SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPS IN ORDERED BANACH SPACES AND APPLICATIONS

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The existence of maximal and minimal fixed points for various set-valued operators is discussed. This paper presents some new fixed point theorems in ordered Banach spaces. A necessary and sufficient condition for the existence of the fixed point to a class of multivalued maps has been obtained. The uniqueness of the positive fixed point has been discussed. The results extend and improve the corresponding results. As an application, we utilize the results to study the existence and uniqueness of positive fixed points for a class of convex operators. In the end, we give a simple application to certain integral equations.

1. Introduction

This paper presents some new fixed point theorems for multivalued maps in ordered Banach spaces. Our arguments rely on the properties of closed convex cone in real Banach spaces (see [6, 8]) and the Kuratowski measure of noncompactness [9]. Our results extend and improve the corresponding results presented in the literature (see [1], [5, Chapter 3]).

We proceed as follows. In Section 2, we present basic concepts and results in ordered Banach spaces. In Section 3, the main results and proofs are given. The results in [1] and [5, Chapter 3] turn out to be special cases of our main results.

Motivated by the ideas and results in [11], we give in Section 4 a necessary and sufficient condition for the existence of the fixed point to a class of multivalued maps which satisfy certain conditions. At the end of Section 4, the uniqueness of the positive fixed point has been discussed sketchily. To demonstrate the applicability of our results, we study the existence and uniqueness of positive fixed points for a class of convex operators and give in the final section of the paper a simple application to certain integral equations.

2. Preliminaries

In this section, we summarize some basic concepts and results in real Banach spaces.

Let X be a real Banach space and let S be a nonempty subset of X. Suppose that $F : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then F is said to be upper

semi-continuous at $x_0 \in S$ if for any $x_n \in S$, $\{x_n\} \to x_0$ (as $n \to \infty$) and $y_n \in F(x_n)$, $\{y_n\} \to y_0$ (as $n \to \infty$) implies that $y_0 \in F(x_0)$. If F is upper semicontinuous at each point of S, then F is called upper semi-continuous on S. Point $x \in S$ is called a fixed point of F if $x \in F(x)$.

If A is a subset of X and c is a real number, we denote $cA = \{ca : a \in A\}$. Let Ω_X be bounded subsets of X, the Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, +\infty)$ defined by (here, $D \in \Omega_X$)

$$\alpha(D) = \inf\left\{r > 0 : D \subset \bigcup_{i=1}^{n} D_i \text{ and } \operatorname{diam}(D_i) \le r\right\}. \tag{2.1}$$

Then we have (see [10, 12, 13])
  (1) $\alpha(A) = 0 \Leftrightarrow A$ is a relatively compact set;
  (2) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$, here A, B are bounded subsets of X; and
    (i) $F : S \to 2^X$ is k-set contractive (here $k \ge 0$) if $\alpha(F(W)) \le k\alpha(W)$ for all non-empty, bounded sets W of S;
    (ii) $F : S \to 2^X$ is condensing if F is 1-set contractive and $\alpha(F(W)) < \alpha(W)$ for all nonempty, bounded sets W of S with $\alpha(W) \ne 0$.

Following the terminology in [10], we say that a nonempty subset $P \subset X$ is a convex cone if $P + P = P$, $\lambda P \subset P$ for all real number $\lambda \ge 0$. If the cone is pointed, (i.e., $P \cap -P = \{0\}$) such a cone P induces an ordering on X. For $x, y \in X$, we write $x \ge y$ if $x - y \in P$. If $x \le y$, the set $[x, y] = \{z \in X \mid x \le z \le y\}$ is called the order interval between x and y.

Putting $\mathring{P} = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if its interior $\mathring{P}$ is nonempty. In the case $y - x \in \mathring{P}$, we write $x \ll y$.

For all $x, y \in X$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$; clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set

$$P_h = \{x \in X \mid \text{there exist } \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \le x \le \mu(x)h\}, \tag{2.2}$$

and it is easy to see that $P_h \subset P$. These concepts discussed above can be found in [5, 6].

We will need the following definition [7, 8].

Definition 2.1. Given a closed convex cone P in the real Banach space X, the following exist.
  (i) P is normal if there exists $b > 0$, for all $x, y \in P : x \le y \Rightarrow \|x\| \le b\|y\|$.
  (ii) P is regular if every monotonically decreasing sequence $x_1 \ge x_2 \ge \ldots$, which is bounded from below by some element $\bar{x} \in X$, converges in norm.
  (iii) P is strongly minimal if it follows from any bounded above set $D \subset X$ that D has supremum.

Definition 2.2 (see [14]). Let A be a positive single-valued mapping on $\mathring{P}$ and let $\alpha \in \mathbb{R}$. Then, say A is α-concave (α-convex) if and only if $A(tx) \ge t^\alpha Ax(A(tx) \le t^\alpha Ax)$ for all $x \in \mathring{P}$ and $t \in (0, 1]$.

