

# FURTHER REMARKS ON SYSTEMS OF INTERLOCKING EXACT SEQUENCES

PETER HILTON AND C. JOANNA SU

*Received 10 May 2004*

In a system of interlocking sequences, the assumption that three out of the four sequences are exact does not guarantee the exactness of the fourth. In 1967, Hilton proved that, with the additional condition that it is differential at the crossing points, the fourth sequence is also exact. In this note, we trace such a diagram and analyze the relation between the kernels and the images, in the case that the fourth sequence is not necessarily exact. Regarding the exactness of the fourth sequence, we remark that the exactness of the other three sequences does guarantee the exactness of the fourth at noncrossing points. As to a crossing point  $p$ , we need the extra criterion that the fourth sequence is differential. One notices that the condition, for the fourth sequence, that  $\text{kernel} \supseteq \text{image}$  at  $p$  turns out to be equivalent to the “opposite” condition  $\text{kernel} \subseteq \text{image}$ . Next, for the kernel and the image at  $p$  of the fourth sequence, even though they may not coincide, they are not far different—they always have the same cardinality as sets, and become isomorphic after taking quotients by a subgroup which is common to both. We demonstrate these phenomena with an example.

## 1. The exactness of systems of interlocking sequences

It is well known that, in topology, the homotopy sequence of a triple  $(X, A, B)$ , where  $X$ ,  $A$ , and  $B$  are topological spaces with base points and  $B \subseteq A \subseteq X$ , is exact. In addition, the homotopy exact sequences of the pairs  $(A, B)$ ,  $(X, A)$ , and  $(X, B)$ , respectively, and the homotopy exact sequence of the triple  $(X, A, B)$  knot together in a one-of-a-kind commutative diagram, thus,

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & \pi_{n+1}(A, B) & \xrightarrow{1} & \pi_n(B) & \xrightarrow{3} & \pi_n(X) & \xrightarrow{2} & \pi_n(X, A) & \rightarrow \cdots \\
 & \nearrow & & & \nearrow & & \nearrow & & \nearrow & \\
 & & \pi_{n+1}(X, B) & & \pi_n(A) & & \pi_n(X, B) & & & \\
 & \nwarrow & & & \nwarrow & & \nwarrow & & \nwarrow & \\
 \cdots & \rightarrow & \pi_{n+1}(X) & \xrightarrow{2} & \pi_{n+1}(X, A) & \xrightarrow{4} & \pi_n(A, B) & \xrightarrow{1} & \pi_{n-1}(B) & \rightarrow \cdots
 \end{array} \tag{1.1}$$

In [1], Hilton analyzed the general case of (1.1); namely, a commutative diagram that interlocks four sequences of which three are known to be exact, and proved the following theorem.

THEOREM 1.1 [1]. *Suppose given 4 sequences*

$$\begin{aligned}
 \alpha: & \quad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots, \\
 \beta: & \quad \cdots \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \cdots, \\
 \gamma: & \quad \cdots \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \cdots, \\
 \partial: & \quad \cdots \dashrightarrow \bullet \dashrightarrow \bullet \dashrightarrow \bullet \dashrightarrow \cdots,
 \end{aligned} \tag{1.2}$$

*of which 3 are long exact, forming a commutative diagram*

*then the fourth is also long exact, provided it is differential; that is,  $\partial\partial = 0$ , at its crossing points  $p$ .*

Note that one derives Theorem 1.1 through standard diagram chasing. However, the criterion that the fourth sequence is differential at its crossing points is not needed when showing the exactness at noncrossing points; thus we have the following.

