A DEGREE CONDITION FOR THE EXISTENCE OF k-FACTORS WITH PRESCRIBED PROPERTIES

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Received 27 July 2004 and in revised form 26 December 2004

Let k be an integer such that $k \geq 3$, and let G be a 2-connected graph of order n with $n \geq 4k+1$, kn even, and minimum degree at least $k+1$. We prove that if the maximum degree of each pair of nonadjacent vertices is at least $n/2$, then G has a k-factor excluding any given edge. The result of Nishimura (1992) is improved.

1. Introduction and result

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we write $N_G(v)$ for the set of vertices of $V(G)$ adjacent to v, $N_G[v]$ for $N_G(v) \bigcup \{v\}$, and $d_G(v) = |N_G(v)|$ for the degree of v in G. If S and T are disjoint subsets of $V(G)$, then $e_G(S,T)$ denotes the number of edges that join S and T, and $G-S$ denotes the subgraph of G obtained from G by deleting the vertices in S together with the edges incident with them. A k-factor of G is a spanning subgraph F of G such that $d_F(x) = k$ for every $x \in V(F)$. If G and H are disjoint graphs, then the join and the union are denoted by $G+H$ and $G \bigcup H$, respectively. Other terminology and notation not defined here can be found in [1].

The following theorems of k-factors in terms of degree conditions are known.

Theorem 1.1 (Nishimura [4]). *Let k be an integer such that $k \geq 3$, and let G be a connected graph of order n with $n \geq 4k-3$, kn even, and minimum degree at least k. Suppose that $\max(d_G(u), d_G(v)) \geq n/2$ for each pair of nonadjacent vertices u, v of $V(G)$. Then G has a k-factor.*

Theorem 1.2 (Iida and Nishimura [3]). *Let k be a positive integer, and let G be a graph of order n with $n \geq 4k-5$, kn even, and minimum degree at least k. If the degree sum of each pair of nonadjacent vertices is at least n, then G has a k-factor.*

Theorem 1.3 (Egawa and Enomoto [2]). *Let k be a positive integer, and let G be a graph of order n with $n \geq 4k-5$, kn even, and minimum degree at least $n/2$. Then G has a k-factor.*

The main result of this paper is the following theorem.

Theorem 1.4. *Let k be an integer such that $k \geq 3$, and let G be a 2-connected graph of order n with $n \geq 4k+1$, kn even, and minimum degree at least $k+1$. Suppose that*

International Journal of Mathematics and Mathematical Sciences 2005:6 (2005) 863–873
DOI: 10.1155/IJMMS.2005.863

$\max(d_G(u), d_G(v)) \geq n/2$ for each pair of nonadjacent vertices u, v of $V(G)$. Then for any $e \in E(G)$, $G - e$ has a k-factor.

The assumptions in Theorem 1.4 cannot be weakened any further. We discuss them in the last section.

2. Proof of Theorem 1.4

In order to prove Theorem 1.4, the following definitions are needed.

Let G be a graph, and $S, T \subseteq V(G)$ with $S \bigcap T = \varnothing$. For an integer $k \geq 1$, a component C of $G - (S \bigcup T)$ is called a k-odd component or k-even component according to whether $k|V(C)| + e_G(V(C), T)$ is odd or even. Assume that e is a cut edge of $G - (S \bigcup T)$ and $C(e)$ is the component of $G - (S \bigcup T)$ which contains e. We say that e is a k-odd cut edge or k-even cut edge according to parity, that is, whether both components of $C(e) - e$ are k-odd components or k-even components of $(G - e) - (S \bigcup T)$. (Note that $C(e)$ must be a k-even component of $G - (S \bigcup T)$ in both cases.) We write

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - h_G(S, T), \tag{2.1}$$

where $h_G(S, T)$ is the number of k-odd components of $G - (S \bigcup T)$.

Lemma 2.1 (Tutte [5]). *Let G be a graph and k a positive integer. For all disjoint subsets S and T of $V(G)$, G has a k-factor if and only if*

(i) $\delta_G(S, T) \geq 0$,

(ii) $\delta_G(S, T) \equiv kn \pmod 2$.