Let A be a positive mapping on $\mathring{P}$ which is α-concave ($\alpha \in [0, 1]$), and choose $h \in \mathring{P}$, then $P_h = \mathring{P}$. So there exist $\lambda, \mu \in \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the positive reals) such that $\lambda h \le Ax \le \mu h$

and $A(tx) \geq t^{\alpha}Ax \geq tAx$ for $t \in (0,1]$. Thus α-concave mappings are concave in the sense of Martin (see [12, Page 185]). Similar remarks apply to α-convex mappings. Note also that A is α-concave (α-convex) if and only if $A(sx) \leq s^{\alpha}Ax(A(sx) \geq s^{\alpha}Ax)$ for all $x \in \overset{\circ}{P}$ and $s \geq 1$.

The following statements play a very important role in the proofs of main results. For $A, B \subset X$, we write $A \leq^{s} B$ ($A \geq^{s} B$) if $a \leq b$ ($a \geq b$) for any $a \in A$, $b \in B$. A map $F : S \to 2^{X}$ is said to be increasing if $x \leq y$ implies that $F(x) \leq^{s} F(y)$. As a direct consequence, we have that if $A \leq^{s} A$, then A is a singleton.

3. Maximal and minimal fixed point theorems

Throughout this section, we always assume that X is a real Banach space with a partial order introduced by a closed convex cone P of X. Take $u_0, v_0 \in X$, $u_0 < v_0$ and $D = [u_0, v_0]$.

THEOREM 3.1. *Let $F : D \to 2^{D}$ be an upper semicontinuous, increasing mapping with nonempty value and let P be a regular cone. Then F has maximal and minimal fixed points on $[u_0, v_0]$.*

Proof. Since $F(u_0)$ is nonempty, there exists $u_1 \in F(u_0)$ such that $u_1 \geq u_0$. If $u_1 = u_0$, then u_0 is a fixed point of F. So we can assume without loss of generality that $u_1 > u_0$, therefore, $F(u_0) \leq^{s} F(u_1)$. So there is $u_2 \in F(u_1)$ such that $u_2 \geq u_1$, if $u_2 = u_1$, then u_1 is a fixed point. Also, we can assume that $u_2 > u_1$. In general, having defined $u_n \in F(u_{n-1})$ such that $u_{n-1} \leq u_n$ if $u_n = u_{n-1}$, then u_{n-1} is a fixed point of F. Without loss of generality, assume that $u_{n-1} < u_n$. Repeating this process, we can obtain an increasing sequence as follows:

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_0. \tag{3.1}$$

Since P is regular, there exists u^* such that $u_n \to u^*$ as $n \to \infty$. On the other hand, from the facts $u_n \in F(u_{n-1})$ and F is upper semicontinuous, it follows that $u^* \in F(u^*)$. That is to say, u^* is a fixed point of F.

Similarly, we can prove that there exists v^* such that $v^* \in F(v^*)$.

In the following, we will show that u^*, v^* are the maximal and minimal fixed points of F on the order interval $[u_0, v_0]$, respectively.

Suppose that $u_0 \leq \bar{x} \leq v_0$ with $\bar{x} \in F(\bar{x})$. Then $F(u_0) \leq^{s} F(\bar{x}) \leq^{s} F(v_0)$, so we obtain $u_1 \leq \bar{x} \leq v_1$. Repeating this process, we have $F(u_{n-1}) \leq^{s} F(\bar{x}) \leq^{s} F(v_{n-1})$, hence, $u_n \leq \bar{x} \leq v_n$, passing the limit, $u^* \leq \bar{x} \leq v^*$. This completes the proof. □

It is of interest to remark on the single-valued version of the Theorem 3.1.

Remark 3.2. (i) The conditions in Theorem 3.1 imply that $\{u_0\} \leq^{s} F(u_0), F(v_0) \leq^{s} \{v_0\}$.

(ii) If $u_0 \in F(u_0)$, then u_0 is the minimal fixed point of F in D. Otherwise, if $u_i \in F(u_i)$ for certain $i \neq 0$, we can prove that u_i is the minimal fixed point of F in D. In fact, supposing that $\bar{u} \in F(\bar{u})$, $\bar{u} \in D$, $\bar{u} \neq u_i$, then $u_0 < \bar{u}$. Similar to the proof of Theorem 3.1, we have $u_i \leq \bar{u}$. Similarly, if $v_j \in F(v_j)$ for certain j, then v_j is the maximal fixed point of F in D.

(iii) If F is a singled-valued upper semicontinuous mapping, then it is continuous.

In addition, we have the following corollary.

COROLLARY 3.3 (see [5]). *Let cone P be regular and let $F : D \to X$ be a single-valued continuous, increasing mapping, assume that*

$$u_0 \leq F(u_0), \qquad F(v_0) \leq v_0. \tag{3.2}$$

Then F has maximal and minimal fixed points x^*, x_* on $[u_0, v_0]$ and

$$x^* = \lim_{n \to \infty} v_n, \qquad x_* = \lim_{n \to \infty} u_n, \tag{3.3}$$

where $v_n = F(v_{n-1})$, $u_n = F(u_{n-1})$ $(n = 1, 2, \dots)$ satisfy

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.4}$$

Proof. Since $F : D \to X$ is a single-valued increasing mapping, and the facts $u_0 \leq F(u_0)$, $F(v_0) \leq v_0$, we can obtain

$$u_0 \leq F(u_0) \leq F(x) \leq F(v_0) \leq v_0, \quad \forall x \in [u_0, v_0]. \tag{3.5}$$

