

# EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF INFINITE-HORIZON SYSTEMS DERIVED FROM OPTIMAL CONTROL

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This paper deals with the existence and uniqueness of solutions for a class of infinite-horizon systems derived from optimal control. An existence and uniqueness theorem is proved for such Hamiltonian systems under some natural assumptions.

## 1. Introduction

We begin with a simple example to introduce the background of the considered problem. Let  $U$  be a bounded closed subset of  $\mathbb{R}^m$  and let functions  $f : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \rightarrow \mathbb{R}^n$ ,  $L : \mathbb{R}^n \times \mathbb{R}^m \times [a, \infty) \rightarrow \mathbb{R}$  be differentiable with respect to the first variable. Consider an optimal control system of the form

$$\text{Minimize } J[u(\cdot)] = \int_a^\infty L(x(t), u(t), t) dt \quad (1.1)$$

over all admissible controls  $u(\cdot) \in L^2([a, \infty); U)$ , where the trajectories  $x : [a, \infty) \rightarrow \mathbb{R}^n$  are differentiable on  $[a, \infty)$  and satisfy the dynamic system

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(a) = x_0. \quad (1.2)$$

From control theory, the well-known Pontryagin maximum principle, an important necessary optimality condition, is usually applied to get optimal controls for this system. By doing this, the following infinite-horizon Hamiltonian system is derived:

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H(x(t), p(t), t)}{\partial p}, \\ x(a) &= x_0, \\ \dot{p}(t) &= \frac{-\partial H(x(t), p(t), t)}{\partial x}, \\ x(\cdot) &\in L^2([a, \infty); \mathbb{R}^n), \quad p(\cdot) \in L^2([a, \infty); \mathbb{R}^n). \end{aligned} \quad (1.3)$$

Here,  $H(x, p, t) = \lambda L(x, \bar{u}, t) + \langle p, f(x, \bar{u}, t) \rangle$  is the Hamiltonian function for (1.1)-(1.2),  $\langle \cdot, \cdot \rangle$  stands for inner product in  $\mathbb{R}^n$ ,  $\bar{u}$  is an optimal control, and  $x(t)$  is the optimal trajectory corresponding to the optimal control  $\bar{u}$ .

The existence and uniqueness of solutions for system (1.3) is a very interesting question; if solutions to (1.3) are unique, then the optimal control for system (1.1)-(1.2) can be solved analytically or numerically through (1.3). When we consider the generalization of (1.3) in infinite-dimensional spaces, the following Hamiltonian system is obtained:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + F(x(t), p(t), t), \\ x(a) &= x_0, \\ \dot{p}(t) &= -A^*(t)p(t) + G(x(t), p(t), t), \\ p(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X), \end{aligned} \tag{1.4}$$

where both  $x(t)$  and  $p(t)$  take values in a Hilbert space  $X$  for  $a \leq t < \infty$ . It is always assumed that  $F, G : X \times X \times [a, \infty) \rightarrow X$  are nonlinear operators, that  $A(t)$  is a closed operator for each  $t \in [a, \infty)$ , and that  $A^*(t)$  is the adjoint operator of  $A(t)$ .

The following system is called a linear Hamiltonian system, which is a special case of (1.4),

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)p(t) + \varphi(t), \\ x(a) &= x_0, \\ \dot{p}(t) &= -A^*(t)p(t) + C(t)x(t) + \psi(t), \\ p(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X), \end{aligned} \tag{1.5}$$

where  $\varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X)$ , and  $B(t), C(t)$  are selfadjoint linear operators from  $X$  to  $X$  for all  $t \in [a, \infty)$ .

In [2], Lions has discussed the existence and uniqueness of solutions for system (1.5) and gave an existence and uniqueness result. In [1], Hu and Peng considered the existence and uniqueness of solutions for a class of nonlinear forward-backward stochastic differential equations similar to (1.3) but on finite horizon, they provided an existence and uniqueness theorem for (1.3). Peng and Shi in [3] dealt with the existence and uniqueness of solutions for (1.3) using the techniques developed in [1]. In this paper, we consider the existence and uniqueness of solutions for infinite-dimensional system (1.4).