Lemma 2.2. *A graph G has a k-factor excluding any given edge if and only if $\delta_G(S, T) \geq \varepsilon(S, T)$ for all disjoint subsets S and T of $V(G)$, where $\varepsilon(S, T) = 2$ if $G[T]$ has an edge, or $G - (S \bigcup T)$ has a k-odd cut edge, or $G - (S \bigcup T)$ has a k-even component C such that $e_G(V(C), T) \geq 1$; otherwise, $\varepsilon(S, T) = 0$.*

Proof. A graph G has a k-factor excluding any given edge if and only if $G - e$ has a k-factor for every $e \in E(G)$. By Lemma 2.1, $G - e$ has a k-factor if and only if $\delta_{G-e}(S, T) \geq 0$ for all disjoint subsets S and T of $V(G)$. So, a graph G has a k-factor excluding any given edge if and only if for all disjoint subsets S and T of $V(G)$,

$$\min_{e \in E(G)} \delta_{G-e}(S, T) \geq 0. \tag{2.2}$$

Note that $\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - h_G(S, T)$ and $\delta_{G-e}(S, T) = k|S| + \sum_{x \in T} d_{G-e-S}(x) - k|T| - h_{G-e}(S, T)$. By the definition of $\varepsilon(S, T)$, we know that

$$\begin{aligned} \varepsilon(S, T) &= \max_{e \in E(G)} \left[\left(\sum_{x \in T} d_{G-S}(x) - \sum_{x \in T} d_{G-e-S}(x) \right) + \left(h_{G-e}(S, T) - h_G(S, T) \right) \right] \\ &= \max_{e \in E(G)} \left(\delta_G(S, T) - \delta_{G-e}(S, T) \right) \\ &= \delta_G(S, T) - \min_{e \in E(G)} \delta_{G-e}(S, T). \end{aligned} \tag{2.3}$$

So,

$$\min_{e\in E(G)} \delta_{G-e}(S,T) = \delta_G(S,T) - \varepsilon(S,T), \tag{2.4}$$

which completes the proof. □

Lemma 2.3. *Let G be a graph G and $k \geq 1$. Assume that there exists a real number θ and disjoint subsets S and T of $V(G)$ satisfying*

(i) $\delta_G(S,T) < \theta$,

(ii) $|S\bigcup T|$ *is as large as possible.*

Then $d_{G-S}(u) \geq k+1$ and $e_G(u,T) \leq k-1$ for all $u \in V(G)-(S\bigcup T)$. Moreover, the order of each component of $G-(S\bigcup T)$ is at least 3.

Proof. If there is $u^* \in V(G)-(S\bigcup T)$ such that $d_{G-S}(u^*) \leq k$. Set $S^* = S$, $T^* = T\bigcup\{u^*\}$, we have

$$\begin{aligned} \delta_G(S^*,T^*) &= k\,|\,S^*\,| + \sum_{x\in T^*} d_{G-S}(x) - k|T^*| - h_G(S^*,T^*) \\ &= k|S| + \sum_{x\in T} d_{G-S}(x) + d_{G-S}(u^*) - k|T| - k - h_G(S,T^*) \\ &\leq k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - \big(h_G(S,T)-1\big) \leq \delta_G(S,T)+1. \end{aligned} \tag{2.5}$$

Therefore, $\delta_G(S^*,T^*) \leq \delta_G(S,T)$ by Lemma 2.1(ii), which contradicts the maximum of $|S\bigcup T|$.

