SPACES OF D_{L^p} TYPE AND A CONVOLUTION PRODUCT ASSOCIATED WITH THE SPHERICAL MEAN OPERATOR

M. DZIRI, M. JELASSI, AND L. T. RACHDI

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We define and study the spaces $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq \infty$, that are of D_{L^p} type. Using the harmonic analysis associated with the spherical mean operator, we give a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ and describe its bounded subsets. Next, we define a convolution product in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p < \infty$, and prove some new results.

1. Introduction

The spherical mean operator $\mathcal{R}$ is defined, for a function f on $\mathbb{R}^{n+1}$, even with respect to the first variable, by

$$\mathcal{R}(f)(r, x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi), \quad (r, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.1)$$

where S^n is the unit sphere $\{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n : \eta^2 + \|\xi\|^2 = 1\}$ in $\mathbb{R}^{n+1}$ and σ_n is the surface measure on S^n normalized to have total measure one.

This operator plays an important role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data (see [7, 8]), or in the linearized inverse scattering problem in acoustics [6]. In [10], the authors associate to the operator $\mathcal{R}$ a Fourier transform and a convolution product and have established many results of harmonic analysis (inversion formula, Paley-Wiener and Plancherel theorems, etc.).

In [11], the authors define and study Weyl transforms related to the mean operator $\mathcal{R}$ and have proved that these operators are compact. The spaces D_{L^p} , $1 \leq p \leq \infty$, have been studied by many authors [1, 2, 4, 5, 12, 13]. In this work, we introduce the function spaces $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq \infty$, similar to D_{L^p} , but replace the usual derivatives by the operator

$$L = I + \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^2, \quad (1.2)$$

where l is the Bessel operator defined on $]0, +\infty[$ by

$$l = \left(\frac{\partial}{\partial r} \right)^2 + \frac{n}{r} \frac{\partial}{\partial r}. \quad (1.3)$$

The main result of this paper gives a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ of the space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ and a description of its bounded subsets. More precisely, in Section 2, we recall some harmonic results related to a convolution product and the Fourier transform connected with the spherical mean operator, that we use in the following sections.

In the Section 3, we define the space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq \infty$, to be the space of measurable functions f on $]0, +\infty[\times \mathbb{R}^{n+1}$ such that for all $k \in \mathbb{N}$, $L^k f$ belongs to the space $L^p(d\nu)$ (the space of functions of p th power integrable on $[0, +\infty[\times \mathbb{R}^{n+1}$ with respect to the measure $r^n dr \otimes dx$). We give some properties of this space, in particular we prove that it is a Frechet space.

Section 4 is consecrated to the study of the dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. We give a nice description of the elements of this space and we characterize its bounded subsets.

In the last section, we define and study a convolution product in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p < \infty$, where $M_r(\mathbb{R} \times \mathbb{R}^n)$ is the closure of the Schwartz space $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $\mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n)$.

2. Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see [3, 6, 10, 11]. We denote by

- (A) $\mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable,
- (B) S^n the unit sphere in $\mathbb{R} \times \mathbb{R}^n$,

$$S^n = \{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n; \eta^2 + \|\xi\|^2 = 1\}, \quad (2.1)$$

where for $\xi = (\xi_1, \dots, \xi_n)$, we have $\|\xi\|^2 = \xi_1^2 + \dots + \xi_n^2$,

- (C) $d\sigma$ the normalized surface measure on S^n .

Definition 2.1. The spherical mean operator is defined on $\mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$ by

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}^n, \quad \mathcal{R}f(r, x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi). \quad (2.2)$$

For $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$, we put

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}^n, \quad \varphi_{\mu, \lambda}(r, x) = \mathcal{R}(\cos(\mu \cdot) e^{-i(\lambda/\cdot)}) (r, x). \quad (2.3)$$

We have

$$\varphi_{\mu, \lambda}(r, x) = j_{(n-1)/2} \left(r \sqrt{\mu^2 + \lambda^2} \right) e^{-i(\lambda/x)}, \quad (2.4)$$

where $j_{(n-1)/2}$ is the normalized Bessel function defined by

$$\begin{aligned} j_{(n-1)/2}(x) &= 2^{(n-1)/2} \Gamma \frac{n+1}{2} \frac{J_{(n-1)/2}(z)}{z^{(n-1)/2}} \\ &= \Gamma \frac{n+1}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma((2k+1+n)/2)} \left(\frac{z}{2}\right)^{2k} \end{aligned} \quad (2.5)$$

with $J_{(n-1)/2}$ the Bessel function of first kind and index $(n-1)/2$ [9, 15], and if $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we put $\lambda^2 = \lambda_1^2 + \dots + \lambda_n^2$ and $\langle \lambda/x \rangle = \lambda_1 x_1 + \dots + \lambda_n x_n$.

The normalized Bessel function $j_{(n-1)/2}$ has the following Mehler integral representation:

$$\forall r \in \mathbb{R}, \quad j_{(n-1)/2}(r) = \frac{2\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \int_0^1 (1-t^2)^{n/2-1} \cos(tr) dt, \quad (2.6)$$

and therefore

$$\forall k \in \mathbb{N}, \forall r \in \mathbb{R}, \quad |j_{(n-1)/2}^{(k)}(r)| \leq 1. \quad (2.7)$$

Moreover, for all $\lambda \in \mathbb{C}$, the function

$$r \mapsto j_{(n-1)/2}(\lambda r) \quad (2.8)$$

is the unique solution of the differential equation

$$\begin{aligned} lu(r) &= -\lambda^2 u(r), \\ u(0) &= 1, \quad u'(0) = 0, \end{aligned} \quad (2.9)$$

where l is the Bessel operator defined on $]0, +\infty[$ by (1.3).

On the other hand, the function $\varphi_{\mu, \lambda}$ is the unique solution of the system

$$\begin{aligned} D_j v(r, x) &= -i\lambda_j v(r, x), \quad j = 1, 2, \dots, n, \\ (l - \Delta)v(r, x) &= -\mu^2 v(r, x), \\ v(0, 0) &= 1; \quad \frac{\partial v}{\partial r}(0, x) = 0 \quad \forall x \in \mathbb{R}^n, \end{aligned} \quad (2.10)$$

where $D_j = \partial/\partial x_j$, and Δ is the Laplacien operator on $\mathbb{R}^n$:

$$\Delta = \sum_{j=1}^n D_j^2. \quad (2.11)$$

Now let Γ be the set

$$\Gamma = \mathbb{R} \times \mathbb{R}^n \cup \{(it, x); (t, x) \in \mathbb{R} \times \mathbb{R}^n, |t| \leq \|x\|\}. \quad (2.12)$$

We have for all $(\mu, \lambda) \in \Gamma$,

$$\sup_{(r,x) \in \mathbb{R} \times \mathbb{R}^n} |\varphi_{\mu,\lambda}(r,x)| = 1. \quad (2.13)$$

In the following, we will define a convolution product and the Fourier transform associated with the spherical mean operator. For this, we use the product formula for the functions $\varphi_{\mu,\lambda}$. For all $(r,x), (s,y) \in \mathbb{R} \times \mathbb{R}^n$,

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \int_0^\pi \varphi_{\mu,\lambda}(\sqrt{r^2+s^2+2rs\cos\theta}, x+y) (\sin\theta)^{n-1} d\theta. \quad (2.14)$$

We denote by (see [11])