Thus, $F : D \to D$. Note that $F : D \to X$ is continuous, we know that $F : D \to D$ is also upper semicontinuous. By Theorem 3.1, F has maximal and minimal fixed points x^*, x_* on $[u_0, v_0]$. From the proof of Theorem 3.1, (3.2) and (3.3) can be easily proved.   □

THEOREM 3.4. *Let $F : D \to 2^D$ be an upper semicontinuous, increasing, condensing mapping with nonempty value and let P be a normal cone. Then F has maximal and minimal fixed points on $[u_0, v_0]$.*

Proof. From the proof of Theorem 3.1, we obtain

$$\begin{aligned} u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_0, \\ v_0 \geq v_1 \geq \cdots \geq v_n \geq \cdots \geq u_0, \end{aligned} \tag{3.6}$$

where $u_n \in F(u_{n-1})$, $v_n \in F(v_{n-1})$ $(n = 1, 2, \dots)$, $u_n \leq v_n$. Evidently, $\{u_n\}, \{v_n\} \subset [u_0, v_0]$. Since P is normal, then $\{u_n\}, \{v_n\}$ are bounded sequences.

Suppose that $\alpha(\{u_n\}) > 0$, it follows from $u_n \in F(u_{n-1})$ that $\{u_n\} \subset F(\{u_n\}) \subset [u_0, v_0]$. Thus $\alpha(\{u_n\}) \leq \alpha(F(\{u_n\}))$. Since F is condensing, we have $\alpha(F(\{u_n\})) < \alpha(\{u_n\})$. This is a contradiction. So $\alpha(\{u_n\}) = 0$, thus $\{u_n\}$ is relatively compact in D. Therefore, there exists a convergent subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u^* \in D$ such that $u_{n_k} \to u^*$ as $k \to \infty$.

Now we prove that $u_n \to u^*$. Suppose that this is not the case, then there exist a $\varepsilon_0 > 0$ and a subsequence $\{u_{n_i'}\}$ of $\{u_n\}$ such that

$$\|u_{n_i'} - u^*\| \geq \varepsilon_0 \quad (i = 1, 2, \dots). \tag{3.7}$$

By the relative compactness of $\{u_n\}$, we know that there exists a subsequence $\{u_{n_{i_j}'}\}$ of $\{u_{n_i'}\}$ such that $u_{n_{i_j}'} \to \bar{u}$ $(j \to \infty)$. Hence, for any given k, it follows from the increasing of $\{u_n\}$ that when j is large enough, $u_{n_k} \leq u_{n_{i_j}'}$.

First letting $j \to \infty$ and then letting $k \to \infty$, we have $u^* \leq \bar{u}$.

Similarly, we can also prove that $\bar{u} \leq u^*$. Hence $u^* = \bar{u}$. This implies that when j is large enough, we have

$$\|u_{n'_{i_j}} - u^*\| < \varepsilon_0. \tag{3.8}$$

This contradicts

$$\|u_{n'_i} - u^*\| \geq \varepsilon_0 \quad (\forall i). \tag{3.9}$$

This completes the proof of $u_n \to u^* (n \to \infty)$. □

Similarly, we can prove that there exists v^* such that $v_n \to v^*$ $(n \to \infty)$.

The remainder is same as Theorem 3.1. We omit it.

COROLLARY 3.5 (see [5]). *Let P be a normal cone and let $F : D \to X$ be a single-valued increasing, condensing mapping (condensing mapping F possesses that F is continuous). Assume that (3.2) holds. Then F has maximal and minimal fixed points x^*, x_* on $[u_0, v_0]$ such that (3.3), (3.4) hold.*

Proof. Similar to the proof of Corollary 3.3, we have $F : D \to D$. Since $F : D \to X$ is a condensing mapping, we know that $F : D \to D$ is continuous. Thus, $F : D \to D$ is upper semicontinuous. So all the conditions of Theorem 3.4 are satisfied. The conclusion follows from Theorem 3.4. □

THEOREM 3.6. *Let cone P be strongly minimal and let $F : D \to 2^D$ be an increasing mapping with nonempty value. Then F has fixed points $\bar{x}$, $\underline{x}$ on $[u_0, v_0]$, where*

$$\begin{aligned}
\bar{x} &= \sup\{x \mid u_0 \leq x \leq v_0,\ u \geq x,\ \forall u \in F(x)\}, \\
\underline{x} &= \inf\{x \mid u_0 \leq x \leq v_0,\ v \leq x,\ \forall v \in F(x)\}.
\end{aligned} \tag{3.10}$$

Proof. Let $\Omega = \{x \mid u_0 \leq x \leq v_0,\ u \geq x,\ \forall u \in F(x)\}$. Evidently, Ω is nonempty ($u_0 \in \Omega$) and Ω has an upper bound v_0. From P being strongly minimal, we know that

$$\bar{x} = \sup \Omega \text{ exists, thus } u_0 \leq \bar{x} \leq v_0. \tag{3.11}$$

In the following, we prove that $\bar{x}$ is the fixed point of F on $[u_0, v_0]$.

