A NOTE ON ASYMPTOTIC STABILITY CONDITIONS FOR DELAY DIFFERENCE EQUATIONS

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We obtain necessary and sufficient conditions for the asymptotic stability of the linear delay difference equation $x_{n+1} + p\sum_{j=1}^{N} x_{n-k+(j-1)l} = 0$, where $n = 0,1,2,\ldots$, p is a real number, and k, l, and N are positive integers such that $k > (N-1)l$.

1. Introduction

In [4], the asymptotic stability condition of the linear delay difference equation

$$x_{n+1} - x_n + p\sum_{j=1}^{N} x_{n-k+(j-1)l} = 0, \tag{1.1}$$

where $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, p is a real number, and k, l, and N are positive integers with $k > (N-1)l$ is given as follows.

Theorem 1.1. *Let k, l, and N be positive integers with $k > (N-1)l$. Then the zero solution of (1.1) is asymptotically stable if and only if*

$$0 < p < \frac{2\sin(\pi/2M)\sin(l\pi/2M)}{\sin(Nl\pi/2M)}, \tag{1.2}$$

where $M = 2k + 1 - (N-1)l$.

Theorem 1.1 generalizes asymptotic stability conditions given in [1, page 87], [2, 3, 5], and [6, page 65]. In this paper, we are interested in the situation when (1.1) does not depend on x_n, namely we are interested in the asymptotic stability of the linear delay difference equation of the form

$$x_{n+1} + p\sum_{j=1}^{N} x_{n-k+(j-1)l} = 0, \tag{1.3}$$

where $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, p is a real number, and k, l, and N are positive integers with $k \geq (N-1)l$. Our main theorem is the following.

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THEOREM 1.2. *Let k, l, and N be positive integers with $k \geq (N-1)l$. Then the zero solution of (1.3) is asymptotically stable if and only if*

$$-\frac{1}{N} < p < p_{\min}, \tag{1.4}$$

where $p_{\min}$ is the smallest positive real value of p for which the characteristic equation of (1.3) has a root on the unit circle.

2. Proof of theorem

The characteristic equation of (1.3) is given by

$$F(z) = z^{k+1} + p\left(z^{(N-1)l} + \cdots + z^l + 1\right) = 0. \tag{2.1}$$

For $p = 0$, $F(z)$ has exactly one root at 0 of multiplicity $k+1$. We first consider the location of the roots of (2.1) as p varies. Throughout the paper, we denote the unit circle by C and let $M = 2k + 2 - (N-1)l$.

PROPOSITION 2.1. *Let z be a root of (2.1) which lies on C. Then the roots z and p are of the form*

$$z = e^{w_m i}, \tag{2.2}$$

$$p = (-1)^{m+1}\frac{\sin(lw_m/2)}{\sin(Nlw_m/2)} \equiv p_m \tag{2.3}$$

for some $m = 0, 1, \ldots, M-1$, where $w_m = (2m/M)\pi$. Conversely, if p is given by (2.3), then $z = e^{w_m i}$ is a root of (2.1).

Proof. Note that $z = 1$ is a root of (2.1) if and only if $p = -1/N$, which agrees with (2.2) and (2.3) for $w_m = 0$. We now consider the roots of (2.1) which lie on C except the root $z = 1$. Suppose that the value z satisfies $z^{Nl} = 1$ and $z^l \neq 1$. Then $z^{Nl} - 1 = (z^l - 1)(z^{(N-1)l} + \cdots + z^l + 1) = 0$ which gives $z^{(N-1)l} + \cdots + z^l + 1 = 0$, and hence z is not a root of (2.1). As a result, to determine the roots of (2.1) which lie on C, it suffices to consider only the value z such that $z^{Nl} \neq 1$ or $z^l = 1$. For these values of z, we may write (2.1) as

$$p = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}. \tag{2.4}$$

Since p is real, we have

$$p = -\frac{\overline{z}^{k+1}}{\overline{z}^{(N-1)l} + \cdots + \overline{z}^l + 1} = -\frac{z^{-k-1+(N-1)l}}{z^{(N-1)l} + \cdots + z^l + 1}, \tag{2.5}$$

where $\overline{z}$ denotes the conjugate of z. It follows from (2.4) and (2.5) that

$$z^{2k+2-(N-1)l} = 1 \tag{2.6}$$

which implies that (2.2) is valid for $m = 0, 1, \ldots, M-1$ except for those integers m such that $e^{Nlw_m i} = 1$ and $e^{lw_m i} \neq 1$. We now show that p is of the form stated in (2.3). There are two cases to be considered as follows.