PROPOSITION 1.2. *Suppose 4 sequences of (1.2) are given of which 3 are long exact, forming the commutative diagram (1.3), then the fourth is automatically exact at all points other than the crossing points  $p$ .*

*Proof.* In the commutative diagram of sequences

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_{i-1} & \xrightarrow{\gamma_{i-1}} & C_i & \xrightarrow{\beta_i} & C_{i+1} & \xrightarrow{\partial_{i+2}} & \cdots \\
 & & \searrow \alpha_{i-2} & & \nearrow \beta_{i-1} & & \searrow \beta_{i+1} & & \nearrow \alpha_{i+2} \\
 & & p \bullet B_{i-1} & & p \bullet B_i & & p \bullet & & \\
 & & \nearrow \beta_{i-2} & & \searrow \alpha_{i-1} & & \nearrow \alpha_{i+1} & & \searrow \beta_{i+2} \\
 \cdots & \longrightarrow & A_{i-1} & \xrightarrow{\partial_{i-1}} & A_i & \xrightarrow{\alpha_i} & A_{i+1} & \xrightarrow{\gamma_{i+2}} & \cdots
 \end{array}
 \quad (1.4)$$

assume, without real loss of generality, that the  $\alpha$ -,  $\beta$ -, and  $\gamma$ -sequences are exact.

To show the exactness of the  $\partial$ -sequence at  $A_i$ , first one sees that  $\partial_i \partial_{i-1} = \partial_i \alpha_{i-1} \beta_{i-2} = \gamma_i \beta_{i-1} \beta_{i-2} = 0$ . Conversely, to show that  $\ker \partial_i \subseteq \text{image } \partial_{i-1}$ , suppose given  $\partial_i(a) = 0$  where  $a \in A_i$ , then  $\alpha_i(a) = \gamma_{i+1} \partial_i(a) = 0$  so there exists a  $b \in B_{i-1}$  such that  $\alpha_{i-1}(b) = a$ , because the  $\alpha$ -sequence is exact. Since  $\gamma_i \beta_{i-1}(b) = \partial_i \alpha_{i-1}(b) = \partial_i(a) = 0$ ,  $\beta_{i-1}(b) \in \ker \gamma_i$ . Due to the fact that the  $\gamma$ -sequence is exact, there is a  $c \in C_{i-1}$  such that  $\gamma_{i-1}(c) = \beta_{i-1}(b)$ . Next, since the element  $b - \alpha_{i-2}(c) \in B_{i-1}$  satisfies  $\beta_{i-1}(b - \alpha_{i-2}(c)) = \beta_{i-1}(b) - \beta_{i-1} \alpha_{i-2}(c) = \beta_{i-1}(b) - \gamma_{i-1}(c) = 0$  and the  $\beta$ -sequence is exact, there is an  $a' \in A_{i-1}$  such that  $\beta_{i-2}(a') = b - \alpha_{i-2}(c)$ . Finally, this element  $a' \in A_{i-1}$  fulfills that  $\partial_{i-1}(a') = \alpha_{i-1} \beta_{i-2}(a') = \alpha_{i-1}(b - \alpha_{i-2}(c)) = \alpha_{i-1}(b) - \alpha_{i-1} \alpha_{i-2}(c) = \alpha_{i-1}(b) = a$ ; thus,  $\ker \partial_i \subseteq \text{image } \partial_{i-1}$ .