(A) $d\nu(r,x)$ the measure defined on $[0, +\infty[\times \mathbb{R}^n$ by

$$d\nu(r,x) = k_n r^n dr \otimes dx \quad (2.15)$$

with

$$k_n = \frac{1}{2^{(n-1)/2} \Gamma((n+1)/2) (2\pi)^{n/2}}; \quad (2.16)$$

(B) $L^p(d\nu)$, $1 \leq p \leq +\infty$, the space of measurable functions on $[0, +\infty[\times \mathbb{R}^n$, satisfying

$$\begin{aligned} \|f\|_{p,\nu} &= \left(\int_{\mathbb{R}^n} \int_0^\infty |f(r,x)|^p d\nu(r,x) \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\infty,\nu} &= \operatorname{ess\,sup}_{(r,x) \in [0, +\infty[\times \mathbb{R}^n} |f(r,x)| < \infty, \quad p = +\infty; \end{aligned} \quad (2.17)$$

(C) $d\gamma(\mu,\lambda)$ the measure defined on the set Γ by

$$\begin{aligned} \int_\Gamma f(\mu,\lambda) d\gamma(\mu,\lambda) &= k_n \left\{ \int_{\mathbb{R}^n} \int_0^\infty f(\mu,\lambda) (\mu^2 + \|\lambda\|^2)^{(n-1)/2} \mu d\mu d\lambda \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \int_0^{\|\lambda\|} f(i\mu,\lambda) (\|\lambda\|^2 - \mu^2)^{(n-1)/2} \mu d\mu d\lambda \right\}; \end{aligned} \quad (2.18)$$

(D) $L^p(d\gamma)$, $1 \leq p \leq +\infty$, the space of measurable functions on Γ , satisfying

$$\begin{aligned} \|f\|_{p,\gamma} &= \left(\int_\Gamma |f(\mu,\lambda)|^p d\gamma(\mu,\lambda) \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\infty,\gamma} &= \operatorname{ess\,sup}_{(\mu,\lambda) \in \Gamma} |f(\mu,\lambda)| < \infty, \quad p = +\infty. \end{aligned} \quad (2.19)$$

Definition 2.2. (i) The translation operator associated with the spherical mean operator is defined on $L^1(d\nu)$ by for all $(r,x), (s,y) \in [0, +\infty[\times \mathbb{R}^n$,

$$\tau_{(r,x)} f(s,y) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y) (\sin\theta)^{n-1} d\theta. \quad (2.20)$$

(ii) A convolution product associated with the spherical mean operator of $f, g \in L^1(d\nu)$ is defined by for all $(r, x) \in [0, +\infty[\times \mathbb{R}^n$,

$$f * g(r, x) = \int_{\mathbb{R}^n} \int_0^\infty f(s, y) \tau_{(r, -x)} \check{g}(s, y) d\nu(s, y), \quad (2.21)$$

where

$$\check{g}(r, x) = g(r, -x). \quad (2.22)$$

We have the following properties.

- (A) $\tau_{(r, x)} \varphi_{\mu, \lambda}(s, y) = \varphi_{\mu, \lambda}(r, x) \varphi_{\mu, \lambda}(s, y)$.
 (B) If $f \in L^p(d\nu)$, $1 \leq p \leq +\infty$, then for all $(s, y) \in [0, +\infty[\times \mathbb{R}^n$, the function $\tau_{(s, y)} f \in L^p(d\nu)$, and we have

$$\|\tau_{(s, y)} f\|_{p, \nu} \leq \|f\|_{p, \nu}. \quad (2.23)$$

- (C) Let $1 \leq p, q, r \leq +\infty$ such that $1/r = 1/p + 1/q - 1$, then for all $f \in L^p(d\nu)$ and all $g \in L^q(d\nu)$, the function $f * g \in L^r(d\nu)$, and we have

$$\|f * g\|_{r, \nu} \leq \|f\|_{p, \nu} \|g\|_{q, \nu}. \quad (2.24)$$

Definition 2.3. The Fourier transform associated with the spherical mean operator is defined on $L^1(d\nu)$ by

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}f(\mu, \lambda) = \int_{\mathbb{R}^n} \int_0^\infty f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu(r, x). \quad (2.25)$$

We have the following properties.

- (A) For all $(\mu, \lambda) \in \Gamma$,

$$\mathcal{F}f(\mu, \lambda) = Bo\tilde{\mathcal{F}}f(\mu, \lambda), \quad (2.26)$$

where for all $(\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n$,

$$\begin{aligned} \tilde{\mathcal{F}}f(\mu, \lambda) &= \int_{\mathbb{R}^n} \int_0^\infty f(r, x) j_{(n-1)/2}(r\mu) e^{-i\langle \lambda/x \rangle} d\nu(r, x), \\ \forall (\mu, \lambda) \in \Gamma, \quad Bf(\mu, \lambda) &= f(\sqrt{\mu^2 + \lambda^2}, \lambda). \end{aligned} \quad (2.27)$$

- (B) For $f \in L^1(d\nu)$ such that $\mathcal{F}f \in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost every $(r, x) \in [0, +\infty[\times \mathbb{R}^n$,

$$f(r, x) = \iint_{\Gamma} \mathcal{F}f(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma(\mu, \lambda). \quad (2.28)$$

(C) Let f be in $L^1(d\nu)$. For all $(s, y) \in [0, +\infty[\times \mathbb{R}^n$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}(\tau_{(s, -y)}f)(\mu, \lambda) = \varphi_{\mu, \lambda}(s, y) \mathcal{F}f(\mu, \lambda). \quad (2.29)$$

(D) For $f, g \in L^1(d\nu)$, we have

$$\forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}(f * g)(\mu, \lambda) = \mathcal{F}f(\mu, \lambda) \mathcal{F}g(\mu, \lambda). \quad (2.30)$$

(E) For all $p \in [1, +\infty]$ and $f \in L^p(d\nu)$,

$$Bf \in L^p(d\gamma), \quad \|Bf\|_{p, \gamma} = \|f\|_{p, \nu}. \quad (2.31)$$

In particular, the mapping B is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. The mapping $\tilde{\mathcal{F}}$ is also an isometric isomorphism from $L^2(d\nu)$ onto itself. Consequently, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

Thus,

$$\forall f \in L^2(d\nu), \quad \mathcal{F}f \in L^2(d\gamma), \quad \|\mathcal{F}f\|_{2, \gamma} = \|f\|_{2, \nu}. \quad (2.32)$$

PROPOSITION 2.4 (see [11]). *Let f be in $L^p(d\nu)$, with $p \in [1, 2]$. Then $\mathcal{F}f \in L^{p'}(d\gamma)$, with $1/p + 1/p' = 1$, and*

$$\|\mathcal{F}f\|_{p', \gamma} \leq \|f\|_{p, \nu}. \quad (2.33)$$

We denote by

- (A) $S_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives;
- (B) $S_*(\Gamma)$ the space of infinitely differentiable functions on Γ , even with respect to the first variable, rapidly decreasing together with all their derivatives; that means for all $k_1, k_2 \in \mathbb{N}$, for all $\alpha \in \mathbb{N}^n$,

$$\sup \left\{ (1 + |\mu|^2 + \|\lambda\|^2)^{k_1} \left| \left(\frac{\partial}{\partial \mu} \right)^{k_2} D_\lambda^\alpha f(\mu, \lambda) \right|; (\mu, \lambda) \in \Gamma \right\} < +\infty, \quad (2.34)$$

where

$$\begin{aligned} \frac{\partial f}{\partial \mu}(\mu, \lambda) &= \begin{cases} \frac{\partial}{\partial r}(f(r, \lambda)) & \text{if } \mu = r \in \mathbb{R}, \\ \frac{1}{i} \frac{\partial}{\partial t}(f(it, \lambda)) & \text{if } \mu = it, |t| \leq \|\lambda\|, \end{cases} \\ D_\lambda^\alpha &= \left(\frac{\partial}{\partial \lambda_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial \lambda_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial \lambda_n} \right)^{\alpha_n}, \end{aligned} \quad (2.35)$$

(see [10]);

(C) $S'_*(\mathbb{R} \times \mathbb{R}^n)$ and $S'_*(\Gamma)$ are, respectively, the dual spaces of $S_*(\mathbb{R} \times \mathbb{R}^n)$ and $S_*(\Gamma)$. Each of these spaces is equipped with its usual topology.