Case 1. z is of the form $e^{w_m i}$ for some $m = 1,2,\dots,M-1$ and $z^{Nl} \neq 1$.

From (2.4), we have

$$\begin{aligned}
p &= -\frac{z^{k+1}(z^l - 1)}{z^{Nl} - 1} = -\frac{e^{(k+1)w_m i}(e^{l w_m i} - 1)}{e^{N l w_m i} - 1} \\
&= -\frac{e^{(k+1-(N-1)(l/2))w_m i}(e^{l w_m i/2} - e^{-l w_m i/2})}{e^{N l w_m i/2} - e^{-N l w_m i/2}} \\
&= -e^{(k+1-(N-1)(l/2))w_m i}\frac{\sin(l w_m/2)}{\sin(N l w_m/2)} \\
&= -e^{m\pi i}\frac{\sin(l w_m/2)}{\sin(N l w_m/2)} = (-1)^{m+1}\frac{\sin(l w_m/2)}{\sin(N l w_m/2)} \equiv p_m.
\end{aligned} \tag{2.7}$$

Case 2. z is of the form $e^{w_m i}$ for some $m = 1,2,\dots,M-1$ and $z^l = 1$.

In this case, we have $l w_m = 2q\pi$ for some positive integer q. Then taking the limit of p_m as $l w_m \to 2q\pi$, we obtain

$$p = -\frac{(-1)^{m+q(N-1)}}{N}. \tag{2.8}$$

From these two cases, we conclude that p is of the form in (2.3) for $m = 1,2,\dots,M-1$ except for those m such that $e^{N l w_m i} = 1$ and $e^{l w_m i} \neq 1$.

Conversely, if p is given by (2.3), then it is obvious that $z = e^{w_m i}$ is a root of (2.1). This completes the proof of the proposition. □

From Proposition 2.1, we may consider p as a holomorphic function of z in a neighborhood of each z_m. In other words, in a neighborhood of each z_m, we may consider p as a holomorphic function of z given by

$$p(z) = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}. \tag{2.9}$$

Then we have

$$\frac{dp(z)}{dz} = -\frac{(k+1)z^k}{z^{(N-1)l} + \cdots + z^l + 1} + \frac{z^k\{(N-1)l z^{(N-1)l} + \cdots + l z^l\}}{(z^{(N-1)l} + \cdots + z^l + 1)^2}. \tag{2.10}$$

From this, we have the following lemma.

Lemma 2.2. *$dp/dz|_{z=e^{w_m i}} \neq 0$. In particular, the roots of (2.1) which lie on C are simple.*

Proof. Suppose on the contrary that $dp/dz|_{z=e^{w_m i}} = 0$. We divide (2.10) by $p(z)/z$ to obtain

$$k+1 - \frac{l\{(N-1)z^{(N-1)l} + \cdots + z^l\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0. \tag{2.11}$$

Substituting z by $1/\bar{z}$ in (2.10), we obtain

$$k+1 - \frac{l\{(N-1) + (N-2)z^l + \cdots + z^{(N-2)l}\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0. \tag{2.12}$$

By adding (2.11) and (2.12), we obtain

$$2k+2-(N-1)l=0 \tag{2.13}$$

which contradicts $k \geq (N-1)l$. This completes the proof. □

From Lemma 2.2, there exists a neighborhood of $z = e^{w_m i}$ such that the mapping $p(z)$ is one to one and the inverse of $p(z)$ exists locally. Now, let z be expressed as $z = re^{i\theta}$. Then we have

$$\frac{dz}{dp} = \frac{z}{r}\left\{\frac{dr}{dp} + ir\frac{d\theta}{dp}\right\} \tag{2.14}$$

which implies that

$$\frac{dr}{dp} = \text{Re}\left\{\frac{r}{z}\frac{dz}{dp}\right\} \tag{2.15}$$

as p varies and remains real. The following result describes the behavior of the roots of (2.1) as p varies.