As to the exactness of the  $\partial$ -sequence at  $C_{i+1}$ , first it is differential because  $\partial_{i+2} \partial_{i+1} = \alpha_{i+2} \beta_{i+1} \partial_{i+1} = \alpha_{i+2} \alpha_{i+1} \gamma_{i+1} = 0$ . To show that  $\ker \partial_{i+2} \subseteq \text{image } \partial_{i+1}$ , let  $\partial_{i+2}(c) = 0$  where  $c \in C_{i+1}$ . Since  $\alpha_{i+2} \beta_{i+1}(c) = \partial_{i+2}(c) = 0$ ,  $\beta_{i+1}(c) \in \ker \alpha_{i+2}$  so there exists an  $a \in A_{i+1}$  such that  $\alpha_{i+1}(a) = \beta_{i+1}(c)$ , by the exactness of the  $\alpha$ -sequence. Similarly, since  $\gamma_{i+2}(a) = \beta_{i+2} \alpha_{i+1}(a) = \beta_{i+2} \beta_{i+1}(c) = 0$ , we find a  $b \in B_i$  satisfying  $\gamma_{i+1}(b) = a$ . Next, the condition that  $\beta_{i+1}(-c + \partial_{i+1}(b)) = -\beta_{i+1}(c) + \beta_{i+1} \partial_{i+1}(b) = -\beta_{i+1}(c) + \alpha_{i+1} \gamma_{i+1}(b) = -\beta_{i+1}(c) + \alpha_{i+1}(a) = 0$  forces  $-c + \partial_{i+1}(b) \in \ker \beta_{i+1}$ , so that, by the exactness of the  $\beta$ -sequence, there exists a  $c' \in C_i$  such that  $\beta_i(c') = -c + \partial_{i+1}(b)$ . Now the element  $b - \gamma_i(c') \in B_i$  satisfies  $\partial_{i+1}(b - \gamma_i(c')) = \partial_{i+1}(b) - \partial_{i+1} \gamma_i(c') = c + \beta_i(c') - \partial_{i+1} \gamma_i(c') = c + \beta_i(c') - \beta_i(c') = c$ , so  $\ker \partial_{i+2} \subseteq \text{image } \partial_{i+1}$ , and thus the  $\partial$ -sequence is exact at  $C_{i+1}$ .  $\square$

Hereafter, it remains to examine the exactness of the  $\partial$ -sequence at the crossing points  $p$ , in diagram (1.3). Theorem 1.1 says that, with the assumption that the  $\alpha$ -,  $\beta$ -, and  $\gamma$ -sequences are exact, the fourth sequence, the  $\partial$ -sequence, is also exact if it is differential at  $p$ . We trace the diagram and make the following expansion.

**THEOREM 1.3.** *In the commutative diagram of sequences (1.4) assume that the  $\alpha$ -,  $\beta$ -, and  $\gamma$ -sequences are exact. Then  $\text{image } \partial_i \subseteq \ker \partial_{i+1}$  if and only if  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ . Thus, the following are equivalent:*

- (i) *the  $\partial$ -sequence is exact at the crossing points  $p$ ,*
- (ii) *the  $\partial$ -sequence is differential at  $p$ ; that is,  $\ker \partial_{i+1} \supseteq \text{image } \partial_i$ ,*
- (iii)  *$\ker \partial_{i+1} \subseteq \text{image } \partial_i$ .*

Before proving Theorem 1.3, we note that it is easy to verify that the fourth sequence in diagram (1.1) is differential at the crossing points; hence the sequence is exact: for the composite homomorphism  $\pi_n(A, B) \rightarrow \pi_n(X, B) \rightarrow \pi_n(X, A)$  is induced by the composite map

$$\begin{array}{ccccc} B & \xlongequal{\quad} & B & \xhookrightarrow{\quad} & A \\ \downarrow & & \downarrow & & \downarrow \\ A & \xhookrightarrow{\quad} & X & \xlongequal{\quad} & X \end{array} \quad (1.5)$$

which coincides with the composite map

$$\begin{array}{ccccc} B & \xhookrightarrow{\quad} & A & \xlongequal{\quad} & A \\ \downarrow & & \parallel & & \downarrow \\ A & \xlongequal{\quad} & A & \xhookrightarrow{\quad} & X \end{array} \quad (1.6)$$

but  $\pi_n(A, A) = 0$ .