Similarly, we can prove that $e_G(u,T) \leq k-1$ for each $u \in V(G)-(S\bigcup T)$. □

Lemma 2.4 (see [4]). *Let m, n, s, t, and ω_0 be nonnegative integers. Suppose that $m \geq 3$, $\omega_0 \geq 4$, and $m(\omega_0-1) \leq n-s-t-3$. Then it holds that*

$$m-1+s+t \leq \frac{1}{3}\left[n+2(s+t+1-\omega_0)\right]. \tag{2.6}$$

Proof of Theorem 1.4. If G contains a complete bipartite graph $K_{n/2,n/2}$ as a subgraph when n is even, then Theorem 1.4 holds by [1, Theorems 8.9 and 8.12]. So we may suppose that G does not contain a complete bipartite graph as a subgraph when n is even.

Suppose that there exists an edge e such that $G-e$ has no k-factor. By Lemma 2.2, there exists $S_0, T_0 \subseteq V(G)$ with $S_0\bigcap T_0 = \varnothing$ such that $\delta_G(S_0,T_0) < \varepsilon(S_0,T_0)$. Clearly $S_0\bigcup T_0 \neq \varnothing$. Otherwise, $\delta_G(\varnothing,\varnothing) < \varepsilon(\varnothing,\varnothing) = 0$ implies $\delta_G(\varnothing,\varnothing) \leq -2$ by Lemma 2.1(ii) which contradicts the fact that G is 2-connected. Set $\theta = \varepsilon(S_0,T_0)$; obviously, $\theta = 2$. We choose disjoint subsets S and T of $V(G)$ such that S and T satisfy the condition of Lemma 2.3. It is easy to check that $S\bigcup T \neq \varnothing$.

By Theorem 1.1 and Lemma 2.1, we have $\delta_G(S,T) = 0$. Therefore,

$$\omega \geq k|S| + \sum_{x\in T} d_{G-S}(x) - k|T|, \tag{2.7}$$

where ω denotes the number of components in $U := G\setminus(S\bigcup T)$.

If $U \neq \emptyset$, let $C_1, C_2, \ldots, C_\omega$ be the components of U, labelled in such a way that their orders $m_1, m_2, \ldots, m_\omega$ are nondecreasing. By Lemma 2.3, we have $m_j \geq 3$ $(1 \leq j \leq \omega)$.

Let $s = |S|$, $t = |T|$. Note that if $U \neq \emptyset$, then

$$|U| = n - s - t \geq 3\omega, \tag{2.8}$$

and

$$d_G(u) \leq m_j - 1 + s + t \tag{2.9}$$

for every $u \in C_j$ $(1 \leq j \leq \omega)$. In particular, we note that when $\omega \geq 2$,

$$m_1 \leq \frac{n-s-t}{\omega}, \qquad m_2 \leq \frac{n-s-t-3}{\omega-1}. \tag{2.10}$$

If $T \neq \emptyset$, we define

$$h_1 := \min\{d_{G-S}(v) \mid v \in T\} \tag{2.11}$$

and let $x_1 \in T$ be a vertex satisfying $d_{G-S}(x_1) = h_1$ for which $|N_T[x_1]|$ is as small as possible. Further, if $T \setminus N_T[x_1] \neq \emptyset$, we define

$$h_2 := \min\{d_{G-S}(v) \mid v \in T \setminus N_T[x_1]\} \tag{2.12}$$

and let $x_2 \in T \setminus N_T[x_1]$ be a vertex satisfying $d_{G-S}(x_2) = h_2$. Obviously, we have that

$$h_1 \leq h_2, \tag{2.13}$$

$$d_G(x_i) \leq s + h_i \quad (i = 1, 2). \tag{2.14}$$

By (2.7), we have that

$$\omega \geq ks + (h_1 - k)|N_T[x_1]| + (h_2 - k)|T \setminus N_T[x_1]|. \tag{2.15}$$

We need to find a pair of nonadjacent vertices u, v in G such that

$$\max(d_G(u), d_G(v)) < \frac{n}{2}. \tag{2.16}$$

In fact, it suffices to prove that at least one of the following three statements holds in each case.

(A) $\omega \geq 2$ and $m_2 - 1 + s + t < n/2$. (Then (2.16) holds for any $u \in V(C_1)$, $v \in V(C_2)$, by (2.9) and $m_1 \leq m_2$.)