Throughout the paper, the following basic assumptions hold.

(I) There exists a real number  $L > 0$  such that

$$\begin{aligned} \|F(x_1, p_1, t) - F(x_2, p_2, t)\| &\leq L(\|x_1 - x_2\| + \|p_1 - p_2\|), \\ \|G(x_1, p_1, t) - G(x_2, p_2, t)\| &\leq L(\|x_1 - x_2\| + \|p_1 - p_2\|) \end{aligned} \tag{1.6}$$

for all  $x_1, p_1, x_2, p_2 \in X$  and  $t \in [a, \infty)$ .

(II) There exists a real number  $\alpha > 0$  such that

$$\begin{aligned} &\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2 \rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2 \rangle \\ &\leq -\alpha(\|x_1 - x_2\| + \|p_1 - p_2\|) \end{aligned} \tag{1.7}$$

for all  $x_1, p_1, x_2, p_2 \in X$  and  $t \in [a, \infty)$ .

### 2. Lemmas

Two lemmas are given in this section. They are essential to prove the main theorem.

LEMMA 2.1. *Consider the Hamiltonian system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + F_\beta(x, p, t) + \varphi(t), \\ x(a) &= x_0, \\ \dot{p}(t) &= -A^*(t)p(t) + G_\beta(x, p, t) + \psi(t), \\ x(\cdot) \in L^2([a, \infty); X), \quad p(\cdot) &\in L^2([a, \infty); X), \end{aligned} \tag{2.1}$$

where  $\varphi(\cdot), \psi(\cdot) \in L^2([a, \infty); X)$ . The functions  $F_\beta$  and  $G_\beta$  are defined as

$$\begin{aligned} F_\beta(x, p, t) &:= -(1 - \beta)\alpha p + \beta F(x, p, t), \\ G_\beta(x, p, t) &:= -(1 - \beta)\alpha x + \beta G(x, p, t). \end{aligned} \tag{2.2}$$

Assume that (2.1) has a unique solution for some real number  $\beta = \beta_0 \geq 0$  and any  $\varphi(t), \psi(t)$ . There exists a real number  $\delta > 0$ , which is independent of  $\beta_0$ , such that (2.1) has a unique solution for any  $\varphi(t), \psi(t)$ , and  $\beta \in [\beta_0, \beta_0 + \delta]$ .

*Proof.* For any given  $\varphi(\cdot), \psi(\cdot), x(\cdot), p(\cdot) \in L^2([a, \infty); X)$  and  $\delta > 0$ , construct the following Hamiltonian system:

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + F_{\beta_0}(X, P, t) + F_{\beta_0+\delta}(x, p, t) - F_{\beta_0}(x, p, t) + \varphi(t), \\ X(a) &= x_0, \\ \dot{P}(t) &= -A^*(t)P(t) + G_{\beta_0}(X, P, t) + G_{\beta_0+\delta}(x, p, t) - G_{\beta_0}(x, p, t) + \psi(t), \\ X(\cdot) \in L^2([a, \infty); X), \quad P(\cdot) &\in L^2([a, \infty); X). \end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned} &F_{\beta_0+\delta}(x, p, t) - F_{\beta_0}(x, p, t) \\ &= -(1 - \beta_0 - \delta)\alpha p + (\beta_0 + \delta)F(x, p, t) + (1 - \beta_0)\alpha p - \beta_0 F(x, p, t) \\ &= \alpha\delta p + \delta F(x, p, t), \\ &G_{\beta_0+\delta}(x, p, t) - G_{\beta_0}(x, p, t) \\ &= -(1 - \beta_0 - \delta)\alpha x + (\beta_0 + \delta)G(x, p, t) + (1 - \beta_0)\alpha x - \beta_0 G(x, p, t) \\ &= \alpha\delta x + \delta G(x, p, t). \end{aligned} \tag{2.4}$$