Remark 2.5. From [10], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $S_*(\mathbb{R} \times \mathbb{R}^n)$ onto $S_*(\Gamma)$. The inverse mapping is given by for all $(r, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\mathcal{F}^{-1}f(r, x) = \int_{\Gamma} f(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma(\mu, \lambda). \quad (2.36)$$

Definition 2.6. The Fourier transform $\mathcal{F}$ is defined on $S'_*(\mathbb{R} \times \mathbb{R}^n)$ by

$$\forall T \in S'_*(\mathbb{R} \times \mathbb{R}^n), \quad \langle \mathcal{F}(T), \varphi \rangle = \langle T, \mathcal{F}^{-1}(\varphi) \rangle, \quad \varphi \in S_*(\Gamma). \quad (2.37)$$

Since the Fourier transform $\mathcal{F}$ is an isomorphism from $S_*(\mathbb{R} \times \mathbb{R}^n)$ onto $S_*(\Gamma)$, we deduce that $\mathcal{F}$ is also an isomorphism from $S'_*(\mathbb{R} \times \mathbb{R}^n)$ onto $S'_*(\Gamma)$.

3. The space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$

We denote by

(A) L the partial differential operator defined by

$$L = -\left(\frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r}\right) - \sum_{j=0}^n \frac{\partial^2}{\partial x_j^2}; \quad (3.1)$$

(B) for $f \in L^p(d\nu)$, $p \in [1, \infty]$, T_f is the element of $S'_*(\mathbb{R} \times \mathbb{R}^n)$ defined by

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} \int_0^\infty f(r, x) \varphi(r, x) d\nu(r, x), \quad \varphi \in S_*(\mathbb{R} \times \mathbb{R}^n); \quad (3.2)$$

(C) for $g \in L^p(d\gamma)$, $p \in [1, \infty]$, T_g is the element of $S'_*(\Gamma)$ defined by

$$\langle T_g, \psi \rangle = \int_{\Gamma} g(\mu, \lambda) \psi(\mu, \lambda) d\gamma(\mu, \lambda), \quad \psi \in S_*(\Gamma). \quad (3.3)$$

From Proposition 2.4 and Remark 2.5, we deduce that for all $f \in L^p(d\nu)$, $1 \leq p \leq 2$, $\mathcal{F}f$ belongs to the space $L^{p'}(d\gamma)$ and we have

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)}. \quad (3.4)$$

Definition 3.1. Let $p \in [1, \infty]$. We define $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ to be the set of measurable functions f on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, and such that for all $k \in \mathbb{N}$ there exists $g_k \in L^p(d\nu)$ satisfying

$$L^k T_f = T_{g_k}. \quad (3.5)$$

The space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ is equipped with the topology generated by the family of norms

$$\gamma_{m,p}(f) = \max_{0 \leq k \leq m} \|g_k\|_{p,\nu}, \quad m \in \mathbb{N}, \quad (3.6)$$

where $g_k, k \in \mathbb{N}$, is the function given by the relation (3.5). Let

$$\begin{aligned} d_p : \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \times \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) &\longrightarrow [0, \infty[, \\ (f, g) &\longmapsto d_p(f, g) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\gamma_{m,p}(f - g)}{1 + \gamma_{m,p}(f - g)}. \end{aligned} \quad (3.7)$$

Then d_p is a distance on $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$. Moreover the sequence $(f_k)_{k \in \mathbb{N}}$ converges to 0 in $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ if and only if

$$\forall m \in \mathbb{N}, \quad \gamma_{m,p}(f_k) \xrightarrow[k \rightarrow \infty]{} 0. \quad (3.8)$$

In the following, we will give some properties of the space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$.

PROPOSITION 3.2. $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ is a Frechet space.

Proof. Let $(f_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ and let $(g_{m,k})_{m \in \mathbb{N}} \subset L^p(d\nu)$ such that

$$L^k T_{f_m} = T_{g_{m,k}}, \quad k \in \mathbb{N}. \quad (3.9)$$

Then for all $k \in \mathbb{N}$, $(g_{m,k})_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^p(d\nu)$. We put

$$\begin{aligned} f &= g_0 = \lim_{m \rightarrow \infty} f_m, \\ g_k &= \lim_{m \rightarrow \infty} g_{m,k}, \quad k \in \mathbb{N}^*, \end{aligned} \quad (3.10)$$

in $L^p(d\nu)$. Thus

$$\forall k \in \mathbb{N}, \quad T_{g_{m,k}} \xrightarrow[m \rightarrow \infty]{} T_{g_k}, \quad (3.11)$$

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$. Since L^k is a continuous operator from $S'_*(\mathbb{R} \times \mathbb{R}^n)$ into itself, we deduce that

$$L^k T_{f_m} \xrightarrow[m \rightarrow \infty]{} L^k T_f, \quad (3.12)$$

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$.

From relations (3.9) and (3.11), we deduce that

$$\forall k \in \mathbb{N}, \quad L^k T_f = T_{g_k}. \quad (3.13)$$

This proves that $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ and

$$f_m \xrightarrow[m \rightarrow \infty]{} f \quad (3.14)$$

in $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), d_p)$. □

PROPOSITION 3.3. Let $p \in [1, 2]$ and $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, then

(i) for all $k \in \mathbb{N}$, the function

$$(\mu, \lambda) \longrightarrow (1 + \mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f)(\mu, \lambda) \quad (3.15)$$

belongs to the space $L^{p'}(d\nu)$ with $p' = p/(p-1)$;

(ii) $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n) \subset \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$, where $\mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R} \times \mathbb{R}^n$ even with respect to the first variable.

Proof. (i) Let $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq 2$, and $g_k \in L^p(d\nu)$ such that

$$L^k T_f = T_{g_k} \quad k \in \mathbb{N}. \quad (3.16)$$

From relation (3.4), we have

$$\mathcal{F}(T_{g_k}) = T_{\mathcal{F}(g_k)}, \quad (3.17)$$

which gives

$$\mathcal{F}(L^k T_f) = T_{\mathcal{F}(g_k)}. \quad (3.18)$$

On the other hand

$$\mathcal{F}(L^k T_f) = (\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(T_f) = T_{(\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f)}, \quad (3.19)$$

hence

$$(\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f) = \mathcal{F}(g_k). \quad (3.20)$$

This equality, together with the fact that the function $\mathcal{F}(g_k)$ belongs to the space $L^{p'}(d\nu)$ implies (i).

(ii) Let $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$. From the assertion (i) and relations (2.26) and (2.31), we deduce that for all $k \in \mathbb{N}$, the function

$$(r, x) \longrightarrow (r^2 + \|x\|^2)^k \tilde{\mathcal{F}}(f) \quad (3.21)$$

belongs to the space $L^{p'}(d\nu)$, in particular $\tilde{\mathcal{F}}(f) \in L^1(d\nu) \cap L^2(d\nu)$.