Proposition 2.3. *The moduli of the roots of (2.1) at $z = e^{w_m i}$ increase as $|p|$ increases.*

Proof. Let r be the modulus of z. Let $z = e^{w_m i}$ be a root of (2.1) on C. To prove this proposition, it suffices to show that

$$\frac{dr}{dp} \cdot p\bigg|_{z=e^{w_m i}} > 0. \tag{2.16}$$

There are two cases to be considered.
Case 1 ($z^{Nl} \neq 1$). In this case, we have

$$p(z) = -\frac{z^{k+1}(z^l-1)}{z^{Nl}-1} = -\frac{z^k f(z)}{z^{Nl}-1}, \tag{2.17}$$

where $f(z) = z(z^l - 1)$. Then

$$\frac{dp}{dz} = -\frac{z^{k-1}g(z)}{(z^{Nl}-1)^2}, \tag{2.18}$$

where $g(z) = (kf(z) + zf'(z))(z^{Nl}-1) - Nlz^{Nl}f(z)$. Letting $w(z) = -(z^{Nl}-1)^2/(z^k g(z))$, we obtain

$$\frac{dr}{dp} = \text{Re}\left(\frac{r}{z}\frac{dz}{dp}\right) = r\text{Re}(w). \tag{2.19}$$

We now compute $\text{Re}(w)$. We note that

$$f(\bar{z}) = -\frac{f(z)}{z^{l+2}}, \qquad f'(\bar{z}) = \frac{h(z)}{z^l}, \tag{2.20}$$

where $h(z) = l+1-z^l$. From the above equalities and as $z^M = 1$, we have

$$\begin{aligned} \bar{z}^k g(\bar{z}) &= \frac{1}{z^k}\left\{\left(kf(\bar{z})+\frac{1}{z}f'(\bar{z})\right)\left(\frac{1}{z^{Nl}}-1\right)-\frac{Nl}{z^{Nl}}f(\bar{z})\right\} \\ &= \frac{(-kf(z)+zh(z))(1-z^{Nl})+Nlf(z)}{z^{Nl+l+2+k}} \\ &= \frac{(-kf(z)+zh(z))(1-z^{Nl})+Nlf(z)}{z^{2Nl-k}}. \end{aligned} \tag{2.21}$$

It follows that

$$\begin{aligned} \operatorname{Re}(w) &= \frac{w+\bar{w}}{2} \\ &= -\frac{1}{2}\left\{\frac{(z^{Nl}-1)^2}{z^k g(z)}+\frac{(\bar{z}^{Nl}-1)^2}{\bar{z}^k g(\bar{z})}\right\} \\ &= -\frac{1}{2}\left\{\frac{\bar{z}^k g(\bar{z})(z^{Nl}-1)^2+z^k g(z)(\bar{z}^{Nl}-1)^2}{|g(z)|^2}\right\} \\ &= -\frac{1}{2|g(z)|^2}\left\{\frac{(-kf(z)+zh(z))(1-z^{Nl})+Nlf(z)}{z^{2Nl-k}}\cdot (z^{Nl-1})^2\right. \\ &\qquad \left. +z^k((kf(z)+zf'(z))(z^{Nl}-1)-Nlz^{Nl}f(z))\left(\frac{1}{z^{Nl}}-1\right)^2\right\} \\ &= -\frac{(z^{Nl}-1)^2 z^k}{2z^{2Nl}|g(z)|^2}\left\{(kf(z)-zh(z))(z^{Nl}-1)+Nlf(z)\right. \\ &\qquad \left. +((kf(z)+zf'(z))(z^{Nl}-1))-Nlz^{Nl}f(z)\right\} \\ &= -\frac{(z^{Nl}-1)^3 z^k}{2z^{2Nl}|g(z)|^2}\{2kf(z)+z(f'(z)-h(z))-Nlf(z)\}. \end{aligned} \tag{2.22}$$

Since

$$2kf(z)+z(f'(z)-h(z))-Nlf(z) = Mf(z), \tag{2.23}$$

we obtain

$$\operatorname{Re}(w) = \frac{(z^{Nl}-1)^4 M}{2z^{2Nl}|g(z)|^2}\cdot\frac{-z^k f(z)}{z^{Nl}-1} = \frac{(z^{Nl}-1)^4 Mp}{2z^{2Nl}|g(z)|^2}. \tag{2.24}$$

The value of $\operatorname{Re}(w)$ at $z = e^{w_m i}$ is

$$\operatorname{Re}(w) = \frac{(z^{Nl}-1)^4}{z^{2Nl}}\cdot\frac{Mp}{2|g(z)|^2} = (2\cos Nlw_m-2)^2\cdot\frac{Mp}{2|g(z)|^2} > 0. \tag{2.25}$$

Therefore,

$$\frac{dr}{dp} = \frac{2r(\cos Nlw_m - 1)^2 Mp}{|g(z)|^2} \tag{2.26}$$

and it follows that (2.16) holds at $z = e^{w_m i}$.