*Proof of Theorem 1.3.* First we assume that the fourth sequence is differential at  $B_i$  and show that  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ . If  $\partial_{i+1}(b) = 0$  where  $b \in B_i$ , then  $\alpha_{i+1}\gamma_{i+1}(b) = \beta_{i+1}\partial_{i+1}(b) = 0$  so  $\gamma_{i+1}(b) \in \ker \alpha_{i+1}$ . Thus, there exists an  $a \in A_i$  such that  $\alpha_i(a) = \gamma_{i+1}(b)$ , because the  $\alpha$ -sequence is exact. Since  $\gamma_{i+1}(b - \partial_i(a)) = \gamma_{i+1}(b) - \gamma_{i+1}\partial_i(a) = \gamma_{i+1}(b) - \alpha_i(a) = 0$ , one finds an element  $c \in C_i$  such that  $\gamma_i(c) = b - \partial_i(a)$ . Next, by the assumption that  $\partial_{i+1}\partial_i = 0$ ,  $\beta_i(c) = \partial_{i+1}\gamma_i(c) = \partial_{i+1}(b - \partial_i(a)) = \partial_{i+1}(b) - \partial_{i+1}\partial_i(a) = \partial_{i+1}(b) = 0$  so there is a  $b' \in B_{i-1}$  such that  $\beta_{i-1}(b') = c$ , because the  $\beta$ -sequence is exact. Then the result that the element  $\alpha_{i-1}(b') + a \in A_i$  satisfies  $\partial_i(\alpha_{i-1}(b') + a) = \partial_i\alpha_{i-1}(b') + \partial_i(a) = \gamma_i\beta_{i-1}(b') + \partial_i(a) = \gamma_i(c) + \partial_i(a) = b$  shows that  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ .

Conversely, assume that  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ , we show that  $\text{image } \partial_i \subseteq \ker \partial_{i+1}$ : let  $a \in A_i$ . Since the  $\alpha$ -sequence is exact,  $\beta_{i+1}\partial_{i+1}\partial_i = \alpha_{i+1}\gamma_{i+1}\partial_i = \alpha_{i+1}\alpha_i = 0$ . Thus,  $\text{image}(\partial_{i+1}\partial_i) \subseteq \ker \beta_{i+1}$ , which forces  $\text{image}(\partial_{i+1}\partial_i) \subseteq \text{image } \beta_i$  because the  $\beta$ -sequence is exact. Hence, there exists a  $c \in C_i$  such that

$$\partial_{i+1}\partial_i(a) = \beta_i(c). \quad (1.7)$$

Since the element  $\partial_i(a) - \gamma_i(c) \in B_i$  satisfies the criterion that  $\partial_{i+1}(\partial_i(a) - \gamma_i(c)) = \partial_{i+1}\partial_i(a) - \partial_{i+1}\gamma_i(c) = \partial_{i+1}\partial_i(a) - \beta_i(c) = 0$ ,  $\partial_i(a) - \gamma_i(c) \in \ker \partial_{i+1}$ . By the assumption that  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ , we find an  $a' \in A_i$  such that

$$\partial_i(a) - \gamma_i(c) = \partial_i(a'). \quad (1.8)$$

Furthermore,  $\alpha_i(a') = \gamma_{i+1}\partial_i(a') = \gamma_{i+1}(\partial_i(a) - \gamma_i(c)) = \gamma_{i+1}\partial_i(a) - \gamma_{i+1}\gamma_i(c) = \gamma_{i+1}\partial_i(a) = \alpha_i(a)$ . Thus,  $\alpha_i(-a' + a) = 0$ , and since the  $\alpha$ -sequence is exact, there exists a  $b \in B_{i-1}$  such that

$$\alpha_{i-1}(b) = -a' + a. \quad (1.9)$$

Combining (1.8) and (1.9), we next consider the element  $c - \beta_{i-1}(b) \in C_i$  and derive that  $\gamma_i(c - \beta_{i-1}(b)) = \gamma_i(c) - \gamma_i\beta_{i-1}(b) = \gamma_i(c) - \partial_i\alpha_{i-1}(b) = \gamma_i(c) - \partial_i(-a' + a) = \gamma_i(c) - \gamma_i(c) = 0$ . Thus, there is a  $c' \in C_{i-1}$  such that

$$\gamma_{i-1}(c') = c - \beta_{i-1}(b), \quad (1.10)$$

due to the facts that  $c - \beta_{i-1}(b) \in \ker \gamma_i$  and the  $\gamma$ -sequence is exact. Finally, the calculation that

