NOTE ON WEIGHTED CARLEMAN-TYPE INEQUALITY

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Received 30 October 2003 and in revised form 19 August 2004

A double inequality involving the constant e is proved by using an inequality between the logarithmic mean and arithmetic mean. As an application, we generalize the weighted Carleman-type inequality.

1. Introduction

Let $p > 1$ and $a_n \geq 0$ with $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1.1)$$

The constant $(p/(p-1))^p$ is the best possible.

Inequality (1.1) is due to Hardy [6, page 239].

Replacing a_n in (1.1) by $a_n^{1/p}$ for $n \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{\infty} \left(\frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n. \quad (1.2)$$

In (1.2), letting $p \rightarrow \infty$, then the following Carleman inequality [6, page 249] is deduced:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (1.3)$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. The constant e is the best possible.

Carleman's inequality (1.3) was generalized in [6, page 256] by Hardy as follows. Let $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^n \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1.4)$$

Note that inequality (1.4) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [5], Hardy himself said that it was Pölya who pointed out this inequality to him.

In several recent papers [2, 4, 11, 12, 13, 14, 15], some strengthened and generalized results of (1.3) and (1.4) have been given by estimating the weight coefficient $(1 + 1/n)^n$.

For information about the history of both Hardy's inequality and Carleman-type inequalities, please refer to [7, 9].

In this note, we will give a generalization of (1.4) as follows.

THEOREM 1.1. *Let $0 < \lambda_{n+1} \leq \lambda_n$ with $\Lambda_n = \sum_{m=1}^n \lambda_m \geq 1$ and $\lim_{n \rightarrow \infty} \Lambda_n = \infty$, and let $a_n \geq 0$ for $n \in \mathbb{N}$ satisfying $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. Then for $0 < p \leq 1$,*

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & \leq \frac{1}{p} \sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{\Lambda_n/\lambda_n} \right)^{p\Lambda_n/\lambda_n} \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right], \end{aligned} \quad (1.5)$$

in particular,

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & < \frac{e^p}{p} \sum_{n=1}^{\infty} \left[\left(1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \right], \end{aligned} \quad (1.6)$$

where

$$c_k^{\lambda_k} = \frac{(\Lambda_{k+1})^{\Lambda_k}}{(\Lambda_k)^{\Lambda_{k-1}}}. \quad (1.7)$$

Remark 1.2. In particular, taking in (1.6) $p = 1$, we obtain the following strengthened Hardy's inequality:

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right) \lambda_n a_n. \quad (1.8)$$

Taking in (1.8) $\lambda_n \equiv 1$, we obtain the following strengthened Carleman's inequality:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1-2/e}{n} \right) a_n. \quad (1.9)$$

2. Lemma

The well-known arithmetic mean $A(a, b)$ and logarithmic mean $L(a, b)$ of two positive numbers a and b are defined, respectively, for $a = b$ by $A(a, b) = L(a, b) = a$ and for $a \neq b$

by

$$A(a, b) = \frac{a+b}{2}, \quad L(a, b) = \frac{b-a}{\ln b - \ln a}. \quad (2.1)$$

For $a \neq b$, we have

$$L(a, b) < A(a, b). \quad (2.2)$$

See [1] and the references therein.

LEMMA 2.1. *Let $x \geq 1$ be a real number. Then*

$$e\left(1 - \frac{1/2}{x}\right) < \left(1 + \frac{1}{x}\right)^x \leq e\left(1 - \frac{1-2/e}{x}\right). \quad (2.3)$$

The constants $1/2$ and $1 - 2/e$ are best possible.

Proof. Inequality (2.3) is equivalent to

$$1 - \frac{2}{e} \leq x \left[1 - \frac{1}{e} \left(1 + \frac{1}{x} \right)^x \right] < \frac{1}{2}. \quad (2.4)$$

Define a function f for $x > 0$ by

$$f(x) = x \left[1 - \frac{1}{e} \left(1 + \frac{1}{x} \right)^x \right]. \quad (2.5)$$

In order to prove (2.4), it is sufficient to show that the function f is strictly increasing on $[1, \infty)$ and with

$$f(1) = 1 - \frac{2}{e}, \quad \lim_{x \rightarrow \infty} f(x) = \frac{1}{2}. \quad (2.6)$$

The following proof shows that in fact $f'(x) > 0$ holds on $(0, \infty)$.