(B) $T \neq \emptyset$, $\omega > 0$, $d_G(x_1) < n/2$, there exists a vertex $u \in V(U) \setminus N(x_1)$ such that $d_G(u) < n/2$. (Then (2.16) holds with $u = u$ and $v = x_1$.)

(C) $T \neq \emptyset$, $T \setminus N_T[x_1] \neq \emptyset$, and $s + h_2 < n/2$. (Then (2.16) holds with $u = x_1$ and $v = x_2$, by (2.14) and $h_1 \leq h_2$.)

Case 1 ($T = \emptyset$). Then $s \geq 1$ since $S \bigcup T \neq \emptyset$. We have $\omega \geq ks \geq 3s \geq 3$ by (2.7). So $s \geq 2$ and $\omega \geq 3s > 4$. Otherwise, when $s = 1$, it holds that $\omega \geq 3$, which contradicts the fact that G is 2-connected. By (2.7) and (2.8), we get $s \leq n/(3k+1)$. This inequality, together with (2.10), gives

$$\begin{aligned} m_2 - 1 + s + t &\leq \frac{n-s-3}{\omega - 1} - 1 + \frac{n}{3k+1} \\ &\leq \frac{n-5}{2k-1} - 1 + \frac{n}{3k+1} < \frac{n}{2}. \end{aligned} \tag{2.17}$$

This shows (A) in this case.

Therefore we may assume $T \neq \emptyset$.

Case 2 ($T \neq \emptyset$).

Case 2.1 ($h_1 \geq k+2$). Let

$$\omega_0 := ks + (h_1 - k)t. \tag{2.18}$$

When $s = 0$ and $t = 1$, we have $\omega \geq \omega_0 \geq 2$, which contradicts that G is 2-connected. Suppose that $s \neq 0$ or $t \geq 2$, then $\omega_0 \geq 4$, and so, by Lemma 2.4, with $m = m_2$,

$$\begin{aligned} m_2 - 1 + s + t &\leq \frac{1}{3}[n + 2(s + t + 1 - ks - h_1 t + kt)] \\ &= \frac{1}{3}[n - 2(k-1)s - 2(h_1 - k - 1) + 2] < \frac{n}{3}, \end{aligned} \tag{2.19}$$

which shows that (A) holds in this case.

Case 2.2 ($h_1 = k+1$). Let $\omega_0 := ks + (h_1 - k)t$, and suppose that $s \neq 0$ or $t \geq 4$. Then $\omega \geq \omega_0 \geq 4$, using the same arguments as in Case 2.1, we get that

$$m_2 - 1 + s + t < \frac{n+2}{3} < \frac{n}{2}. \tag{2.20}$$

This also shows (A) in this case. Thus we may consider the following three cases.

(i) $s = 0$ and $t = 1$.

Clearly, $\delta_G(S,T) = 0 = \epsilon(S,T)$. According to the choice of S and T, when $T \neq \emptyset$ and $|S \bigcup T| \geq 2$, we have $\delta_G(S,T) \geq 2$, which is a contradiction by Lemma 2.2.

(ii) $s = 0$ and $t = 3$.

Then by (2.7) we have $\omega \geq \omega_0 \geq 3$. By (2.14), we have

$$m_2 - 1 + s + t \leq \frac{n-s-t-3}{\omega - 1} - 1 + s + t \leq \frac{n-6}{2} - 1 + 3 < \frac{n}{2}, \tag{2.21}$$

which shows that (A) holds in this case.

(iii) $s = 0$ and $t = 2$.

Let $T = \{y_1, y_2\}$, and $d_G(y_i) = k+1$, for $i = 1,2$. Otherwise, $\omega \geq 3$ since $\delta(G) \geq k+1$. We prove in this case that (A) holds in a similar way to that in (ii). Since $k \geq 3$ and $n \geq 4k+1$, we have that $k - 1 < n/2$. So, we may suppose that y_1 and y_2 are adjacent, otherwise, (2.16) holds with $u = y_1$ and $v = y_2$. Since G is 2-connected and $\omega \geq \omega_0 = 2$, we may assume that y_1 is adjacent to some vertex of a component, say C_2, where $C_1, C_2, \ldots, C_\omega$ are the components of $G - (S \bigcup T)$. Thus we have the following claim.