$$\begin{aligned} \partial_{i+1}\partial_i(a) &= \beta_i(c) && \text{(by (1.7))} \\ &= \beta_i(\gamma_{i-1}(c') + \beta_{i-1}(b)) && \text{(by (1.10))} \\ &= \beta_i\gamma_{i-1}(c') + \beta_i\beta_{i-1}(b) \\ &= \beta_i\gamma_{i-1}(c') && \text{(because the } \beta\text{-sequence is exact)} \\ &= \partial_{i+1}\gamma_i\beta_{i-1}\alpha_{i-2}(c') && \text{(because } \beta_i = \partial_{i+1}\gamma_i \text{ and } \gamma_{i-1} = \beta_{i-1}\alpha_{i-2}) \\ &= \partial_{i+1}\partial_i\alpha_{i-1}\alpha_{i-2}(c') && \text{(because } \gamma_i\beta_{i-1} = \partial_i\alpha_{i-2}) \\ &= 0 && \text{(because the } \alpha\text{-sequence is exact)} \end{aligned} \quad (1.11)$$

completes the proof that the  $\partial$ -sequence is differential at the crossing points.

Now we have shown that  $\text{image } \partial_i \subseteq \ker \partial_{i+1}$  if and only if  $\ker \partial_{i+1} \subseteq \text{image } \partial_i$ , the equivalence of the three assertions of Theorem 1.3 is clear.  $\square$

Hence, in a system of interlocking sequences, (1.3), the fact that three out of the four sequences are exact does not guarantee the exactness of the fourth at the crossing points  $p$ . One needs the condition that the sequence is differential at  $p$  to assure its exactness. The next couple of examples demonstrates the necessity of this extra criterion.

*Example 1.4* [1].

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{\text{the identity map}} & \mathbb{Z}_2 \\ & \searrow \iota_1 \quad \nearrow \partial = \partial_{i+1} & \\ & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \\ & \nwarrow \iota_2 = \partial_i \quad \searrow \epsilon_2 & \\ \mathbb{Z}_2 & \xrightarrow{\text{the identity map}} & \mathbb{Z}_2 \end{array} \quad (1.12)$$

where  $\iota_1, \iota_2$  are the inclusions into the first and the second factors, respectively,  $\epsilon_2$  is the projection onto the second factor, and  $\partial = \langle 1_{\mathbb{Z}_2}, 1_{\mathbb{Z}_2} \rangle$ ; that is,  $\partial(a, b) = a + b$  where  $a, b \in \mathbb{Z}_2$ .

*Example 1.5.* Let  $G$  be an arbitrary nontrivial abelian group

$$\begin{array}{ccccc}
 G & \xrightarrow{\text{the identity map}} & G & & \\
 \searrow \iota_1 & & \nearrow \epsilon_1 = \partial_{i+1} & & \\
 & G \oplus G & & & \\
 \nearrow \Delta = \partial_i & & \searrow \epsilon_2 & & \\
 G & \xrightarrow{\text{the identity map}} & G & & 
 \end{array} \quad (1.13)$$

where  $\iota_1$  is the inclusion into the first factor,  $\epsilon_1, \epsilon_2$  are the projections onto the first and the second factors, respectively, and  $\Delta = \{1_G, 1_G\}$ ; that is,  $\Delta(g) = (g, g)$  where  $g \in G$ .

Note that, in either example, it is easy to verify all the necessary commutativity relations; yet the exactness of the fourth sequence  $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$  fails at the crossing point because none of these equivalent relations

$$\begin{aligned}
 \ker \partial_{i+1} &= \text{image } \partial_i, \\
 \ker \partial_{i+1} &\supseteq \text{image } \partial_i, \\
 \ker \partial_{i+1} &\subseteq \text{image } \partial_i
 \end{aligned} \quad (1.14)$$

holds.