Easy computation yields

$$ef'(x) = e - [1 + xg(x)] \left(1 + \frac{1}{x} \right)^x, \quad (2.7)$$

where

$$g(x) = \ln \left(1 + \frac{1}{x} \right) - \frac{1}{x+1} = \frac{1}{L(x, x+1)} - \frac{1}{x+1}. \quad (2.8)$$

Now we are in a position to prove $f'(x) > 0$, which is equivalent to

$$h(x) = [1 + xg(x)] \left(1 + \frac{1}{x} \right)^x < e. \quad (2.9)$$

Differentiation yields

$$h'(x) = \left[xg^2(x) + 2g(x) - \frac{1}{(x+1)^2} \right] \left(1 + \frac{1}{x} \right)^x. \quad (2.10)$$

In the following we show $h'(x) > 0$. Clearly, the equation

$$xt^2 + 2t - \frac{1}{(x+1)^2} = 0 \quad (2.11)$$

has two roots

$$t_{1,2} = \frac{-(x+1) \pm \sqrt{(x+1)^2 + x}}{x(x+1)}. \quad (2.12)$$

To prove $h'(x) > 0$, it is sufficient to show that

$$\frac{-(x+1) + \sqrt{(x+1)^2 + x}}{x(x+1)} = t_2 < g(x) = \frac{1}{L(x, x+1)} - \frac{1}{x+1}, \quad (2.13)$$

which is equivalent to

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{1}{L(x, x+1)}. \quad (2.14)$$

Inequality (2.14) holds based on the following fact:

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{2}{2x+1} = \frac{1}{A(x, x+1)} < \frac{1}{L(x, x+1)}. \quad (2.15)$$

Hence, the function h is increasing on $(0, \infty)$, and then $h(x) < \lim_{x \rightarrow \infty} h(x) = e$. This means $f'(x) > 0$, and then

$$1 - \frac{2}{e} = f(1) < \lim_{x \rightarrow \infty} f(x). \quad (2.16)$$

Using Maclaurin formula

$$(1+t)^{1/t} = e - \frac{e}{2}t + o(t), \quad (2.17)$$

we have

$$\lim_{n \rightarrow \infty} f(n) = \lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f\left(\frac{1}{t}\right) = \lim_{t \rightarrow 0^+} \frac{(et)/2 + o(t)}{et} = \frac{1}{2}. \quad (2.18)$$

The proof of Lemma 2.1 is complete. $\square$

Remark 2.2. There are other very sharp estimates of the crucial factor $(1 + 1/n)^n$ in [8] and the references therein.

3. Proof of Theorem 1.1

By the power mean inequality, we have

$$\prod_{m=1}^n \alpha_m^{q_m} \leq \left(\sum_{m=1}^n q_m \alpha_m^p \right)^{1/p}, \quad (3.1)$$

where $p \geq 0$, $\alpha_m \geq 0$, and $q_m > 0$ for $m \in \mathbb{N}$ with $\sum_{m=1}^n q_m = 1$.

Let $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m / \Lambda_m$, then we obtain

$$(c_1 a_1)^{\lambda_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n / \Lambda_n} \leq \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p}. \quad (3.2)$$

Further, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n / \Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \\ &\leq \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p}. \end{aligned} \quad (3.3)$$

By the following inequality (see [3, 10])

$$\left(\sum_{m=1}^n z_m \right)^t \leq t \sum_{m=1}^n z_m \left(\sum_{k=1}^m z_k \right)^{t-1}, \quad (3.4)$$

where $t \geq 1$ is constant and $z_m \geq 0$ for $m \in \mathbb{N}$, it is easy to see that

$$\begin{aligned} \left(\frac{1}{\Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} &\leq \frac{1}{\Lambda_n} \left(\sum_{m=1}^n \lambda_m (c_m a_m)^p \right)^{1/p} \\ &\leq \frac{1}{p \Lambda_n} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}, \end{aligned} \quad (3.5)$$

where $\Lambda_n \geq 1$ and $0 < p \leq 1$. Thus, we obtain from (3.3) and (3.5) that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ &\leq \frac{1}{p} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \sum_{m=1}^n \lambda_m (c_m a_m)^p \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ &= \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{\lambda_{n+1}}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \right) \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.6)$$

Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n}$ for $n \in \mathbb{N}$ and setting $\Lambda_0 = 0$, we get from $0 < \lambda_{n+1} \leq \lambda_n$ that

$$c_n = \left[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}} \right]^{1/\Lambda_n} = \left(1 + \frac{\lambda_{n+1}}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \Lambda_n \leq \left(1 + \frac{\lambda_n}{\Lambda_n} \right)^{\Lambda_n/\lambda_n} \Lambda_n. \quad (3.7)$$

This implies that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & \leq \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n \Lambda_{n+1}} \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \frac{1}{\Lambda_m} \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ & \leq \frac{1}{p} \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m} \right)^{p\Lambda_m/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left(\sum_{k=1}^m \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.8)$$

Hence, we obtain from the above inequality and Lemma 2.1 that

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \\ & < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \frac{1-2/e}{\Lambda_n/\lambda_n} \right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^n \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \quad (3.9)$$

The last inequality holds strictly since the right-hand inequality of (2.3) is valid if and only if $n = 1$. The proof is complete.

Acknowledgments. The authors are indebted to the anonymous referees for their much detailed comments and suggestions to improve this note. The first and third authors were supported in part by the National Natural Science Foundation of China Grant 10001016, Science Foundation for the Prominent Youth of Henan Province Grant 0112000200, Science Foundation of Henan Innovation Talents at Universities, and the Doctor Fund of Henan Polytechnic University, China. The second author was supported in part by the Research Grants Council of the Hong Kong SAR (project no. HKU7040/03P), China.

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