On the other hand, the transform $\tilde{\mathcal{F}}$ is an isometric isomorphism from $L^2(d\nu)$ onto itself, then from the inversion formula for $\tilde{\mathcal{F}}$ and using the continuity of the function f , we have for all $(r, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$f(r, x) = \int_{\mathbb{R}^n} \int_0^\infty \tilde{\mathcal{F}}f(\mu, \lambda) j_{(n-1)/2}(r\mu) e^{i\langle \lambda/x \rangle} d\nu(\mu, \lambda). \quad (3.22)$$

Consequently, (ii) follows from relation (2.7) and the fact that for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, the function

$$(\mu, \lambda) \longrightarrow \mu^k \lambda^\alpha \tilde{\mathcal{F}}(f)(\mu, \lambda) \quad (3.23)$$

belongs to the space $L^1(d\nu)$. □

PROPOSITION 3.4. *Let $p \in [1, 2]$, then, for all $r \in [2, \infty]$,*

$$\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n) \subset \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n). \quad (3.24)$$

Proof. Let $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$, $p \in [1, 2]$, $r \geq 2$, and $r' = r/(r-1)$. From Proposition 3.3, we deduce that $f \in \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$ and for all $k \in \mathbb{N}$, the function (3.21) belongs to the space $L^{p'}(d\nu)$. By applying Holder's inequality, it follows that this last function belongs to the space $L^{r'}(d\nu)$. On the other hand, for all $(r, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\begin{aligned} L^k f(r, x) &= \int_{\mathbb{R}^n} \int_0^\infty (\mu^2 + \|\lambda\|^2)^k \tilde{\mathcal{F}}(f)(\mu, \lambda) j_{(n-1)/2}(r\mu) e^{i(\lambda/x)} d\nu(\mu, \lambda) \\ &= \tilde{\mathcal{F}}\left((\mu^2 + \|\lambda\|^2)^k \tilde{\mathcal{F}}(f)\right)(r, x). \end{aligned} \quad (3.25)$$

From Proposition 2.4 and the fact that

$$\|\mathcal{F}(g)\|_{r, y} = \|\tilde{\mathcal{F}}(g)\|_{r, y}, \quad g \in L^{r'}(d\nu), \quad (3.26)$$

we deduce that, for all $k \in \mathbb{N}$, the function $L^k f$ belongs to the space $L^r(d\nu)$. $\square$

4. The dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$

In this section, we will give a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ of $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$. We recall that for every $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, the family $\{V_{m, p, \varepsilon}(f), m \in \mathbb{N}, \varepsilon > 0\}$ is a basis of neighborhoods of f in $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), d_p)$, where

$$V_{m, p, \varepsilon}(f) = \{g \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \gamma_{m, p}(f - g) < \varepsilon\}. \quad (4.1)$$

In addition, $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ if and only if there exist $m \in \mathbb{N}$ and $c > 0$ such that

$$\forall f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \quad |\langle T, f \rangle| \leq c \gamma_{m, p}(f). \quad (4.2)$$

For $f \in L^{p'}(d\nu)$ and $\varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, we put

$$\langle L^k(T_f), \varphi \rangle = \int_{\mathbb{R}^n} \int_0^\infty f(r, x) \psi_k(r, x) d\nu(r, x) \quad (4.3)$$

with $L^k T_\varphi = T_{\psi_k}$. Then

$$|\langle L^k(T_f), \varphi \rangle| \leq \|f\|_{p', \nu} \|\psi_k\|_{p, \nu} \leq \|f\|_{p', \nu} \gamma_{k, p}(\varphi). \quad (4.4)$$

This proves that for all $f \in L^{p'}(d\nu)$ and $k \in \mathbb{N}$, the functional $L^k T_f$ defined by the relation (4.3) belongs to the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$.

In the following, we will prove that every element of $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ is also of this type.

THEOREM 4.1. *Let $T \in S'_*(\mathbb{R} \times \mathbb{R}^n)$. Then $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p < \infty$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0, \dots, f_m\} \subset L^{p'}(d\nu)$ such that*

$$T = \sum_{k=0}^m L^k T_{f_k}, \quad (4.5)$$

where $L^k T_{f_k}$ is given by relation (4.3).

Proof. It is clear that if

$$T = \sum_{k=0}^m L^k T_{f_k}, \quad \{f_0, \dots, f_m\} \subset L^{p'}(d\nu), \quad (4.6)$$

then T belongs to the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$.

Conversely, suppose that $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. From relation (4.2) there exist $m \in \mathbb{N}$ and $c > 0$ such that

$$\forall \varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \quad |\langle T, \varphi \rangle| \leq c \gamma_{m,p}(\varphi). \quad (4.7)$$

Let

$$(L^p(d\nu))^{m+1} = \{(f_0, \dots, f_m), f_k \in L^p(d\nu), 0 \leq k \leq m\} \quad (4.8)$$

equipped with the norm

$$\|(f_0, \dots, f_m)\|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \|f_k\|_{p,\nu}. \quad (4.9)$$

We consider the mappings

$$\begin{aligned} \mathcal{A} : \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) &\longrightarrow (L^p(d\nu))^{m+1}, \\ \varphi &\longmapsto (\varphi, g_1, \dots, g_m), \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} L^k T_\varphi &= T_{g_k}, \quad k \geq 1, \\ \mathcal{B} : \text{Im}(\mathcal{A}) &\longrightarrow \mathbb{C}, \\ \mathcal{B}(\mathcal{A}\varphi) &= \langle T, \varphi \rangle. \end{aligned} \quad (4.11)$$

From relation (4.2) we deduce that

$$|\mathcal{B}(\mathcal{A}(\varphi))| = |\langle T, \varphi \rangle| \leq c \|\mathcal{A}(\varphi)\|_{(L^p(d\nu))^{m+1}}. \quad (4.12)$$

This means that $\mathcal{B}$ is a continuous functional on the subspace $\text{Im}(\mathcal{A})$ of the space $(L^p(d\nu))^{m+1}$. From Hahn-Banach theorems, there exists a continuous extension of $\mathcal{B}$ to $(L^p(d\nu))^{m+1}$, denoted again by $\mathcal{B}$.

By Riez's theorem there exist $(f_0, \dots, f_m) \in (L^{p'}(d\nu))^{m+1}$ such that for all $(\varphi_0, \dots, \varphi_m) \in (L^p(d\nu))^{m+1}$,

$$\mathcal{B}(\varphi_0, \dots, \varphi_m) = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_k(r, x) \varphi_k(r, x) d\nu(r, x). \quad (4.13)$$

By means of relation (4.3), we deduce that for $\varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, we have

$$\langle T, \varphi \rangle = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_k(r, x) \varphi_k(r, x) d\nu(r, x) = \sum_{k=0}^m \langle L^k T_{f_k}, \varphi \rangle. \quad (4.14)$$

This completes the proof of Theorem 4.1. $\square$

PROPOSITION 4.2. *Let $p \geq 2$. Then for all $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$, there exist $m \in \mathbb{N}$ and $F \in L^p(d\gamma)$ such that*

$$\mathcal{F}(T) = T_{(1+\mu^2+2\|\lambda\|^2)^m F}. \quad (4.15)$$

Proof. Let $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. From Theorem 4.1 there exist $m \in \mathbb{N}$ and $(f_0, \dots, f_m) \in (L^{p'}(d\nu))^{m+1}$, $p' = p/(p-1)$, such that

$$T = \sum_{k=0}^m L^k T_{f_k}. \quad (4.16)$$

Consequently

$$\mathcal{F}(T) = \sum_{k=0}^m \mathcal{F}(L^k T_{f_k}) = \sum_{k=0}^m (\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(T_{f_k}). \quad (4.17)$$

By using relation (3.4) we get (4.15), where

$$F = \sum_{k=0}^m \frac{(\mu^2 + 2\|\lambda\|^2)^k}{(1 + \mu^2 + 2\|\lambda\|^2)^m} \mathcal{F}(f_k), \quad (4.18)$$

which proves the result. $\square$

PROPOSITION 4.3. *Let $T \in \mathcal{S}'_*(\mathbb{R} \times \mathbb{R}^n)$, then $T \in \mathcal{M}'_2(\mathbb{R} \times \mathbb{R}^n)$ if and only if there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma)$ such that (4.15) holds.*