## 2. Kernels versus images at the crossing points

As discussed in Section 1, in a commutative diagram such as (1.4), where the  $\alpha$ -,  $\beta$ -, and  $\gamma$ -sequences are exact, the fourth sequence, the  $\partial$ -sequence, may not be exact at the crossing points  $p$ . Nevertheless, in Example 1.4, Example 1.5, and a few other examples, we notice that even though  $\text{image } \partial_i$  and  $\ker \partial_{i+1}$  might not coincide, they seem to be isomorphic in most cases and they always have the same cardinality as sets. The following theorem assures our assertion and shows the close relation between  $\text{image } \partial_i$  and  $\ker \partial_{i+1}$ .

**THEOREM 2.1.** *In the commutative diagram of sequences (1.4) assume that the  $\alpha$ -,  $\beta$ -, and  $\gamma$ -sequences are exact. Then  $\text{image } \partial_i / \text{image } \gamma_i \beta_{i-1} \cong \ker \partial_{i+1} / \text{image } \gamma_i \beta_{i-1}$ . Thus,  $\text{image } \partial_i$  and  $\ker \partial_{i+1}$  always have the same cardinality as sets.*

*Proof.* Suppose given an equivalence class  $[b] \in \ker \partial_{i+1} / \text{image } \gamma_i \beta_{i-1}$ , then  $b \in \ker \partial_{i+1}$ . Since  $\alpha_{i+1} \gamma_{i+1}(b) = \beta_{i+1} \partial_{i+1}(b) = 0$ ,  $\gamma_{i+1}(b) \in \ker \alpha_{i+1}$  so there exists an  $a \in A_i$  such that  $\alpha_i(a) = \gamma_{i+1}(b)$ , because the  $\alpha$ -sequence is exact. Then,  $\partial_i(a) \in \text{image } \partial_i$ ; thus, we define the map  $\chi : \ker \partial_{i+1} / \text{image } \gamma_i \beta_{i-1} \rightarrow \text{image } \partial_i / \text{image } \gamma_i \beta_{i-1}$  by  $\chi([b]) = [\partial_i(a)]$  and show that it is an isomorphism as follows.

To assure that  $\chi$  is well defined, assume that there are elements  $a$  and  $a'$  in  $A_i$  such that  $\alpha_i(a) = \gamma_{i+1}(b) = \alpha_i(a')$ . Then  $\alpha_i(a - a') = 0$ , so there exists an  $x \in B_{i-1}$  such that  $a - a' = \alpha_{i-1}(x)$ , since the  $\alpha$ -sequence is exact. This yields  $[\partial_i(a)] = [\partial_i(\alpha_{i-1}(x) + a')] = [\partial_i \alpha_{i-1}(x) + \partial_i(a')] = [\gamma_i \beta_{i-1}(x) + \partial_i(a')] = [\partial_i(a')]$  in  $\text{image } \partial_i / \text{image } \gamma_i \beta_{i-1}$ . Thus,  $\chi([b]) = [\partial_i(a)]$  is independent of the choice of  $a$ . The proof that  $\chi([b]) = [\partial_i(a)]$  is independent of the choice of  $b$  is similar.

We show that the map  $\chi$  is monomorphic: suppose given  $[b] \in \ker \partial_{i+1}/\text{image } \gamma_i \beta_{i-1}$  such that  $\chi([b]) = [\partial_i(a)] = 0$  in  $\text{image } \partial_i/\text{image } \gamma_i \beta_{i-1}$ , where  $a \in A_i$  and  $\alpha_i(a) = \gamma_{i+1}(b)$ . Since  $\gamma_{i+1}(b - \partial_i(a)) = \gamma_{i+1}(b) - \gamma_{i+1}\partial_i(a) = \gamma_{i+1}(b) - \alpha_i(a) = 0$ ,  $b - \partial_i(a) \in \ker \gamma_{i+1}$  so there exists a  $c \in C_i$  such that

$$\gamma_i(c) = b - \partial_i(a), \quad (2.1)$$

because the  $\gamma$ -sequence is exact. In addition, since  $[\partial_i(a)] = 0$  in  $\text{image } \partial_i/\text{image } \gamma_i \beta_{i-1}$ , there is an  $x \in \text{image } \gamma_i \beta_{i-1}$  such that