Claim 1. y_1 can be adjacent to at most $k-1$ vertices of C_1.

Note that $m_1 \le (n-2)/2$. Suppose that $m_1 = (n-2)/2$. Since $(n-2)/2 \ge k+1$, there exists at least one vertex u_1 of $V(C_1)$ not adjacent to y_1, and $d_G(u_1) \le m_1 - 1 + 1 < n/2$. Thus (2.16) holds with $u = u_1$ and $v = y_1$. Thus we assume that $m_1 < (n-2)/2$. Clearly, $m_1 \ge k$ since $\delta(G) \ge k+1$. Note that $d_G(u) \le m_1 - 1 + 2 < n/2$ for every $u \in V(C_1)$. By Claim 1, there exists at least $u_2 \in V(C_1)$ not adjacent to y_1. Thus (2.16) holds with $u = u_2$ and $v = y_1$.

For the case where $0 \le h_1 \le k$, since $\min\{d_G(u) \mid u \in V(G)\} \ge k+1$ by the hypothesis of the theorem, it holds that

$$s \ge k - h_1 + 1. \tag{2.22}$$

We will prove that $d_G(x_1) < n/2$ in the case where $0 \le h_1 \le k$.

Case A ($h_1 = 0$). By (2.7), we have $0 \ge ks - kt - \omega$, $G[T]$ is an isolated set if the equality holds. By (2.8), we have $ks - kt - \omega \ge ks - k(n-s)$, and $n - s - t = \omega = 0$ when the equality holds. Thus,

$$0 \ge ks - kt - \omega \ge ks - k(n-s). \tag{2.23}$$

It follows from the inequality above that $G[T]$ is an isolated set and $n - s - t = \omega = 0$ if none of the inequalities in (2.23) is strict. Moreover, in this case, we have $s = t = n/2$. From (2.14) we get

$$d_G(x_1) \le h_1 + s = \frac{n}{2}. \tag{2.24}$$

If $d_G(x_1) = n/2$, it is easy to see that each vertex in T is adjacent to all vertices in S by the choice of x_1. Therefore, G contains $K_{n/2,n/2}$ as a subgraph, which is a contradiction. So $d_G(x_1) < n/2$.

If one of the inequalities in (2.23) is strict, we can get $s < n/2$ from (2.23), thus $d_G(x_1) \le s + h_1 < n/2$ by (2.14).

Case B ($h_1 = 1$). In this case, it follows from (2.7) and (2.8) that

$$0 \ge ks + (1-k)t - \omega \ge ks + (1-k)(n-s). \tag{2.25}$$

Thus by (2.14) we have that

$$d_G(x_1) \le h_1 + s \le 1 + \frac{(k-1)n}{2k-1} < \frac{n}{2}. \tag{2.26}$$

Case C ($2 \le h_1 \le k-1$). It follows from (2.7) And (2.8) that

$$0 \ge ks + (h_1 - k)t - \omega \ge ks + (h_1 - k)(n-s), \tag{2.27}$$

thus $s \le n - kn/(2k - h_1)$. Suppose that $d_G(x_1) \ge n/2$, by (2.14), we have that

$$\frac{n}{2} \le s + h_1 \le n - \frac{kn}{2k - h_1} + h_1. \tag{2.28}$$

So $n \le 4k - 2h_1 \le 4k - 4$, which contradicts the fact that $n \ge 4k+1$.

Case D ($h_1 = k$). Thus $s \geq 1$ by (2.22). From (2.7) we have that $\omega \geq ks$. Suppose that $d_G(x_1) \geq n/2$, by (2.14), we have that

$$ks \geq k + s - 1 \geq d_G(x_1) - 1 \geq \frac{n-2}{2}. \tag{2.29}$$

Thus, by (2.8), we have that

$$n - s - t \geq 3\omega \geq 3ks \geq \frac{3(n-2)}{2}, \tag{2.30}$$

which is a contradiction.