Proof. From Proposition 4.2, we deduce that if $T \in \mathcal{M}'_2(\mathbb{R} \times \mathbb{R}^n)$, then there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma)$ verifying (4.15). Conversely, suppose that (4.15) holds with $F \in L^2(d\gamma)$. Since $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$, then there exists $G \in L^2(d\nu)$ such that $\mathcal{F}(G) = F$ and from relation (3.4) we have

$$\mathcal{F}(T_G^\vee) = T_F. \quad (4.19)$$

Consequently

$$\mathcal{F}(T) = \mathcal{F}((I+L)^m T_{\tilde{G}}), \quad (4.20)$$

thus

$$T = \sum_{k=0}^m C_m^k L^k T_{\tilde{G}}, \quad (4.21)$$

and Theorem 4.1 implies that $T \in \mathcal{M}'_2(\mathbb{R} \times \mathbb{R}^n)$. $\square$

We denote by

- (A) $\mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and with compact support, equipped with its usual topology;
- (B) for $a > 0$, $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ the subspace of $\mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$ consisting of function f such that $\text{supp } f \subset B(0, a) = \{(r, x) \in \mathbb{R} \times \mathbb{R}^n, r^2 + \|x\|^2 \leq a^2\}$;
- (C) for $a > 0$, $\mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ the dual space of $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$;
- (D) for $a > 0$ and $m \in \mathbb{N}$, $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ the space of function $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ of class C^{2m} on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and with support in $B(0, a)$, normed by

$$N_{\infty, m}(f) = \max_{0 \leq k \leq m} \|L^k(f)\|_{\infty, \gamma}. \quad (4.22)$$

PROPOSITION 4.4. *Let $a > 0$ and $m \in \mathbb{N}$. Then there exists $p_o \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, $p \geq p_o$, it is possible to find $\varphi_p \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ and $\psi_p \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ satisfying*

$$\delta = (I+L)^p T_{\varphi_p} + T_{\psi_p} \quad (4.23)$$

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$.

Proof. Let $p \geq n+1$ and g_p the function defined by

$$\forall (\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n, \quad g_p(\mu, \lambda) = \tilde{\mathcal{F}}\left(\frac{1}{(1+r^2+\|x\|^2)^p}\right)(\mu, \lambda). \quad (4.24)$$

Using relation (2.7), we deduce that there exists $p_o \in \mathbb{N}$ such that for all $p \geq p_o$ the function g_p is of class C^{2m} on $\mathbb{R} \times \mathbb{R}^n$ (e.g., we can choose $p_o = 3n+1+2m$).

Now, we prove that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$. The function g_p can be written as

$$g_p(\mu, \lambda) = \frac{1}{2^{n-1/2}\Gamma(n+1/2)} \int_0^\infty \frac{1}{(1+s^2)^p} j_{n-1/2}\left(s\sqrt{\mu^2 + \|\lambda\|^2}\right) s^{2n} ds. \quad (4.25)$$

By relation (2.6) and Fubini's theorem we get

$$\begin{aligned} g_p(\mu, \lambda) &= \frac{1}{2^{n-1/2}\sqrt{\pi}\Gamma(n)} \int_{-1}^1 (1-t^2)^{n-1} \left[\int_0^\infty \frac{\cos\left(ts\sqrt{\mu^2 + \|\lambda\|^2}\right)}{(1+s^2)^p} s^{2n} ds \right] dt \\ &= \frac{1}{2^{n-3/2}\sqrt{\pi}\Gamma(n)} \int_0^1 (1-t^2)^{n-1} h_p\left(t\sqrt{\mu^2 + \|\lambda\|^2}\right) dt, \end{aligned} \quad (4.26)$$

where

$$h_p(u) = \int_0^\infty \frac{\cos(su)}{(1+s^2)^p} s^{2n} ds = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{isu}}{(1+s^2)^p} s^{2n} ds. \quad (4.27)$$

By standard calculus, we have

$$\int_0^\infty \frac{\cos(su)}{(1+s^2)^p} s^{2n} ds = e^{-u} P(u) \quad (4.28)$$

with

$$P(u) = \frac{\pi}{2^{2p-1}} \sum_{k=0}^{p-1} \frac{C_{2p-2-k}^{p-1}}{k!} (2u)^k. \quad (4.29)$$

On the other hand, we have

$$h_p(u) = (-1)^n \left(\frac{d}{du} \right)^{2n} \left(\frac{1}{2} \int_{-\infty}^\infty \frac{e^{isu}}{(1+s^2)^p} ds \right), \quad (4.30)$$

then, we get

$$\forall u \geq 0, \quad h_p(u) = Q_p(u) e^{-u}, \quad (4.31)$$

where Q_p is a real polynomial. Since h_p is an even function on $\mathbb{R}$, then we deduce that

$$\forall u \in \mathbb{R}, \quad h_p(u) = k_p(|u|), \quad (4.32)$$

where k_p is the infinitely differentiable function defined on $\mathbb{R}$ by

$$k_p(u) = Q_p(u) e^{-u}. \quad (4.33)$$

Now, the function

$$u \longrightarrow F_p(u) = \frac{1}{2^{n-3/2}\sqrt{\pi}\Gamma(n)} \int_0^1 (1-t^2)^{n-1} k_p(tu) dt \quad (4.34)$$

is infinitely differentiable on $\mathbb{R}$ and we have

$$g_p(\mu, \lambda) = F_p\left(\sqrt{\mu^2 + \|\lambda\|^2}\right). \quad (4.35)$$

This shows that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$, even with respect to the first variable.

Let $\gamma \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ such that

$$\forall (r, x) \in \mathbb{R} \times \mathbb{R}^n, \quad r^2 + x^2 \leq \frac{a^2}{4}, \quad \gamma(r, x) = 1. \quad (4.36)$$

Since $(I + L)^p T_{g_p} = \delta$, we get

$$\gamma(I + L)^p T_{g_p} = (I + L)^p T_{g_p} = \delta. \quad (4.37)$$

On the other hand, by using the fact that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$, we deduce that the function

$$\varphi_p(r, x) = (\gamma - 1)(I + L)^p g_p + (I + L)^p ((1 - \gamma)g_p) \quad (4.38)$$

belongs to the space $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$.

Moreover, from relation (4.37), we have

$$T_{(\gamma-1)(I+L)^p g_p} = (\gamma - 1)(I + L)^p T_{g_p} = 0, \quad (4.39)$$

and this implies by using relation (4.38) that

$$T_{\varphi_p} = T_{(I+L)^p ((1-\gamma)g_p)} = (I + L)^p T_{((1-\gamma)g_p)}. \quad (4.40)$$

Hence,

$$T_{\varphi_p} + (I + L)^p T_{\gamma g_p} = (I + L)^p T_{g_p} = \delta, \quad (4.41)$$

and this completes the proof of the proposition by taking $\psi_p = \gamma g_p$. □

To prove the main result of this section, that is, Theorem 4.7, we will define some new families of norms on the space $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$. We use these norms to prove that the elements of all bounded subset $B' \subset \mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$.

For $f \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$, $a > 0$,

- (A) $P_m(f) = \max_{k+|\alpha| \leq m} \|(\partial/\partial r)^k D^\alpha f\|_{\infty, \nu}$,
- (B) $\tilde{P}_m(f) = \max_{k+|\alpha| \leq m} \|l^k D^\alpha f\|_{\infty, \nu}$,
- (C) $N_{p,m}(f) = \max_{0 \leq k \leq m} \|L^k(f)\|_{p, \nu}$, $p \in [1, \infty]$,

where l is defined by relation (1.3).