$$\partial_i(a) = \gamma_i \beta_{i-1}(x). \quad (2.2)$$

Thus,

$$\begin{aligned} \beta_i(c) &= \partial_{i+1}\gamma_i(c) && (\text{because } \beta_i = \partial_{i+1}\gamma_i) \\ &= \partial_{i+1}(b - \partial_i(a)) && (\text{by (2.1)}) \\ &= \partial_{i+1}(b) - \partial_{i+1}\partial_i(a) \\ &= -\partial_{i+1}\partial_i(a) && \left( \text{because } [b] \in \frac{\ker \partial_{i+1}}{\text{image } \gamma_i \beta_{i-1}}, \text{ so } b \in \ker \partial_{i+1} \right) \\ &= -\partial_{i+1}\gamma_i \beta_{i-1}(x) && (\text{by (2.2)}) \\ &= -\beta_i \beta_{i-1}(x) && (\text{because } \beta_i = \partial_{i+1}\gamma_i) \\ &= 0 && (\text{because the } \beta\text{-sequence is exact}). \end{aligned} \quad (2.3)$$

Next, since  $\beta_i(c) = 0$  and the  $\beta$ -sequence is exact, there is a  $y \in B_{i-1}$  such that

$$c = \beta_{i-1}(y). \quad (2.4)$$

Substituting (2.1) with (2.4) and (2.2), we have  $b = \gamma_i(c) + \partial_i(a) = \gamma_i \beta_{i-1}(y) + \gamma_i \beta_{i-1}(x) = \gamma_i \beta_{i-1}(y + x)$ , which means that  $b \in \text{image } \gamma_i \beta_{i-1}$ . Thus,  $[b] = 0$  in  $\ker \partial_{i+1}/\text{image } \gamma_i \beta_{i-1}$ , and therefore, the map  $\chi$  is monomorphic.

Finally, we show that  $\chi$  is epimorphic: suppose given  $[b] \in \text{image } \partial_i/\text{image } \gamma_i \beta_{i-1}$ , then  $b \in \text{image } \partial_i$  so there is an  $a \in A_i$  such that  $b = \partial_i(a)$ . Thus,  $\beta_{i+1}\partial_{i+1}(b) = \beta_{i+1}\partial_{i+1}\partial_i(a) = \alpha_{i+1}\gamma_{i+1}\partial_i(a) = \alpha_{i+1}\alpha_i(a) = 0$ , which says that  $\partial_{i+1}(b) \in \ker \beta_{i+1}$ . Then there exists a  $c \in C_i$  such that  $\beta_i(c) = \partial_{i+1}(b)$ , because the  $\beta$ -sequence is exact. Now we find an element  $b - \gamma_i(c) \in B_i$  satisfying the condition that  $b - \gamma_i(c) \in \ker \partial_{i+1}$  because  $\partial_{i+1}(b - \gamma_i(c)) = \partial_{i+1}(b) - \partial_{i+1}\gamma_i(c) = \partial_{i+1}(b) - \beta_i(c) = 0$ . Hence  $[b - \gamma_i(c)] \in \ker \partial_{i+1}/\text{image } \gamma_i \beta_{i-1}$  and  $\gamma_{i+1}(b - \gamma_i(c)) = \gamma_{i+1}(b) - \gamma_{i+1}\gamma_i(c) = \gamma_{i+1}(b) = \gamma_{i+1}\partial_i(a) = \alpha_i(a)$ . By the construction of  $\chi$ , we assign that  $\chi([b - \gamma_i(c)]) = [\partial_i(a)] = [b]$ . Thus the map  $\chi$  is epimorphic.

