APPLICATION OF UNIFORM ASYMPTOTICS METHOD TO ANALYZING THE ASYMPTOTIC BEHAVIOUR OF THE GENERAL FOURTH PAINLEVÉ TRANSCENDENT

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We use the uniform asymptotics method proposed by A. P. Bassom et al. (1998) to study the general fourth Painlevé transcendent, find a group of its asymptotics and the corresponding monodromic data, and prove its existence and "uniqueness."

1. Introduction

With more and more discovery of their applications in the areas of physics such as quantum mechanics and solitons, Painlevé transcendents have attracted significant attention of many mathematicians in about the last twenty years. One direction of the research in this area is to find the asymptotics behaviour of the Painlevé transcendents as their independent variable approaches the singular points. The work about this can be found in [1, 3, 5, 6, 7, 8, 9, 10, 11] and some other papers. But there are not many results about the asymptotics of the fourth Painlevé equation

(PIV)

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{1.1}$$

where α and β are constant parameters. In [3], Clarkson and McLeod studied a special case of (PIV) when $\beta = 0$ and obtained a group of asymptotics to its solutions. In [1], Abdullayev applied the classical successive approximation method to (PIV) when $\beta = 0$ and proved the existence of one of the asymptotic expressions obtained by Clarkson and McLeod in [3]. In [10], Lu studied the asymptotic behaviour of the solutions to (PIV) when $\beta > 0$ and $i\alpha$ is real, and obtained a group of asymptotic expressions of the real solutions when x approaches infinity along the ray $\arg(x) = \pi/4$.

In this paper, we study the behaviour of the real solutions of (PIV) when $\beta > 0$ and $\alpha > 0$, and obtain the following result about the asymptotics of its real solutions. Noticing that (PIV) does not change when we change x to $-x$ and y to $-y$, we only need to consider the asymptotics of the solutions in one side of the x-axis based on the result of Lemma 2.1 in Section 2.

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THEOREM 1.1. *If $\beta > 0$, the solutions of (PIV) cannot cross the x-axis. Furthermore, if $\alpha > 0$, the only negative solution of the Painlevé equation (PIV) that does not blow up at any finite point when x goes to positive infinity is oscillating as $x \to +\infty$ and it satisfies the following.*

(1) *As $x \to +\infty$,*

$$y = -\frac{2}{3}x \pm d\cos\phi + O(x^{-1}),$$
$$y' = \frac{2\sqrt{3}x}{3} d\sin\phi + O(1), \tag{1.2}$$

where $\phi = (\sqrt{3}/3)x^2 - (\sqrt{3}/4)d^2\log x + \phi_0 + O(x^{-1})$, d and ϕ_0 are real parameters.

(2) *As x goes to negative infinity, y blows up at a finite point of x.*

It is important to point out that we care about both the existence and uniqueness of the asymptotic expression in (1.2). There are several major differences between the results in [3] and ours. First, Clarkson and McLeod proved that, as $\beta = 0$, (PIV) has a solution approaching 0 when x goes to negative infinity. Because of the addition of the β term, not only can we prove this solution does not exist any more, but we can also prove that every solution blows up in that direction. Second, Clarkson and McLeod obtained three different kinds of asymptotics when x goes to positive infinity. Here with the extra term, we can prove that there is no solution approaching $-2x$ any more. Our major result is that we can rigorously prove the existence, " uniqueness," and the differentiability of the asymptotic expression in (1.2) by applying the uniform asymptotics method to this problem. Of course, it is also possible to use the successive approximation method used by Abdullayev [1] to prove the existence of the asymptotic expressions in (1.2). A very important side product of this paper is the monodromic data corresponding to the asymptotic expression (1.2) that may be used to find the necessary connection formula in the future.

This paper is planned as follows. In Section 2, we use elementary analysis to prove the general part of the theorem that states the rough behaviour of the solutions of (PIV) and the general form of the asymptotics of its negative solution as x approaches positive infinity (or the general asymptotics of its positive solution as x approaches negative infinity). Section 3 is the major part of this paper in which we apply the uniform asymptotics method to the general form of the asymptotics obtained in Section 2, find the corresponding monodromic data, and then use the monodromic data theory to refine the general asymptotics expression found in Section 2. In Section 4, we get back to elementary asymptotics analysis and prove the statement (2) of Theorem 1.1.

2. The general form of the asymptotics as $x \to +\infty$

To find a general form of the asymptotics as $x \to +\infty$, we first use the transformation $x = \sqrt{t}$ and $y = xz = \sqrt{t}z$ to change (PIV) into the following equation:

$$8zz'' - 4z'^2 + \frac{8zz'}{t} - 3z^4 - 8xz^3 - 4z^2 + \left(\frac{4\alpha}{t} - \frac{1}{t^2}\right)z^2 - \frac{2\beta}{t^2} = 0. \tag{2.1}$$

LEMMA 2.1. *If $\beta > 0$, t_0 is a positive constant and z is a solution of (2.1) satisfying $z(t_0) \neq 0$, then $z \neq 0$ for all $t > 0$.*

Proof. Suppose that there exists a real number $t_1 > 0$ such that $z(t_1) = 0$. Because $z(t)$ is analytic nearby t_1, we can expand it into a power series

$$z(t) = C(t - t_1)^n + O\Big((t - t_1)^{n+1}\Big), \tag{2.2}$$

where $C \neq 0$ and $n \geq 1$. Substituting expression (2.2) into (2.1), we obtain

$$8Cn(n-1)(t-t_1)^{n-2} - 4Cn^2(t-t_1)^{n-2} - \frac{2\beta}{t^2C}(t-t_1)^{-n} + O\Big((t-t_1)^{n-1}\Big) = 0. \tag{2.3}$$