LEMMA 4.5. (i) For all $m \in \mathbb{N}$, there exists $c_1 > 0$ such that

$$\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad P_m(\varphi) \leq c_1 \tilde{P}_m(\varphi). \quad (4.42)$$

(ii) For all $m \in \mathbb{N}$, there exist $c_2 > 0$ and $m' \in \mathbb{N}$ such that

$$\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad \tilde{P}_m(\varphi) \leq c_2 N_{p,m'}(\varphi). \quad (4.43)$$

Proof. (i) Let $m \in \mathbb{N}$, and $\varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$. By induction on k we have

$$\left(\frac{\partial}{\partial r}\right)^k D^\alpha \varphi(r, x) = \sum_{s=0}^k P_s(r) \left(\frac{\partial}{\partial r^2}\right)^s D^\alpha \varphi(r, x), \quad (4.44)$$

where P_s is a real polynomial. On the other hand, and also by induction, we deduce that for all $s \geq 1$,

$$\left(\frac{\partial}{\partial r^2}\right)^s D^\alpha \varphi(r, x) = \int_0^1 \cdots \int_0^1 t^s D^\alpha \varphi(rt_1, \dots, t_s, x) t_1^{n+2(s-1)} \cdots t_s^n dt_1, \dots, dt_s. \quad (4.45)$$

From relations (4.44) and (4.45), it follows that there exists $c_{a,m} > 0$ satisfying

$$P_m(\varphi) \leq c_{a,m} \tilde{P}_m(\varphi). \quad (4.46)$$

(ii) Let $p \in [1, \infty]$, $m \in \mathbb{N}$, and $m_1 \in \mathbb{N}$ such that

$$\left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1,v} < \infty, \quad (4.47)$$

then, for all $(k, \alpha) \in \mathbb{N} \times \mathbb{N}^n$, $k + |\alpha| \leq m$, we have

$$\begin{aligned} \|l^k D^\alpha \varphi\|_{\infty, v} &= \|\tilde{\mathcal{F}}^{-1}(\tilde{\mathcal{F}}(l^k D^\alpha \varphi))\|_{\infty, v} \\ &\leq \|\tilde{\mathcal{F}}(l^k D^\alpha \varphi)\|_{1, v} \\ &\leq \|\mu^{2k} \lambda^\alpha \tilde{\mathcal{F}}(\varphi)\|_{1, v} \\ &\leq \|(1 + \mu^2 + \|\lambda\|^2)^m \tilde{\mathcal{F}}(\varphi)\|_{1, v} \\ &= \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \tilde{\mathcal{F}}((I + L)^{m+m_1} \varphi) \right\|_{1, v} \\ &\leq \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1, v} \|\tilde{\mathcal{F}}((I + L)^{m+m_1} \varphi)\|_{\infty, v} \\ &\leq \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1, v} \|(I + L)^{m+m_1} \varphi\|_{1, v}, \end{aligned} \quad (4.48)$$

and by Holder's inequality, we get

$$\begin{aligned} \|l^k D^\alpha \varphi\|_{\infty, v} &\leq \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1, v} (\nu(B(0, a)))^{1/p'} \|(I + L)^{m+m_1} \varphi\|_{p, v} \\ &\leq \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1, v} (\nu(B(0, a)))^{1/p'} 2^{m+m_1} N_{p, m+m_1}(\varphi), \end{aligned} \quad (4.49)$$

which implies that

$$\tilde{P}_m(\varphi) \leq 2^{m+m_1} (\nu(B(0, a)))^{1/p'} \left\| \frac{1}{(1 + \mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1, v} N_{p, m+m_1}(\varphi), \quad (4.50)$$

and the proof of the lemma is complete. $\square$

THEOREM 4.6. *Let $a > 0$ and B' a weakly* bounded set of $\mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$. Then, there exists $m \in \mathbb{N}$ such that the elements of B' can be continuously extended to $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$. Moreover, the family of these extensions is equicontinuous.*

Proof. Let $p \in [1, \infty]$. Since B' is weakly* bounded in $\mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$, then from [14] and Lemma 4.5 there exist a positive constant c and $m \in \mathbb{N}$ such that for all $T \in B'$, for all $\varphi \in D_{*,a}(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle T, \varphi \rangle| \leq cN_{p,m}(\varphi). \quad (4.51)$$

We consider the mappings

$$\begin{aligned} A : \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n) &\longrightarrow (L^p(d\nu))^{m+1}, \\ \varphi &\longmapsto (L^k \varphi)_{0 \leq k \leq m}, \end{aligned} \quad (4.52)$$

and for all $T \in B'$,

$$\begin{aligned} L_T : A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n)) &\longrightarrow \mathbb{C}, \\ \langle L_T, A\varphi \rangle &= \langle T, \varphi \rangle. \end{aligned} \quad (4.53)$$

From relation (4.51), we deduce that for all $\varphi \in D_{*,a}(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle L_T, A\varphi \rangle| \leq c \|A\varphi\|_{(L^p(d\nu))^{m+1}}. \quad (4.54)$$

This means that L_T is a continuous functional on the subspace $A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))$ of the space $(L^p(d\nu))^{m+1}$ and that for all $T \in B'$,

$$\|L_T\|_{A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))} = \sup_{\|A\varphi\|_{(L^p(d\nu))^{m+1}} \leq 1} |\langle L_T, A\varphi \rangle| \leq c. \quad (4.55)$$

From the Hahn-Banach theorems, L_T can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by L_T . Furthermore, for all $T \in B'$,

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \sup_{\|\psi\|_{(L^p(d\nu))^{m+1}} \leq 1} |\langle L_T, \psi \rangle| = \|L_T\|_{A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))} \leq c. \quad (4.56)$$

Now, from the Riez theorem, there exists $(f_{T,k})_{0 \leq k \leq m} \subset L^{p'}(d\nu)$ such that for all $\psi = (\psi_0, \dots, \psi_m) \in (L^p(d\nu))^{m+1}$,

$$\langle L_T, \psi \rangle = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_{T,k}(r, x) \psi_k(r, x) d\nu \quad (4.57)$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \|f_{T,k}\|_{p', \nu}. \quad (4.58)$$

Thus, from (4.56) it follows that for all $T \in B'$, for all $k \in \mathbb{N}$, $0 \leq k \leq m$,

$$\|f_{T,k}\|_{p', \nu} \leq c. \quad (4.59)$$

In particular, for $\varphi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ we have

$$\langle L_T, A\varphi \rangle = \sum_{k=0}^m \int_{\mathbb{R}} \int_0^\infty f_{T,k}(r, x) L^k(\varphi)(r, x) d\nu(r, x). \quad (4.60)$$

Using Holder's inequality and relation (4.59), we get for all $T \in B'$, for all $\varphi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle L_T, A\varphi \rangle| \leq (m+1)c[\nu(B(0, a))]^{1/p} N_{\infty, m}(\varphi). \quad (4.61)$$

This shows that the mapping $L_T o A$ is a continuous extension of T on $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ and that the family $\{L_T o A\}_{T \in B'}$ is equicontinuous, when applied to $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$. This completes the proof of Theorem 4.6. $\square$

In the following, we will give a new characterization of the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$.

THEOREM 4.7. *Let $T \in S'_*(\mathbb{R} \times \mathbb{R}^n)$, $p \in [1, \infty[$, $p' = p/(p-1)$. Then $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ if and only if for every $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$, where*

$$T * \varphi(r, x) = \langle T, \tau_{(r, -x)} \check{\varphi} \rangle. \quad (4.62)$$

Proof. Let $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. From Theorem 4.1, there exist $m \in \mathbb{N}$ and $f_0, \dots, f_m \in L^{p'}(d\nu)$ such that

$$T = \sum_{k=0}^m L^k T_{f_k}, \quad (4.63)$$

in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. Thus, for every $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$,

$$T * \varphi = \sum_{k=0}^m T_{f_k} * L^k \varphi = \sum_{k=0}^m f_k * L^k \varphi. \quad (4.64)$$

Since, for all $k \in \mathbb{N}$, $0 \leq k \leq m$, $f_k \in L^{p'}(d\nu)$ and $L^k \varphi \in L^1(d\nu)$, then from inequality (2.24), we deduce that $f_k * L^k \varphi \in L^{p'}(d\nu)$. This implies that the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$.

