausdorff, it is not regular.

For $x_0 \in X$ and a nonempty subset A of $X \setminus \{x_0\}$, we have $\rho(x_0,x_0,A) = x_0$. A sequence $\{x_n\}$ in X converges to an element x_0 of X with respect to τ if and only if given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any integer $n \geq N$ either $x_n < \varepsilon$ or $x_n = x_0$. Hence $\{1/2^{2n}\}$ converges to $1/2$ with respect to τ. It can be seen that $\{1/2^{2n}\}$ converges to $1/2$ with respect to τ_1 also. Let A be a nonempty subset of $X \setminus (\{1/2^{2n} : n \in \mathbb{N} \cup \{1/2\}\})$. Then $\{\rho(1/2^{2n},1/2^{2n},A)\} = \{1/2^{2n}\}$ converges to zero with respect to the usual topology of the real line. But $\rho(1/2,1/2,A) = 1/2 \neq 0$. Hence the function $x \mapsto \rho(x,x,A)$ is not continuous when X is equipped with the topology τ or τ_1 and the real line with the usual topology.

Since $\{1/2^{2n}\}$ converges to $1/2$ with respect to τ and $\rho(1/2^{2n},1/2,1/2) = 1/2$ for all $n \in \mathbb{N}$, $\{\rho(1/2^{2n},1/2,1/2)\}$ does not converge to $\rho(1/2,1/2,1/2) = 0$. Hence the D-metric ρ is not sequentially continuous with respect to τ even in a single variable.

Remark 1.12. Example 1.11 shows that [2, Lemma 1.2, Theorems 2.1 and 2.2, and Corollaries 2.1 and 2.2] are false.

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