Case 3 ($0 \leq h_1 \leq k$ and $T = N_T[x_1]$). In this case $t \leq k$ unless $h_1 = k$. Thus it follows from (2.7) and (2.22) that

$$\omega \geq ks + (h_1 - k)t \geq k + (k - h_1)(k - t) \geq k \geq 3. \tag{2.31}$$

Suppose that $V(C_j) \subset N_G(x_1)$ for some j $(1 \leq j \leq \omega)$. Since $|T| = |N_T(x_1)| + 1$ and $|V(C_j)| \leq e(x_1, U)$, we get

$$d_{G/S}(u) \leq |T| + (|V(C_j)| - 1) \leq |N_T(x_1)| + e(x_1, U) = d_{G/S}(x_1) = h_1 \leq k \tag{2.32}$$

for every $u \in C_j$, which contradicts the result of Lemma 2.3. Hence $V(C_j) \not\subset N_G(x_1)$, and so there exists a vertex $u \in C_j$, which is not adjacent to x_1.

Let $u_1 \in V(C_1) \setminus N_G(x_1)$. If $d_G(u_1) < n/2$, then (B) holds. Thus we may assume that $d_G(u_1) \geq n/2$. Note that $d_G(u_1)$ is strictly less than the upper bound in (2.9) because u_1 is not adjacent to all vertices of T. Therefore, we obtain

$$\frac{n}{2} \leq d_G(u_1) \leq m_1 - 1 + s + (t - 1) \leq \frac{n - s - t}{3} + s + t - 2. \tag{2.33}$$

Hence, it follows that

$$4s \geq n - 4t + 12. \tag{2.34}$$

On the other hand, by (2.8) and (2.31), we have that

$$(3k + 1)s \leq n - [3(h_1 - k) + 1]t. \tag{2.35}$$

This inequality, together with (2.34), implies that

$$\begin{aligned} 3(k-1)n &\leq (3k+1)(4s + 4t - 12) - 4n \\ &\leq 12(2k - h_1)t - 36k - 12 \\ &\leq 12(2k - h_1)(h_1 + 1) - 36k - 12 < 12(k-1)^2, \end{aligned} \tag{2.36}$$

which contradicts that $n \geq 4k + 1$.

Thus we may assume that $T \setminus N_T[x_1] \neq \emptyset$. Let $p = |N_T[x_1]|$. We know that $t \geq p + 1$, $h_1 \geq p - 1$.

Case 4 ($0 \leq h_1 \leq k - 1$ and $T \setminus N_T[x_1] \neq \emptyset$).

Subcase 4.1 ($h_1 \le h_2 \le k-1$). Since $\omega \le n-s-t$ and $k-h_2 \ge 1$, it follows by (2.15) that

$$(k-h_2)(n-s-t) \ge \omega \ge ks + (h_1-k)p + (h_2-k)(t-p). \tag{2.37}$$

Therefore

$$(k-h_2)(n-s) - ks \ge (h_1-h_2)p \ge (h_1-h_2)(h_1+1). \tag{2.38}$$

Since $p \le h_1+1$ and, by hypothesis, it holds that $n \ge 4k+1 > 4k$, we get that

$$h_2 \cdot \frac{n}{2} > h_2 \cdot 2k. \tag{2.39}$$

We may suppose that $s \ge n/2 - h_2$, since otherwise (C) holds. So, we have that

$$\left(s - \frac{n}{2}\right)(2k-h_2) \ge -h_2(2k-h_2). \tag{2.40}$$

Adding (2.38), (2.39), and (2.40), we obtain

$$\begin{aligned} 0 &> h_2^2 - h_2(h_1+1) + h_1^2 + h_1 \\ &= \frac{1}{4}(2h_1-h_2)^2 + \frac{3}{4}\left(h_2 - \frac{2}{3}\right)^2 + h_1 - \frac{1}{3}. \end{aligned} \tag{2.41}$$

For nonnegative integers h_1 and h_2, $h_2^2 - h_2(h_1+1) + h_1^2 + h_1 \ge -1/3$ implies that $h_2^2 - h_2(h_1+1) + h_1^2 + h_1 \ge 0$. So, the above inequality is impossible.