Conversely, let $T \in S'_*(\mathbb{R} \times \mathbb{R}^n)$ such that for every $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$ the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$. For φ, ψ in $\mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, we have

$$\langle T_{T*\varphi}, \psi \rangle = \langle T, \varphi * \check{\psi} \rangle = \langle T, \psi * \check{\varphi} \rangle = \langle T_{T*\psi}, \varphi \rangle. \quad (4.65)$$

From Holder's inequality and using the hypothesis, we obtain

$$|\langle T_{T*\varphi}, \psi \rangle| \leq \|T * \psi\|_{p', \nu} \|\varphi\|_{p, \nu}, \quad (4.66)$$

from which we deduce that the set

$$B' = \{T_{T*\varphi}, \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n); \|\varphi\|_{p,\nu} \leq 1\} \quad (4.67)$$

is bounded in $\mathcal{D}'_*(\mathbb{R} \times \mathbb{R}^n)$.

Now, using Theorem 4.6, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \leq 1$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ and the family of these extensions is equicontinuous, which means that there exists $c > 0$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \leq 1$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle T_{T*\varphi}, \psi \rangle| \leq cN_{\infty,m}(\psi). \quad (4.68)$$

This involves that for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle T_{T*\varphi}, \psi \rangle| \leq cN_{\infty,m}(\psi)\|\varphi\|_{p,\nu}. \quad (4.69)$$

On the other hand, we have for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle T_{T*\varphi}, \psi \rangle = \langle T * T_\psi, \check{\varphi} \rangle, \quad (4.70)$$

where for all $\varphi \in S_*(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle T * T_\psi, \varphi \rangle = \langle T, T_\psi * \varphi \rangle = \langle T, \psi * \varphi \rangle. \quad (4.71)$$

Relations (4.69) and (4.70) lead to for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle T * T_\psi, \varphi \rangle| \leq cN_{\infty,m}(\psi)\|\varphi\|_{p,\nu}. \quad (4.72)$$

This last inequality shows that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem, there exists $g \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_g. \quad (4.73)$$

Furthermore, from Proposition 4.4, there exist $s \in \mathbb{N}$, $\psi_s \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$, and $\varphi_s \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ satisfying

$$\delta = (I + L)^s T_{\psi_s} + T_{\varphi_s}, \quad (4.74)$$

then

$$T = (I + L)^s (T * T_{\psi_s}) + T * T_{\varphi_s} = (I + L)^s (T * T_{\psi_s}) + T_{T*\varphi_s}. \quad (4.75)$$

We complete the proof by using the hypothesis, relation (4.73), and Theorem 4.1. $\square$

In the following, we will give a characterization of the bounded sets in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$.

THEOREM 4.8. Let $p \in [1, \infty[$ and let B' be a subset of $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$. The following assertions are equivalent:

- (i) B' is weakly bounded in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$,
- (ii) there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$, it is possible to find $f_{0,T}, \dots, f_{m,T} \subset L^{p'}(d\nu)$ satisfying

$$T = \sum_{k=0}^m L^k T_{f_k} \quad \text{with} \quad \max_{0 \leq k \leq m} \|f_k\|_{p', \nu} \leq c, \quad (4.76)$$

- (iii) for every $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, the set $\{T * \varphi\}_{T \in B'}$ is bounded in $L^{p'}(d\nu)$.

Proof. (1) Suppose that B' is weakly* bounded in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$, then from [14] B' is equicontinuous. There exist $c > 0$ and $m \in \mathbb{N}$ such that

$$\forall T \in B', \forall f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \quad |\langle T, f \rangle| \leq c \gamma_{m,p}(f). \quad (4.77)$$

As in the proof of Theorem 4.6, we consider the mappings

$$\begin{aligned} A : \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) &\longrightarrow (L^p(d\nu))^{m+1}, \\ f &\longmapsto (f, g_1, \dots, g_m) \end{aligned} \quad (4.78)$$

with

$$L^k T_f = T_{g_k}, \quad 0 \leq k \leq m, \quad (4.79)$$

and for all $T \in B'$,

$$\begin{aligned} L_T : A(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)) &\longrightarrow \mathbb{C}, \\ \langle L_T, A(f) \rangle &= \langle T, f \rangle. \end{aligned} \quad (4.80)$$

Then, relation (4.77) implies that for all $\varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$,

$$|L_T(A\varphi)| \leq c \|A\varphi\|_{(L^p(d\nu))^{m+1}}. \quad (4.81)$$

Using Hahn-Banach's theorem and Riez's theorem, we deduce that L_T can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by L_T , and that there exists $(f_{T,k})_{0 \leq k \leq m} \subset L^{p'}(d\nu)$ verifying for all $\psi = (\psi_0, \dots, \psi_m) \in (L^p(d\nu))^{m+1}$,

$$\langle L_T, \psi \rangle = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_{T,k}(r, x) \psi_k(r, x) d\nu(r, x) \quad (4.82)$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \|f_{T,k}\|_{p', \nu} \leq c. \quad (4.83)$$

In particular, if $\psi = A(f)$, $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle L_T, A(f) \rangle = \langle T, f \rangle = \sum_{k=0}^m \langle L^k T_{f_{T,k}}, f \rangle. \quad (4.84)$$

This proves that (i) $\Rightarrow$ (ii).

(2) Suppose that there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$ we can find $f_{0,T}, \dots, f_{m,T} \subset L^{p'}(d\nu)$ satisfying

$$T = \sum_{k=0}^m L^k T_{f_{T,k}}, \quad \max_{0 \leq k \leq m} \|f_{T,k}\|_{p',\nu} \leq c. \quad (4.85)$$

Then for all $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, for all $T \in B'$,

$$\langle T, f \rangle = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_{T,k}(r, x) g_k(r, x) d\nu(r, x), \quad (4.86)$$

consequently, for all $T \in B'$, for all $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$,

$$|\langle T, f \rangle| \leq (m+1)c\gamma_{m,p}(f), \quad (4.87)$$

which means that the set B' is weakly* bounded in $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ and proves that (ii) $\Rightarrow$ (i).

(3) Suppose that (ii) holds. Let $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, then from Theorem 4.7 we know that for all $T \in B'$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$. But

$$T * \varphi = \sum_{k=0}^m T_{f_k} * L^k \varphi, \quad (4.88)$$

consequently, for all $T \in B'$,

$$\|T * \varphi\|_{p',\nu} \leq (m+1)c\gamma_{m,p}(\varphi). \quad (4.89)$$

This shows that the set $\{T * \varphi\}_{T \in B'}$ is bounded in $L^{p'}(d\nu)$ and therefore (ii) involves (iii).

(4) Suppose that (iii) holds. Let $T \in B'$; for all $\varphi, \psi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, we have

$$|\langle T_{T*\varphi}, \psi \rangle| = |\langle T_{T*\psi}, \varphi \rangle| \leq \|T * \psi\|_{p',\nu} \|\varphi\|_{p,\nu}, \quad (4.90)$$

from which we deduce that the set

$$B' = \{T_{T*\varphi}, T \in B', \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n); \|\varphi\|_{p,\nu} \leq 1\} \quad (4.91)$$

is bounded in $\mathcal{D}'_*(\mathbb{R} \times \mathbb{R}^n)$.