For $x \in \Omega$, $u_0 \leq x \leq \bar{x} \leq v_0$ holds. So we obtain

$$\{u_0\} \leq^s F(x) \leq^s F(\bar{x}) \leq^s \{v_0\}. \tag{3.12}$$

Since $\{x\} \leq^s F(x)$, then $\{x\} \leq^s F(\bar{x})$, which implies that $u \geq x$ for all $u \in F(\bar{x})$. That is to say, for all $u \in F(\bar{x})$ is the upper bound of Ω. Therefore, $\bar{x} \leq u$ for all $u \in F(\bar{x})$. Hence, $F(\bar{x}) \leq^s F(u)$ for all $u \in F(\bar{x})$. So $\{u\} \leq^s F(u)$ for $\forall u \in F(\bar{x})$. By the definition of Ω, we have $u \in \Omega$, thus $F(\bar{x}) \subset \Omega$. It follows that $F(\bar{x}) \leq^s \{\bar{x}\}$. So $u \leq \bar{x}$ for all $u \in F(\bar{x})$. Consequently, $\bar{x} = u$, namely, $F(\bar{x}) = \{\bar{x}\}$ (a singleton). This implies that $\bar{x}$ is the fixed point of F on $[u_0, v_0]$.

Similarly, we can prove that

$$\underline{x} = \inf\{x \mid u_0 \le x \le v_0,\ v \le x,\ \forall v \in F(x)\} \tag{3.13}$$

is the fixed point of F on $[u_0, v_0]$.   □

Remark 3.7. Here we remove the assumption that F is upper semicontinuous. If F is a single-valued mapping, then

$$\begin{aligned}
\bar{x} &= \sup\{x \mid u_0 \le x \le v_0,\ F(x) \ge x\}, \\
\underline{x} &= \inf\{x \mid u_0 \le x \le v_0,\ F(x) \le x\}
\end{aligned} \tag{3.14}$$

are the maximal and minimal fixed points of F on $[u_0, v_0]$. In fact, suppose that there is $x^* \in [u_0, v_0]$ such that $F(x^*) = x^*$, then we have $x^* \in \{x \mid u_0 \le x \le v_0,\ F(x) \ge x\}$ and $x^* \in \{x \mid u_0 \le x \le v_0,\ F(x) \le x\}$. Thus, $\underline{x} \le x^* \le \bar{x}$. This shows the following corollary.

COROLLARY 3.8 (see [1]). *Let cone P be strongly minimal and let $F : D \to X$ be a single-valued increasing mapping. In addition, (3.2) holds. Then F has maximal and minimal fixed points on $[u_0, v_0]$.*

4. Necessity and sufficiency

In this section, we present a necessary and sufficient condition for the existence of the fixed point to multivalued maps which satisfy some conditions. In addition, we can obtain the uniqueness result under relatively strong conditions. We also do not assume that the mapping is upper semicontinuous.

THEOREM 4.1. *Let cone P be normal and let $F : P_h \to 2^{P_h}$ be an increasing mapping with nonempty value. In addition, there exist $\mu \in (0, 1)$ such that*

$$F(tx) \ge^s t^\mu F(x), \quad \text{for } t \in (0, 1),\ \forall x \in P_h. \tag{4.1}$$

Then a necessary and sufficient condition for F to have a fixed point in P_h is that the following holds. There exist $u_0, v_0 \in P_h$ with $u_0 < v_0$ such that

$$\{u_0\} \le^s F(u_0), \qquad F(v_0) \le^s \{v_0\}. \tag{4.2}$$

Proof. (1) *Sufficiency.* From the proof of Theorem 3.1, we obtain

$$\begin{aligned}
u_0 &\le u_1 \le \cdots \le u_n \le \cdots \le v_0, \\
v_0 &\ge v_1 \ge \cdots \ge v_n \ge \cdots \ge u_0,
\end{aligned} \tag{4.3}$$

where $u_n \in F(u_{n-1})$, $v_n \in F(v_{n-1})$ $(n = 1, 2, \ldots)$, $u_n \le v_n$. The fact that $u_0, v_0 \in P_h$, $u_0 < v_0$ shows that there is $\lambda_0 \in (0, 1)$ such that $u_0 \ge \lambda_0 v_0$, then we have $\lambda_0{}^\mu F(v_0) \le^s F(u_0)$. This implies that $u_1 \ge \lambda_0{}^\mu v_1$, then $\lambda_0{}^{\mu^2} F(v_1) \le^s F(u_1)$, and so $u_2 \ge \lambda_0{}^{\mu^2} v_2$. Repeating this process, we can obtain $u_n \ge \lambda_0{}^{\mu^n} v_n$. Therefore,

$$0 \le v_n - u_n \le v_n - \lambda_0{}^{\mu^n} v_n = (1 - \lambda_0{}^{\mu^n}) v_n < (1 - \lambda_0{}^{\mu^n}) v_0. \tag{4.4}$$

By the normality of cone P, we have

$$||v_n - u_n|| \leq N(1 - \lambda_0^{\mu^n})||v_0|| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{4.5}$$

here N is the normal constant. Further,

$$\begin{aligned} 0 &\leq u_{n+p} - u_n \leq v_n - u_n, \\ 0 &\leq v_n - v_{n+p} \leq v_n - u_n \quad (p \in N). \end{aligned} \tag{4.6}$$