For the case where $0 \le h_1 \le k-1$ and $h_2 \ge k$, since $t \ge p+1$, we have $n-s-t \le n-s-p-1$. Further, since $h_2 \ge k$, using (2.8), (2.15) we have

$$n-s-p-1 \ge n-s-t \ge 3\omega \ge 3[ks + (h_1-k)p], \tag{2.42}$$

that is,

$$(3k+1)s \le n + (3k-3h_1-1)p - 1. \tag{2.43}$$

Subcase 4.2 ($h_2 = k$). Since $3k-3h_1-1 > 0$ and $h_1 \ge p-1$, it follows by (2.43) that

$$(3k+1)s \le n + (3k-3h_1-1)(h_1+1) - 1. \tag{2.44}$$

By the same reason as in the proof of Subcase 4.1, we may suppose that $s \ge n/2 - h_2 = n/2 - k$. This inequality, together with (2.44), gives

$$\begin{aligned} (3k-1)n &\le (3k+1)(2s+2k) - 2n \\ &\le 2[(3k-3h_1-1)(h_1+1)-1] + 6k^2 + 2k \\ &= -6h_1^2 + (6k-8)h_1 + 6k^2 + 8k - 4 \\ &< -3h_1^2 + (6k-12)h_1 + 6k^2 + 8k - 2 \\ &= -3[h_1 - k + 2]^2 + 9k^2 - 4k + 10 \\ &\le 9k^2 - 4k + 10 < (3k+1)(3k-1), \end{aligned} \tag{2.45}$$

which contradicts that $n \ge 4k+1$, for $k \ge 3$.

Subcase 4.3 ($h_2 \geq k+1$). By (2.15) and (2.22),

$$\omega \geq ks + (h_1 - k)p + (h_2 - k)(t - p) \tag{2.46}$$

$$\geq (k - h_1)(k - p) + t + k - p. \tag{2.47}$$

Subcase 4.3.1 ($p \leq k-1$). Let $\omega_0 = 2k - h_1 - p + t$. Then $\omega \geq \omega_0 \geq 5$ by (2.47). Suppose that $m_2 - 1 + s + t \geq n/2$, since otherwise (A) holds. Hence, by (2.10), the hypotheses of Lemma 2.4 are satisfied for $m = m_2$. Therefore

$$m_2 - 1 + s + t \leq \frac{1}{3}[n + 2(s + t + 1 - 2k + h_1 + p - t)]. \tag{2.48}$$

This, together with the inequality $m_2 - 1 + s + t \geq n/2$, gives

$$n \leq 4(s + h_1 + p - 2k) + 4. \tag{2.49}$$

By (2.43) and (2.49), we obtain

$$\begin{aligned} 3(k-1)n &\leq (3k+1)[4s + 4h_1 + 4p - 8k + 4] - 4n \\ &\leq 4[(3k - 3h_1 - 1)p - 1] + (3k+1)(4h_1 + 4p - 8k + 4) \\ &\leq 4(3k - 3h_1 - 1)(k-1) + (3k+1)(4h_1 - 4k) - 4 \leq -4k - 16. \end{aligned} \tag{2.50}$$

This is obviously impossible.