The assumption of Lemma 2.1 implies that (2.3) has a unique solution for each pair  $(x(\cdot), p(\cdot)) \in L^2([a, \infty); X) \times L^2([a, \infty); X)$ . Therefore, the mapping  $J$ ,

$$L^2([a, \infty); X) \times L^2([a, \infty); X) \longrightarrow L^2([a, \infty); X) \times L^2([a, \infty); X), \quad (2.5)$$

given by

$$J(x(\cdot), p(\cdot)) := (X(\cdot), P(\cdot)) \quad (2.6)$$

is well defined.

Let  $J(x_1(\cdot), p_1(\cdot)) = (X_1(\cdot), P_1(\cdot))$  and  $J(x_2(\cdot), p_2(\cdot)) = (X_2(\cdot), P_2(\cdot))$ . Since  $X_1(\cdot) - X_2(\cdot) \in L^2([a, \infty); X)$  and  $P_1(\cdot) - P_2(\cdot) \in L^2([a, \infty); X)$ , there exists a sequence of real numbers  $a < t_1 < t_2 < \dots < t_k < \dots$  such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  and

$$X_1(t_k) - X_2(t_k) \longrightarrow 0, \quad P_1(t_k) - P_2(t_k) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \quad (2.7)$$

Note that

$$\begin{aligned} & \frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle \\ &= \langle F_{\beta_0}(X_1, P_1, t) - F_{\beta_0}(X_2, P_2, t) + \alpha \delta(p_1 - p_2) + \delta(F(x_1, p_1, t) - F(x_2, p_2, t)), P_1 - P_2 \rangle \\ & \quad + \langle G_{\beta_0}(X_1, P_1, t) - G_{\beta_0}(X_2, P_2, t) + \alpha \delta(x_1 - x_2) + \delta(G(x_1, p_1, t) - G(x_2, p_2, t)), X_1 - X_2 \rangle \\ &:= I_1 + I_2. \end{aligned} \quad (2.8)$$

Since

$$F_{\beta_0}(X_1, P_1, t) - F_{\beta_0}(X_2, P_2, t) = -\alpha(1 - \beta_0)(P_1 - P_2) + \beta_0(F(X_1, P_1, t) - F(X_2, P_2, t)) \quad (2.9)$$

implies that

$$\begin{aligned} I_1 &= -\alpha(1 - \beta_0) \|P_1 - P_2\|^2 + \beta_0 \langle F(X_1, P_1, t) - F(X_2, P_2, t), P_1 - P_2 \rangle \\ & \quad + \alpha \delta \langle p_1 - p_2, P_1 - P_2 \rangle + \delta \langle F(x_1, p_1, t) - F(x_2, p_2, t), P_1 - P_2 \rangle, \end{aligned} \quad (2.10)$$

similarly,

$$G_{\beta_0}(X_1, P_1, t) - G_{\beta_0}(X_2, P_2, t) = -\alpha(1 - \beta_0)(X_1 - X_2) + \beta_0(G(X_1, P_1, t) - G(X_2, P_2, t)) \quad (2.11)$$

implies that

$$\begin{aligned} I_2 &= -\alpha(1 - \beta_0) \|X_1 - X_2\|^2 + \beta_0 \langle G(X_1, P_1, t) - G(X_2, P_2, t), X_1 - X_2 \rangle \\ & \quad + \alpha \delta \langle x_1 - x_2, X_1 - X_2 \rangle + \delta \langle G(x_1, p_1, t) - G(x_2, p_2, t), X_1 - X_2 \rangle. \end{aligned} \quad (2.12)$$