Thus,

$$\begin{aligned} ||u_{n+p} - u_n|| &\leq N||v_n - u_n|| \longrightarrow 0 \quad (n \longrightarrow \infty), \\ ||v_n - v_{n+p}|| &\leq N||v_n - u_n|| \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{4.7}$$

So we can claim that $\{u_n\}$, $\{v_n\}$ are Cauchy sequences. Then there exists u^* such that $u_n \to u^*$ as $n \to \infty$ and $v_n \to u^*$ as $n \to \infty$. It follows that $u_n < u^* < v_n$. Thus, $F(u_n) \leq^s F(u^*) \leq^s F(v_n)$. Since $u_{n+1} \in F(u_n)$, $v_{n+1} \in F(v_n)$, we obtain

$$u_{n+1} \leq u \leq v_{n+1}, \quad \forall u \in F(u^*), \tag{4.8}$$

passing the limit in (4.8), $u = u^*$, namely, $F(u^*) = \{u^*\}$. This implies that u^* is a fixed point of F.

(2) *Necessity.* Let x_0 be the fixed point of F, then $t^\mu F(x_0) \leq^s F(tx_0)$. Take t_0 sufficiently small such that $t_0^\mu > t_0$, then

$$\{t_0 u\} \leq^s \{t_0^\mu u\} \leq^s F(t_0 x_0), \quad \forall u \in F(x_0), \tag{4.9}$$

which implies that

$$\{t_0 u \mid \forall u \in F(x_0)\} \leq^s F(t_0 x_0), \tag{4.10}$$

that is, $t_0 F(x_0) \leq^s F(t_0 x_0)$. It follows from $t^\mu F(x_0) \leq^s F(tx_0)$ that

$$F(x_0) = F\left(t \cdot \frac{1}{t} x_0\right) \geq^s t^\mu F\left(\frac{1}{t} x_0\right). \tag{4.11}$$

So we have

$$F\left(\frac{1}{t} x_0\right) \leq^s \frac{1}{t^\mu} F(x_0), \tag{4.12}$$

and we can choose t_1 sufficiently small satisfying $1/t_1^\mu < 1/t_1$, then

$$F\left(\frac{1}{t_1} x_0\right) \leq^s \left\{\frac{1}{t_1^\mu} u\right\} \leq^s \left\{\frac{1}{t_1} u\right\}, \quad \forall u \in F(x_0). \tag{4.13}$$

That is,

$$F\left(\frac{1}{t_1} x_0\right) \leq^s \left\{\frac{1}{t_1} u \mid \forall u \in F(x_0)\right\} = \frac{1}{t_1} F(x_0). \tag{4.14}$$

Now we write $\bar{t} = \min\{t_0, t_1\} < 1$, $u_0 = \bar{t}x_0$, $v_0 = (1/\bar{t})x_0$. Then

$$u_0 < v_0, \quad \{u_0\} \leq^s F(u_0), \quad F(v_0) \leq^s \{v_0\}. \tag{4.15}$$

Taking $\lambda_0 \in (0, \bar{t}^2)$, we have $u_0 \geq \lambda_0 v_0$. The proof is complete. $\qquad\square$

The uniqueness of the positive fixed point appears somewhat difficult to study. In the following, we give a result under relatively strong conditions.

THEOREM 4.2. *Let cone P be normal and let $F : P_h \to 2^D$ be a strictly increasing mapping (i.e., $x < y$ implies that $F(x) <^s F(y)$) with nonempty value. In addition, assume that*
  (i) $u_0, v_0 \in P_h$ *and* $u_0 \notin F(u_0), v_0 \notin F(v_0)$;
  (ii) *there exists $\mu \in (0,1)$ such that $F(tx) \geq^s t^\mu F(x)$ for $t \in (0,1)$, for all $x \in P$.*
Then F has a unique fixed point in D.

Proof. From (i), we have

$$u_0 < u_1 < \cdots < u_n < \cdots < v_n < \cdots < v_1 < v_0, \tag{4.16}$$

where $u_n \in F(u_{n-1})$, $v_n \in F(v_{n-1})$ $(n = 1, 2, \ldots)$.

From the sufficiency of Theorem 4.1, we know that there exists $u^* \in D$ such that $u_n \to u^*$, $v_n \to u^*$, $u^* \in F(u^*)$.

In the following, we prove that u^* is the unique fixed point of F in D. In fact, suppose that $\bar{u}$ is the fixed point of F with $\bar{u} \neq u^*$, then $u_0 < \bar{u} < v_0$. So $F(u_0) <^s F(\bar{u}) <^s F(v_0)$, thus $u_1 < \bar{u} < v_1$. Repeating this process, we have

$$u_n < \bar{u} < v_n, \tag{4.17}$$

passing the limit in (4.17), $\bar{u} = u^*$. This is a contradiction. This completes the proof. $\quad\square$

From a theoretical viewpoint, it is of interest to discuss the single-valued version of Theorems 4.1 and 4.2, then in that case, the operator is called μ-convex, some results of this type can be found in [4, 14].