Subcase 4.3.2 ($p = k$). In this case $h_1 = k - 1$. Then by (2.22), $s \geq 2$. Since $t \geq p + 1 = k + 1 \geq 4$, by (2.46), we have $\omega \geq ks + t - 2k \geq t$. Let $\omega_0 = t$, then $\omega \geq \omega_0 \geq 4$ by (2.46). Hence, by (2.10), the hypotheses of Lemma 2.4 are satisfied for $m = m_2$. Therefore,

$$m_2 - 1 + s + t \leq \frac{n + 2s + 2}{3}. \tag{2.51}$$

By the same reason as in the proof of Subcase 4.3.1, we may suppose that $m_2 - 1 + s + t \geq n/2$. This, together with (2.51), gives

$$n \leq 4s + 4. \tag{2.52}$$

Further, when $p = k$ and $h_1 = k - 1$, (2.43) is as follows:

$$(3k+1)s \leq n + 2k - 1. \tag{2.53}$$

By (2.52) and (2.53), we get

$$(3k+1)s \leq 4s + 2k + 3, \tag{2.54}$$

which contradicts that $s \geq 2$ in this case.

Case 5 ($h_1 = k$ and $T \setminus N_T[x_1] \neq \emptyset$).

Subcase 5.1 ($k \leq h_2 \leq k+1$). In this case $t \geq 2$, so that $n - 2 - s \geq n - s - t$. Since $h_1 = k$ and $t \geq p + 1$, we have

$$(h_1 - k)p + (h_2 - k)(t - p) \geq h_2 - k. \tag{2.55}$$

By (2.8) and (2.46), we get

$$n-s-2 \geq n-s-t \geq 3[ks+(h_2-k)], \tag{2.56}$$

that is,

$$s \leq \frac{n-2+3(k-h_2)}{3k+1} \leq \frac{n-2}{3k+1}. \tag{2.57}$$

We still suppose that $s+h_2 \geq n/2$, since otherwise (C) holds. Therefore, this, together with (2.57), gives

$$\frac{n}{2} \leq s+h_2 \leq \frac{n-2}{3k+1}+k+1, \tag{2.58}$$

that is,

$$(3k-1)n \leq 6k^2+8k-2 < \left(2k+\frac{11}{3}\right)(3k-1), \tag{2.59}$$

which contradicts that $n \geq 4k+1$, for $k \geq 3$.
Subcase 5.2 ($h_2 \geq k+2$). By (2.22), we have $s \geq 1$. Since $t \geq p+1$ and $p \leq h_1+1 = k+1$, by (2.46), we get $\omega \geq ks+2(t-p)$. Let $\omega_0 = ks+2(t-p)$, then $\omega \geq \omega_0 \geq 5$. By (2.9), the hypotheses of Lemma 2.4 are satisfied for $m = m_2$, we get

$$\begin{aligned} m_2-1+s+t &\leq \frac{1}{3}[n+2(s+t+1-ks-2t+2p)] \\ &\leq \frac{1}{3}[n-2(k-1)s-2(p+1)+2+4p] \\ &\leq \frac{1}{3}[n-2(k-1)s+2(k+1)] \leq \frac{n+4}{3} < \frac{n}{2}. \end{aligned} \tag{2.60}$$

This shows that (A) holds in this case. This completes the proof of Theorem 1.4. □

3. Sharpness of Theorem 1.4

The condition $\delta(G) \geq k+1$ in Theorem 1.4 is necessary. The assumption that G is 2-connected and $n \geq 4k+1$ in Theorem 1.4 cannot be weakened any further. Let k be an odd integer such that $k \geq 3$, and let n be an even integer such that $n \geq 4k+1$. G_1 is a graph obtained by adding an edge e to connect K_{k+2} and K_{n-k-2}. Then G_1 satisfies all the conditions of Theorem 1.4 except that G_1 is 1-connected and G_1 has no k-factors excluding edge e. Let $G_2 = K_{2k-1}+(K_1 \bigcup kK_2)$, then G_2 satisfies all the conditions of Theorem 1.4 except $n = 4k$. Setting $S = V(K_{2k-1})$ and $T = V(K_1 \bigcup kK_2)$, we have $\delta_G(S,T) = 0 < \varepsilon(S,T) = 2$; by Lemma 2.2, Theorem 1.4 does not hold.

Acknowledgment

I would like to thank the referees for their valuable comments which led to a considerable improvement of the original paper.

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