It follows from the estimates for  $I_1, I_2$ , and the assumption (I) that

$$\begin{aligned}
 I_1 + I_2 &\leq -\alpha(\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
 &\quad + \alpha\delta(\|p_1 - p_2\|\|P_1 - P_2\| + \|x_1 - x_2\|\|X_1 - X_2\|) \\
 &\quad + \delta\|F(x_1, p_1, t) - F(x_2, p_2, t)\|\|P_1 - P_2\| \\
 &\quad + \delta\|G(x_1, p_1, t) - G(x_2, p_2, t)\|\|X_1 - X_2\| \\
 &\leq -\alpha(\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
 &\quad + \delta(2L + \alpha)(\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|p_1 - p_2\|^2).
 \end{aligned}
 \tag{2.13}$$

Therefore,

$$\begin{aligned}
 \frac{d}{dt} \langle X_1(t) - X_2(t), P_1(t) - P_2(t) \rangle \\
 \leq -\alpha(\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) \\
 + \delta(2L + \alpha)(\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|p_1 - p_2\|^2).
 \end{aligned}
 \tag{2.14}$$

Integrating between  $a$  and  $t_k$ , we have

$$\begin{aligned}
 \langle X_1(t_k) - X_2(t_k), P_1(t_k) - P_2(t_k) \rangle - \langle X_1(a) - X_2(a), P_1(a) - P_2(a) \rangle \\
 \leq -\alpha \int_a^{t_k} (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) dt + \delta(2L + \alpha) \\
 \times \int_a^{t_k} (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2 + \|x_1 - x_2\|^2 + \|p_1 - p_2\|^2) dt.
 \end{aligned}
 \tag{2.15}$$

Letting  $k \rightarrow \infty$  and noting that (2.7), we obtain

$$\int_a^\infty (\|X_1 - X_2\|^2 + \|P_1 - P_2\|^2) dt \leq \frac{2\delta L + \delta\alpha}{\alpha - 2\delta L - \delta\alpha} \int_a^\infty (\|x_1 - x_2\|^2 + \|p_1 - p_2\|^2) dt.
 \tag{2.16}$$

Choose a small  $\delta$  (independent of  $\beta_0$ ) such that

$$\frac{2\delta L + \delta\alpha}{\alpha - 2\delta L - \delta\alpha} \leq \frac{1}{2},
 \tag{2.17}$$

then  $J$  is a contractive mapping and hence has a unique fixed point. Thus, (2.3) becomes

$$\begin{aligned}
 \dot{x}(t) &= A(t)x(t) + F_{\beta_0+\delta}(x, p, t) + \varphi(t), \\
 x(a) &= x_0, \\
 \dot{p}(t) &= -A^*(t)p(t) + G_{\beta_0+\delta}(x, p, t) + \psi(t), \\
 x(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X).
 \end{aligned}
 \tag{2.18}$$

This shows that system (2.1) has a unique solution on  $[a, \infty)$  for  $\beta \in [\beta_0, \beta_0 + \delta]$ . The proof is complete.  $\square$

LEMMA 2.2. System (2.1) has a unique solution on  $[a, \infty)$  for  $\beta = 0$ , that is, the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) - \alpha p(t) + \varphi(t), \\ x(0) &= x_0, \\ \dot{p}(t) &= -A^*(t)p(t) - \alpha x(t) + \psi(t), \\ x(\cdot) \in L^2([a, \infty); X), \quad p(\cdot) &\in L^2([a, \infty); X), \end{aligned} \tag{2.19}$$

has a unique solution on  $[a, \infty)$ .

For the proof, see [2, Section 6.2, Chapter III].

### 3. Main theorem

THEOREM 3.1. System (1.4) has a unique solution under assumptions (I) and (II).