5. Applications

It is well known that concave and convex operators defined on a cone in a Banach space play an important role in theory of positive operators (see e.g., [7, Chapter 6]). In [14], Potter introduced the definitions of α-concave operators and α-convex operators, he shows that for $\alpha \geq 0$, increasing α-concave and decreasing $(-\alpha)$-convex mappings have contraction ratios less than or equal to α and give the existence of solutions to the nonlinear eigenvalue problem $Ax = \lambda x$. The method is based upon Hilbert's projective metric (see [2] for details). In [15], the author improves the corresponding results presented in [14] by using contraction mapping theorem. In [4], Guo widens the conditions and removes the hypotheses of continuation for operators, and then extends the results of fixed points, eigenvectors for α-concave ($-\alpha$-convex) operators. However, they restrict their attention to $0 < |\alpha| \leq 1$, while for the remaining cases $\alpha > 1$ and $\alpha < -1$, the research proceeds slowly and appears difficult because Hilbert's projective metric is useless for these cases. Up to now, pleasant results are seldom obtained.

The aim of this section is to obtain the existence and uniqueness of positive fixed points for α-convex ($\alpha > 1$) operators by using Theorem 4.1.

THEOREM 5.1. *Let E be a real Banach space and let P be a normal cone in E. Let $A : P_h \to P_h$ be an increasing α-convex operator ($\alpha > 1$), that is, $A(tx) \leq t^\alpha Ax$ for all $x \in P_h$ and $t \in (0,1]$. Assume that there exists a nonempty, totally ordered set $S \subset P_h$ such that*
   (i) *$\lambda S \subset S$ ($\lambda \in (0,1)$), $AS = S$;*
   (ii) *$Av_0 \geq v_0$ for certain $v_0 \in S$.*

Then A has exactly one positive fixed point in S.

Proof.   Now we divide the proof into several steps.

Step 1.   Consider the operator $A : S \to S$. For all $y \in S$, set $A^{-1}y = \{x \in S : Ax = y\}$. Then $A^{-1} : S \to 2^S$ is a set-valued mapping. Consequently, we have the following conclusions.
   (a) A^{-1} is increasing in the sense of set-valued mappings.
   In fact, if $y_1 > y_2$ for all $y_1, y_2 \in S$, then $A^{-1}y_1 \geq^s A^{-1}y_2$. Suppose that this is not true, then we have $x_1 < x_2$ for all $x_1 \in A^{-1}y_1, x_2 \in A^{-1}y_2$. Using the monotonicity of A, we have $Ax_1 \leq Ax_2$, that is to say, $y_1 \leq y_2$, this is a contradiction.
   (b) $A^{-1}(sy) \geq^s s^{1/\alpha}A^{-1}y$ for $y \in S, s \in (0,1)$.
   For $x \in A^{-1}y$, then $Ax = y$ and $A(tx) \leq t^\alpha Ax$, for $t \in (0,1)$. Let $s = t^\alpha$, then we have $A(s^{1/\alpha}x) \leq sAx$, and (a) implies that $A^{-1}(A(s^{1/\alpha}x)) \leq^s A^{-1}(sAx)$. Thus

$$\{s^{1/\alpha}x\} \leq^s A^{-1}(sy), \quad \forall x \in A^{-1}y. \tag{5.1}$$

By the arbitrariness of x, one obtains that

$$\{s^{1/\alpha}x \mid x \in A^{-1}y\} \leq^s A^{-1}(sy), \tag{5.2}$$

namely, $s^{1/\alpha}A^{-1}y \leq^s A^{-1}(sy)$.

Step 2.   For $v_0 \in S, t \in (0,1)$, we have $A(tv_0) \leq t^\alpha Av_0$. Since $Av_0 \in S \subset P_h$, there exist $\lambda, \mu > 0$ such that $\lambda v_0 \leq Av_0 \leq \mu v_0$, so $A(tv_0) \leq t^\alpha Av_0 \leq \mu t^\alpha v_0$. So we can choose t_1 sufficiently small satisfying $\mu t^\alpha < t$, then $A(t_1 v_0) \leq t_1 v_0, t_1 \in (0,1)$.
   Now we write $u_0 = t_1 v_0$, then $Au_0 \leq u_0$ and $u_0 < v_0$. Taking $\lambda_0 = t_1{}^2$, then $\lambda_0 \in (0,1)$, $u_0 = t_1 v_0 \geq t_1{}^2 v_0 = \lambda_0 v_0$.

Step 3.   By Step 2 and (ii), we know that

$$u_0, v_0 \in S, \quad u_0 < v_0, \quad u_0 \geq \lambda_0 v_0, \quad Au_0 \leq u_0, \quad Av_0 \geq v_0. \tag{5.3}$$

Evidently, $A^{-1}u_0 \geq^s \{u_0\}, A^{-1}v_0 \leq^s \{v_0\}$. An application of the sufficiency of Theorem 4.1 shows that A^{-1} has a fixed point, write u^*, which implies that $Au^* = u^*$. That is to say, u^* is the positive fixed point of A.

Step 4.   In the following, we prove that u^* is the unique fixed point of A in S.