Now, using Theorem 4.6, it follows that for all $a > 0$, there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \leq 1$, and $T \in B'$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}'_a(\mathbb{R} \times \mathbb{R}^n)$ and the family of these extensions is equicontinuous,

which means that there exists $c > 0$ satisfying for all $T \in B'$, for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$; for all $\psi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$, (4.69) holds. On the other hand, for every $T \in B'$, we have for all $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$, (4.70) holds. From relations (4.69) and (4.70), we deduce that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem, there exist $g_{T,\psi} \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_{g_{T,\psi}}. \quad (4.92)$$

However, relations (4.69) and (4.70) involve that for all $T \in B'$,

$$\|g_{T,\psi}\|_{p',\nu} \leq cN_{\infty,m}(\psi). \quad (4.93)$$

Again by Proposition 4.4, it follows that there exist $s \in \mathbb{N}$, $\psi_s \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$, and $\varphi_s \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ verifying for all $T \in B'$,

$$T = T * \delta = (I + L)^s (T * T_{\psi_s}) + T_{T * \varphi_s}, \quad (4.94)$$

and by relation (4.92) we get

$$T = (I + L)^s T_{g_{T,s}} + T_{T * \varphi_s}. \quad (4.95)$$

Thus, from the hypothesis we obtain,

$$\forall T \in B', \quad \|T * \varphi_s\|_{p',\nu} \leq c_s, \quad (4.96)$$

and using relation (4.93), we have

$$\forall T \in B', \quad \|g_{T,s}\|_{p',\nu} \leq cN_{\infty,m}(\varphi_s). \quad (4.97)$$

This completes the proof. □

5. Convolution product on the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n)$

In this section, we define and study a convolution product on the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p < \infty$, where $M_r(\mathbb{R} \times \mathbb{R}^n)$ is the closure of the space $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $\mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n)$.

PROPOSITION 5.1. *Let $p \in [1, \infty[$. For every $(r, x) \in [0, \infty[\times \mathbb{R}^n$, the operator $\tau_{(r,x)}$ given by Definition 2.2(i), is a continuous mapping from $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ into itself.*

Proof. Let $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ and $g_k \in L^p(d\nu)$ such that

$$T_{g_k} = L^k T_f, \quad k \in \mathbb{N}. \quad (5.1)$$

Then for all $\varphi \in S_*(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle L^k T_{\tau(r,x)f}, \varphi \rangle = \langle T_{\tau(r,-x)} \check{g}_k, \varphi \rangle. \quad (5.2)$$

Since the operator $\tau_{(r,x)}$ is continuous from $L^p(d\nu)$ into itself, we deduce that for all $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ and $(r,x) \in [0, \infty[\times \mathbb{R}^n$, the function $\tau_{(r,x)}f$ belongs to the space $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$. Moreover,

$$\gamma_{m,p}(\tau_{(r,x)}f) = \max_{0 \leq k \leq m} \|\tau_{(r,-x)} \check{g}_k\|_{p,\nu} \leq \max_{0 \leq k \leq m} \|g_k\|_{p,\nu} = \gamma_{m,p}(f), \quad (5.3)$$

which shows that the operator $\tau_{(r,x)}$ is continuous from $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ into itself. $\square$

Definition 5.2. A convolution product of $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ and $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$ is defined by for all $(r,x) \in [0, \infty[\times \mathbb{R}^n$,

$$T * f(r,x) = \langle T, \tau_{(r,-x)} \check{f} \rangle. \quad (5.4)$$

Let $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$; $T = \sum_{k=0}^m L^k T_{f_k}$ with $\{f_k\}_{0 \leq k \leq m} \subset L^{p'}(d\nu)$ and $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p$, then for all $k \in \mathbb{N}$, there exists $\phi_k \in L^r(d\nu)$ such that $T_{\phi_k} = L^k T_\phi$. From inequality (2.24), it follows that for $0 \leq k \leq m$, the function $f_k * \phi_k$ belongs to the space $L^q(d\nu)$ with $1/q = 1/r + 1/p' - 1 = 1/r - 1/p$ and by using the density of $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $M_r(\mathbb{R} \times \mathbb{R}^n)$, we deduce that the expression $\sum_{k=0}^m f_k * \phi_k$ is independent of the sequence $\{f_k\}_{0 \leq k \leq m}$. Then, we put

$$T * \phi = \sum_{k=0}^m f_k * \phi_k. \quad (5.5)$$

This allows us to say that

$$\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) * M_r(\mathbb{R} \times \mathbb{R}^n) \subset L^q(d\nu). \quad (5.6)$$

LEMMA 5.3. Let $1 \leq r \leq p < \infty$, $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$, and $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$. Then, for all $k \in \mathbb{N}$

$$L^k T_{T*\phi} = T_{T*\phi_k} \quad (5.7)$$

with $T_{\phi_k} = L^k T_\phi$.

Proof. If $\phi \in S_*(\mathbb{R} \times \mathbb{R}^n)$, then the function $T * \phi$ is infinitely differentiable and we have

$$L^k (T_{T*\phi}) = T_{L^k(T*\phi)} = T_{T*L^k\phi}. \quad (5.8)$$

Therefore, the result follows from the density of $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $M_r(\mathbb{R} \times \mathbb{R}^n)$. $\square$

PROPOSITION 5.4. Let $1 \leq r \leq p < \infty$ and $q \in [1, \infty]$ such that

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}. \quad (5.9)$$

Then for every $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$, the mapping

$$\phi \longrightarrow T * \phi \quad (5.10)$$

is continuous from $M_r(\mathbb{R} \times \mathbb{R}^n)$ into $\mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n)$.

Proof. Let $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$; $T = \sum_{k=0}^m L^k T_{f_k}$ with $\{f_k\}_{0 \leq k \leq m} \subset L^{p'}(d\nu)$, then for $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p$, and by using relation (5.5), we get $T * \phi = \sum_{k=0}^m f_k * \phi_k$, where $\phi_k \in L^r(d\nu)$ and

$$T_{\phi_k} = L^k T_{\phi}. \quad (5.11)$$

From Lemma 5.3, we have for all $s \in \mathbb{N}$, for all $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$,

$$L^s T_{T * \phi} = T_{T * \phi_s}. \quad (5.12)$$

Using relation (5.6), we deduce that the function $T * \phi$ belongs to the space $\mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n)$.

On the other hand, from relation (5.12), we obtain

$$\gamma_{l,q}(T * \phi) = \max_{0 \leq s \leq l} \|T * \phi_s\|_{q,\nu}. \quad (5.13)$$

According to relation (5.12), we have

$$T * \phi_s = \sum_{k=0}^m f_k * \phi_{k+s}, \quad (5.14)$$

consequently,

$$\|T * \phi_s\|_{q,\nu} \leq \sum_{k=0}^m \|f_k\|_{p',\nu} \|\phi_{k+s}\|_{r,\nu} \leq \left(\sum_{k=0}^m \|f_k\|_{p',\nu} \right) \gamma_{m+l,r}(\phi). \quad (5.15)$$

Hence

$$\gamma_{l,q}(T * \phi) \leq \left(\sum_{k=0}^m \|f_k\|_{p',\nu} \right) \gamma_{m+l,r}(\phi), \quad (5.16)$$

which proves the result. $\square$

Definition 5.5. Let $1 \leq p, q, r < \infty$ such that (5.9) holds. A convolution product of $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ and $S \in \mathcal{M}'_q(\mathbb{R} \times \mathbb{R}^n)$ is defined by for all $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle S * T, \phi \rangle = \langle S, T * \phi \rangle. \quad (5.17)$$

From this definition and Proposition 5.4 we deduce the following result.

PROPOSITION 5.6. Let $1 \leq p, q, r < f^\infty$ such that (5.9) holds. Then, for all $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ and $S \in \mathcal{M}'_q(\mathbb{R} \times \mathbb{R}^n)$, the functional $S * T$ is continuous on $M_r(\mathbb{R} \times \mathbb{R}^n)$.

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M. Dziri: Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis, 1060 Tunis, Tunisia

E-mail address: moncef.dziri@iscae.rnu.tn

M. Jelassi: Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis, 1060 Tunis, Tunisia

E-mail address: mouradjelassi@fst.rnu.tn

L. T. Rachdi: Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis, 1060 Tunis, Tunisia

E-mail address: lakhdartannech.rachdi@fst.rnu.tn