*Proof.* By Lemma 2.2, system (2.1) has a unique solution on  $[a, \infty)$  in the case  $\beta_0 = 0$ . It follows from Lemma 2.1 that there exists a real number  $\delta > 0$  such that (2.1) has a unique solution on  $[a, \infty)$  for any  $\beta \in [0, \delta]$  and  $\varphi, \psi \in L^2([a, \infty); X)$ . Let  $\beta_0 = \delta$  in Lemma 2.1. Repeating this procedure implies that (2.1) has a unique solution on  $[a, \infty)$  for any  $\beta \in [\delta, 2\delta]$  and  $\varphi, \psi \in L^2([a, \infty); X)$ . After finitely many steps, one can show that system (2.1) has a unique solution for  $\beta = 1$ . Therefore, it is proved that system (1.4) has a unique solution on  $[a, \infty)$  by letting  $\beta = 1$ ,  $\varphi(t) \equiv 0$ , and  $\psi(t) \equiv 0$ . □

Remark 3.2. Consider system (1.5). Note that

$$\begin{aligned} &\langle F(x_1, p_1, t) - F(x_2, p_2, t), p_1 - p_2 \rangle + \langle G(x_1, p_1, t) - G(x_2, p_2, t), x_1 - x_2 \rangle \\ &= \langle B(t)(p_1 - p_2), p_1 - p_2 \rangle + \langle C(t)(x_1 - x_2), x_1 - x_2 \rangle. \end{aligned} \tag{3.1}$$

By Theorem 3.1, system (1.5) has a unique solution if it is assumed that both  $B(t)$  and  $C(t)$  are uniformly negative definite on  $[a, \infty)$ , that is, there exists a real number  $\gamma > 0$  such that  $\langle B(t)x, x \rangle \leq -\gamma\|x\|^2$  and  $\langle C(t)x, x \rangle \leq -\gamma\|x\|^2$  for all  $x \in X$ ,  $x \neq 0$ , and  $t \in [a, \infty)$ .

Remark 3.3. Consider the control system

$$\dot{x}(t) = A(t)x(t) + Bu(t), \quad x(a) = x_0, \tag{3.2}$$

with a quadratic cost functional

$$J[u(\cdot)] = \int_a^\infty [\langle Qx(t), x(t) \rangle + \langle Ru(t), u(t) \rangle] dt, \tag{3.3}$$

where  $u(t)$  and  $x(t)$  take values in Hilbert spaces  $U$  and  $X$ , where  $B \in \mathcal{L}[U, X]$ , and where  $Q \in \mathcal{L}[X, X]$  and  $R \in \mathcal{L}[U, U]$  are selfadjoint operators.

From optimal control theory, the following Hamiltonian system is derived:

$$\begin{aligned}
 \dot{x}(t) &= A(t)x(t) - BR^{-1}Bp(t), \\
 x(a) &= x_0, \\
 \dot{p}(t) &= -A^*(t)p(t) - Qx(t), \\
 x(\cdot) &\in L^2([a, \infty); X), \quad p(\cdot) \in L^2([a, \infty); X).
 \end{aligned} \tag{3.4}$$

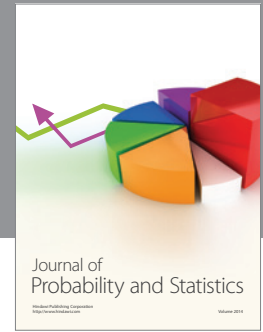
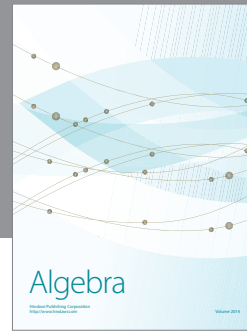
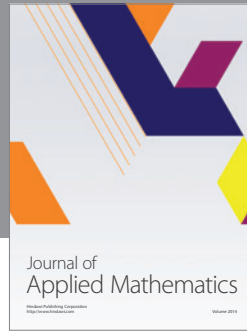
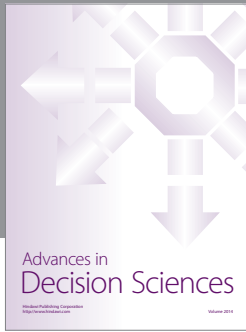
This is a special case of system (1.5). Therefore, system (3.4) has a unique solution if both  $BR^{-1}B$  and  $Q$  are positive definite.

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