In fact, suppose that $\bar{u}$ is the fixed point of A with $\bar{u} \neq u^*$. Evidently, $\bar{u}, u^* > \theta$. Since S is a totally ordered set, we assume without loss of generality that $\bar{u} > u^*$, write $x_1 = u^*$, $x_2 = \bar{u}$, then $x_2 > x_1$. The fact that $x_1, x_2 \in P_h$ shows that there exists $\mu_0 > 0$ such that $x_1 \geq \mu_0 x_2$. Obviously, $\mu_0 < 1$.

If $x_1 = \mu_0 x_2$, then $A x_1 = A(\mu_0 x_2) \leq \mu_0^\alpha A x_2$, that is, $x_1 \leq \mu_0^\alpha x_2 < \mu_0 x_2$, this is a contradiction. Thus $x_1 > \mu_0 x_2$, so $A^{-1} x_1 \geq^s A^{-1}(\mu_0 x_2) \geq^s \mu_0^{1/\alpha} A^{-1} x_2$, this shows that $x_1 \geq \mu_0^{1/\alpha} x_2$. If $x_1 = \mu_0^{1/\alpha} x_2$, then $A(x_1) = A(\mu_0^{1/\alpha} x_2) \leq \mu_0 x_2$, this is a contradiction. Hence $x_1 > \mu_0^{1/\alpha} x_2$, repeating this process, we obtain $x_1 > \mu_0^{1/\alpha^n} x_2$. Consequently,

$$\theta < x_2 - x_1 < x_2 - \mu_0^{1/\alpha^n} x_2 = \left(1 - \mu_0^{1/\alpha^n}\right) x_2, \tag{5.4}$$

by the normality of cone P, we have

$$\|x_2 - x_1\| \leq N\left(1 - \mu_0^{1/\alpha^n}\right)\|x_2\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{5.5}$$

thus, $x_1 = x_2$, this is a contradiction. This completes the proof. □

COROLLARY 5.2. *Let E be a real Banach space and let P be a normal cone in E. Let $A : P_h \to P_h$ be an increasing α-convex operator ($\alpha > 1$), that is, $A(tx) \leq t^\alpha Ax$ for all $x \in P_h$ and $t \in (0,1]$. Assume that there exists a nonempty, totally ordered set $S \subset P_h$ such that $\lambda S \subset S$ ($\lambda > 0$), $AS = S$. Then A has exactly one positive fixed point in S.*

Proof. From the proof of Theorem 5.1, we only have to prove that there exists $v_0 \in S$ such that $Av_0 \geq v_0$.

For all $x_0 \in S$, $s > 1$, we have $A(sx_0) \geq s^\alpha Ax_0$. Since $Ax_0 \in S \subset P_h$, there exist $\mu, \lambda > 0$ such that $\lambda x_0 \leq Ax_0 \leq \mu x_0$, thus $A(sx_0) \geq s^\alpha \lambda x_0$. So we can choose s_0 large enough such that $s_0^\alpha \lambda \geq s_0$. Hence, $A(s_0 x_0) \geq s_0 x_0$. Now we write $v_0 = s_0 x_0$, then $Av_0 \geq v_0$. The proof is complete. □

Remark 5.3. (i) Under the hypotheses of this section, the tool—Hilbert's projective metric—used in those papers [2, 3, 14] cannot be used.

(ii) By using the properties of inverse mapping, (set-valued mapping) we give the similar results to α-concave operators ($0 < \alpha < 1$). So our results compliment the theory of concave and convex operators and the method is new and different from previous ones.

Example 5.4. Theorem 5.1 or Corollary 5.2 can be used to discuss the solution of the following special integral equation:

$$x(t) = \int_0^1 k(t,s) x^\alpha(s) ds, \quad \alpha > 1. \tag{5.6}$$

Suppose that $k(t,s) = h(t)f(s)$, and h, f are nonnegative continuous functions with $f(t) > 0$, $h(t) > 0$, for $t \in [0,1]$. Then, (5.6) has exactly one positive continuous solution.

Proof. First, we have some notation. Put $X = C[0,1]$ (the space of continuous functions defined on $[0,1]$ endowed with supremum norm). Let P be the cone of nonnegative

functions in X and so P is normal, $\overset{\circ}{P}$ is the set of positive functions in X. Noting that P is a closed solid, the norm is monotonic. Consider the integral operator $A : \overset{\circ}{P} \to X$ defined by

$$Ax(t) = \int_0^1 k(t,s)x^\alpha(s)ds, \quad x \in \overset{\circ}{P}. \tag{5.7}$$

Therefore,

$$Ax(t) = h(t)\int_0^1 f(s)x^\alpha(s)ds = ah(t), \tag{5.8}$$

where $a = \int_0^1 f(s)x^\alpha(s)ds = x^\alpha(\xi)\int_0^1 f(s)ds$, for certain $\xi \in (0,1)$. Evidently, $a > 0$, $ah \in \overset{\circ}{P}$, so $A : \overset{\circ}{P} \to \overset{\circ}{P}$. Let $S = \{x : x(t) = ah(t), a \in \mathbb{R}^+\}$, then we have $S \subset \overset{\circ}{P}$ and S is a totally ordered set with $\lambda S \subset S(\lambda > 0)$. In the following, we prove that $AS = S$. For all $y \in S$, $y = ah$, there is $x = (a/H)^{1/\alpha}h \in S$ such that $Ax = y$, where $H = \int_0^1 f(s)h^\alpha(s)ds$. Thus, $AS = S$. Since A is increasing in $\overset{\circ}{P}$, Theorem 5.1 implies that A has exactly one fixed point x^* in S. Further, $x^* = H^{1/1-\alpha}h$. In fact, let $x^* = a_0h$, then $Ax^* = A(a_0h) = a_0^\alpha Ah = a_0^\alpha Hh = a_0h$. So we obtain $a_0 = H^{1/1-\alpha}$, thus (5.6) has one positive solution $x^*(t) = H^{1/1-\alpha}h(t)$. $\square$

Remark 5.5. For (5.6), we can also use the following lemma generalized from [2, 3] to prove the results.