Case 2 ($z^l = 1$). With an argument similar to Case 1, we obtain

$$\frac{dr}{dp} = \frac{2rN^2 Mp}{|(M+1)z - M + 1|^2} \tag{2.27}$$

which implies that (2.16) is valid for $z = e^{w_m i}$.

This completes the proof. □

We now determine the minimum of the absolute values of p_m given by (2.3). We have the following result.

Proposition 2.4. $|p_0| = \min\{|p_m| : m = 0, 1, \dots, M-1\}$.

To prove Proposition 2.4, we need the following lemma, which was proved in [4].

Lemma 2.5. *Let N be a positive integer, then*

$$\left|\frac{\sin Nt}{\sin t}\right| \le N \tag{2.28}$$

holds for all $t \in \mathbb{R}$.

Proof of Proposition 2.4. From (2.3), $p_m = (-1)^{m+1}(\sin(lw_m/2)/\sin(Nlw_m/2))$. For $m = 0$, it follows from L'Hospital's rule that $p_0 = -1/N$. For $m = 1, 2, \dots, M-1$, we have

$$|p_m| = \left|(-1)^{m+1}\frac{\sin(lw_m/2)}{\sin(Nlw_m/2)}\right| \ge \frac{1}{N} \tag{2.29}$$

by Lemma 2.5. This completes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Note that $F(1) = 1 + Np \le 0$ if and only if $p \le -1/N$. Since $\lim_{z\to+\infty} F(z) = +\infty$, it follows that (2.1) has a positive root α such that $\alpha > 1$ when $p \le -1/N$. We claim that if $|p|$ is sufficiently small, then all the roots of (2.1) are inside the unit disk. To this end, we note that when $p = 0$, (2.1) has exactly one root at 0 of multiplicity $k+1$. By the continuity of the roots with respect to p, this implies that our claim is true. By Proposition 2.4, $p_0 = -1/N$ and $|p_m| \ge 1/N$ which implies that $|p_0| = 1/N$ is the smallest positive value of p such that a root of (2.1) intersects the unit circle as $|p|$ increases. Moreover, Proposition 2.3 implies that if $p > p_{\min}$, then there exists a root α of (2.1) such that $|\alpha| \ge 1$, where $p_{\min}$ is the smallest positive real value of p for which (2.1) has a root on C. We conclude that all the roots of (2.1) are inside the unit disk if and only if $-1/N < p < p_{\min}$. In other words, the zero solution of (1.3) is asymptotically stable if and only if condition (1.4) holds. This completes the proof. □

3. Examples

Example 3.1. In (1.3), Let l and k be even positive integers, then we have

$$F(-1) = -1 + pN. \tag{3.1}$$

Thus if $p = 1/N$, then $F(-1) = 0$ and we conclude that (1.3) is asymptotically stable if and only if $-1/N < p < 1/N$.

Example 3.2. In (1.3), let $N = 3$, $l = 3$, and $k = 6$. Then $M = 8$ and we obtain $p_0 = -1/3$, $p_1 = \sin(3/8)\pi/\sin(9/8)\pi$, $p_2 = -\sin(3/4)\pi/\sin(9/4)\pi$, $p_3 = \sin(9/8)\pi/\sin(27/8)\pi$, $p_4 = -\sin(3/2)\pi/\sin(9/2)\pi$, $p_5 = \sin(15/8)\pi/\sin(45/8)\pi$, $p_6 = -\sin(9/4)\pi/\sin(27/4)\pi$, and $p_7 = \sin(21/8)\pi/\sin(63/8)\pi$. Thus, $p_3 = p_5 = \sin(\pi/8)/\sin(3\pi/8)$ is the smallest positive real value of p such that (2.1) has a root on C. We conclude that (1.3) is asymptotically stable if and only if $-1/3 < p < \sin(\pi/8)/\sin(3\pi/8)$.

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