Knowing that  $\chi: \ker \partial_{i+1}/\text{image } \gamma_i \beta_{i-1} \rightarrow \text{image } \partial_i/\text{image } \gamma_i \beta_{i-1}$  is an isomorphism, it is evident that  $\text{image } \partial_i$  and  $\ker \partial_{i+1}$  must have the same cardinality as sets.  $\square$

It is unfortunate that, in (1.3), the fact that three out of the four sequences are exact does not promise the exactness of the fourth. Nevertheless, the only places that we need to examine are the crossing points  $p$ . Even though the kernel and the image may not agree here, they are not far different—they always have the same cardinality as sets, and they

become isomorphic after taking quotients by a subgroup which is common to both. We close the discussion with an example that demonstrates these phenomena.

*Example 2.2.*

$$\begin{array}{ccccc}
 & & \mathbb{Z}_8 & \xrightarrow{\text{mod } 4} & \mathbb{Z}_4 \\
 & \nearrow \times 4 & \searrow \beta_{i-1} & \nearrow \iota_1 & \nearrow \partial_{i+1} \\
 \mathbb{Z}_2 & & & & \\
 & \searrow \times 4 & & & \\
 & & \mathbb{Z}_8 \oplus \mathbb{Z}_4 & \xrightarrow{\epsilon_2} & \mathbb{Z}_4 \\
 & \nearrow \partial_i & \nearrow \gamma_i & \nearrow \partial_{i+1} & \\
 & & \mathbb{Z}_8 & \xrightarrow{\text{mod } 4} & \mathbb{Z}_4
 \end{array} \quad (2.5)$$

where  $\iota_1$  is the inclusion into the first factor,  $\epsilon_2$  is the projection onto the second factor,  $\times 4 : \mathbb{Z}_2 \rightarrow \mathbb{Z}_8$  is the map of multiplication by 4,  $\text{mod } 4 : \mathbb{Z}_8 \rightarrow \mathbb{Z}_4$  is the map of modulo by 4,  $\partial_i = \{1_{\mathbb{Z}_8}, \text{mod } 4\}$ ; that is,  $\partial_i(a) = (a, a(\text{mod } 4))$  where  $a \in \mathbb{Z}_8$ , and  $\partial_{i+1} = \langle \text{mod } 4, 0 \rangle$ ; that is,  $\partial_{i+1}(a, b) = a(\text{mod } 4)$  where  $(a, b) \in \mathbb{Z}_8 \oplus \mathbb{Z}_4$ .

Note that, in (2.5),  $\text{image } \partial_i = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 0), (5, 1), (6, 2), (7, 3)\} = \langle (1, 1) \rangle \cong \mathbb{Z}_8$  and  $\ker \partial_{i+1} = \{(0, 0), (0, 1), (0, 2), (0, 3), (4, 0), (4, 1), (4, 2), (4, 3)\} = \langle (4, 0), (0, 1) \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ , so that they are not isomorphic. However,  $\text{image } \gamma_i \beta_{i-1} = \{(0, 0), (4, 0)\} = \langle (4, 0) \rangle$ , which yields  $\text{image } \partial_i / \text{image } \gamma_i \beta_{i-1} = \{[(0, 0)], [(1, 1)], [(2, 2)], [(3, 3)]\} = \langle [(1, 1)] \rangle \cong \mathbb{Z}_4$  and  $\ker \partial_{i+1} / \text{image } \gamma_i \beta_{i-1} = \{[(0, 0)], [(0, 1)], [(0, 2)], [(0, 3)]\} = \langle [(0, 1)] \rangle \cong \mathbb{Z}_4$ . Thus,  $\text{image } \partial_i / \text{image } \gamma_i \beta_{i-1} \cong \ker \partial_{i+1} / \text{image } \gamma_i \beta_{i-1}$ .

## References

- [1] P. J. Hilton, *On systems of interlocking exact sequences*, Fund. Math. **61** (1967), 111–119.

Peter Hilton: Department of Mathematical Sciences, State University of New York at Binghamton, Binghamton, NY 13902-6000, USA

*E-mail address:* marge@math.binghamton.edu

C. Joanna Su: Department of Mathematics and Computer Science, Providence College, Providence, RI 02918, USA

*E-mail address:* jsu@providence.edu