LEMMA 5.6. *Let the norm in Banach space X be monotonic on cone P, let $A : \overset{\circ}{P} \to \overset{\circ}{P}$ be positive homogeneous of degree p, $0 <| p |< 1$ (i.e., $A(tx) = t^p Ax, \forall x \in \overset{\circ}{P}, t > 0$). In addition, A is increasing $(0 < p < 1)$ or decreasing $(-1 < p < 0)$. Then A has exactly one positive fixed point in $\overset{\circ}{P}$.*

Proof of (5.6). As in the proof above, $A : S \to S$, $AS = S$. For $x_1, x_2 \in S$ with $x_1 \neq x_2$, then $Ax_1 \neq Ax_2$. In fact, let $x_1 = a_1h$, $x_2 = a_2h$, $a_1 \neq a_2$, $a_1, a_2 > 0$. Then $Ax_1 = A(a_1h) = a_1^\alpha Ah$, and $Ax_2 = A(a_2h) = a_2^\alpha Ah$, this together with $a_1^\alpha \neq a_2^\alpha$ implies the conclusion. So A is one-to-one mapping, consequently, $A^{-1} : S \to S$ exists, further,

$$A^{-1}(tx) = t^{1/\alpha}A^{-1}x, \quad t \in (0,1). \tag{5.9}$$

Since A is strictly increasing, we obtain that A^{-1} is also increasing. Otherwise, for $y_1, y_2 \in S$, $y_1 \leq y_2$, we have $A^{-1}y_1 > A^{-1}y_2$, thus, $AA^{-1}y_1 > AA^{-1}y_2$, that is, $y_1 > y_2$. This is a contradiction. An application of Lemma 5.6 implies that A^{-1} has exactly one positive fixed point in S. Since $Ah = Hh$, $A^{-1}(\lambda x) = \lambda^{1/\alpha}A^{-1}x$. We have $A^{-1}h = H^{-1/\alpha}h$. Let $u_0 = h$, $u_n = A^{-1}u_{n-1}$ $(n = 1,2,\dots)$. Then

$$u_1 = A^{-1}u_0 = A^{-1}h = H^{-1/\alpha}h,$$

$$u_2 = A^{-1}u_1 = A^{-1}(H^{-1/\alpha}h) = (H^{-1/\alpha})^{(1+1/\alpha)}h,$$

$$\vdots \tag{5.10}$$

$$u_{n+1} = A^{-1}u_n = (H^{-1/\alpha})^{(1+1/\alpha+\cdots+1/\alpha^n)}h = H^{(1-1/\alpha^n)/(1-\alpha)}h.$$

Thus, we have

$$\|u_{n+1} - H^{1/1-\alpha}h\| = |H^{(1-1/\alpha^n)/(1-\alpha)} - H^{1/1-\alpha}| \|h\| \longrightarrow 0 \quad (n \longrightarrow \infty); \tag{5.11}$$

consequently, $u_n \to H^{1/1-\alpha}h$. In addition,

$$A^{-1}(H^{1/1-\alpha}h) = (H^{1/1-\alpha})^{1/\alpha}H^{-1/\alpha}h = H^{1/1-\alpha}h. \tag{5.12}$$

which implies that $H^{1/1-\alpha}h$ is the fixed point of A^{-1}. Hence, A has exactly one positive fixed point $H^{1/1-\alpha}h$ in S. □

Remark 5.7. For the uniqueness of the solution of (5.6), we can also prove it by using the following method.

Proof. Let x_1, x_2 be the solutions of (5.6). Noting that $x_1 = a_1h$, $x_2 = a_2h$, then $x_1 = a_1/a_2$, $x_2 = ax_2$, where $a = a_1/a_2$, evidently, $0 < a < \infty$.

When $a > 1$, we have $Ax_1 = A(ax_2) = a^\alpha Ax_2 = a^\alpha x_2 > ax_2$, that is, $x_1 > ax_2$. This is a contradiction. When $a < 1$, we have $Ax_1 = A(ax_2) = a^\alpha Ax_2 = a^\alpha x_2 < ax_2$, that is, $x_1 < ax_2$. This is a contradiction. So $a = 1$, and we obtain $x_1 = x_2$. This completes the proof. □

Acknowledgment

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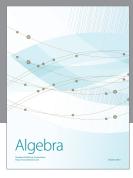
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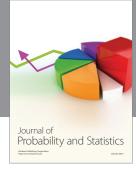*E-mail address*: cbzhai215@sohu.com

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