Comparing the terms with lowest power, we get that $n = 1$ and $2C + (\beta/t_1^2C) = 0$. This is clearly a contradiction to the fact that $C > 0$ and $\beta > 0$. □

Now, we can introduce new variables t and u by the following transformations:

$$x = \sqrt{t}, \qquad y = -xu^2(t). \tag{2.4}$$

The Painlevé equation (PIV) is changed by (2.4) into

$$u'' + t^{-1}u' = \frac{3}{16}u^5 - \frac{1}{2}u^3 + \frac{1}{4}\left(1 - \frac{\alpha}{t}\right)u + \frac{u}{16t^2} + \frac{\beta}{8t^2u^3}. \tag{2.5}$$

Thanks to Lemma 2.1, we can assume that $u > 0$. Because $3u^5 - 8u^3 + 4u = u(3u^2 - 2)(u^2 - 2)$, u may either blow up at a finite point or approach 0, $\sqrt{2}$, or $\sqrt{2/3}$ as $t \to +\infty$ heuristically. Surprisingly, we will be able to prove that it is impossible for u to approach 0 or $\sqrt{2}$.

Lemma 2.2. *If $\alpha > 0$, $\beta > 0$, and u does not blow up at any finite value of t as $t \to +\infty$, u is bounded and satisfies $u < \sqrt{2} + o(1)$.*

Proof. First, we prove that u cannot monotonously go to infinity as $t \to +\infty$. Otherwise, there would be constants C_1 and $t_0 > 0$ such that

$$u''(t) + t^{-1}u'(t) > C_1u^5(t) \quad \text{for } t \geq t_0. \tag{2.6}$$

Applying the transformation $\tau = \ln t$ and $v(\tau) = u(t)$ to the inequality (2.6) yields

$$v'' > C_1e^{2\tau}v^5(\tau) > C_1v^5(\tau) \quad \text{for } \tau \geq \tau_0 = \ln t_0. \tag{2.7}$$

Multiplying both sides of (2.7) by $2v'$ which is positive and integrating it, we obtain

$$v'^2(\tau) > \frac{1}{3}C_1v^6(\tau) - \frac{1}{3}C_1v^6(\tau_0) + v'^2(\tau_0) \quad \text{for } \tau \geq \tau_0. \tag{2.8}$$

Hence, there exist constants C_2 and $\tau_1 > \tau_0$ such that

$$v'(\tau) > C_2v^3(\tau) \quad \text{for } \tau \geq \tau_1. \tag{2.9}$$

Integrating this inequality again, we obtain

$$v^2(\tau) > \frac{1}{v^{-2}(\tau_1) + C_2\tau_1 - C_2\tau} \quad \text{for } \tau \geq \tau_1 \tag{2.10}$$

which forces v to blow up at a finite point.

Using a similar argument as we use to take care of Case 2 in Section 4, we can show that u cannot approach a finite limit larger than $\sqrt{2}$ as t goes to infinity. Assume that u is oscillating when $t \to +\infty$ and t_2 is a large number where u attains its maximum value. Then, $u'(t_2) = 0$, $u''(t_2) \leq 0$, and

$$\frac{3}{16}u^5(t_2) - \frac{1}{2}u^3(t_2) + \frac{1}{4}(1 - \alpha t_2^{-1})u(t_2) \leq -\frac{u(t_2)}{16t_2^2} - \frac{\beta}{8t_2^2 u^3(t_2)} < 0. \tag{2.11}$$

Therefore,

$$\sqrt{\frac{4}{3} - \frac{2}{3}\sqrt{1 + 3\alpha t_2^{-1}}} < u(t_2) < \sqrt{\frac{4}{3} + \frac{2}{3}\sqrt{1 + 3\alpha t_2^{-1}}}, \tag{2.12}$$

and the lemma is proved. □

Lemma 2.3. *If $\alpha > 0$, $\beta > 0$, and u does not blow up at any finite point, then $u = \sqrt{2/3} + O(t^{-1/2})$ and $u' = O(t^{-1/2})$ as t goes to positive infinity.*

Proof. Substituting $u = \sqrt{2/3} + t^{-1/2}w$ into (2.5) yields

$$\begin{aligned} w'' + \frac{1}{3}w = &-\frac{1}{4}\sqrt{\frac{2}{3}}t^{-1/2}(w^2 + \alpha) + \frac{1}{4}t^{-1}(3w^3 - \alpha w) \\ &+ \frac{1}{16}\sqrt{\frac{2}{3}}t^{-3/2}(15w^4 + 1) + \frac{3}{16}t^{-2}(w^5 - w) + \frac{\beta t^{-3/2}}{8u^3}. \end{aligned} \tag{2.13}$$

Multiplying both sides by $2w'$ and integrating it, we have

$$\begin{aligned} &w'^2 + \frac{1}{3}w^2 + \frac{1}{6}\sqrt{\frac{2}{3}}t^{-1/2}w^3 - \frac{3}{8}t^{-1}w^4 - \frac{3}{8}\sqrt{\frac{2}{3}}t^{-3/2}w^5 - \frac{1}{16}t^{-2}w^6 + \frac{1}{4}\alpha\int_{t_0}^{t} t^{-2}w^2 dt \\ &= -\frac{1}{12}\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-3/2}(w^3 + 3\alpha w)dt + \frac{3}{8}\int_{t_0}^{t} t^{-2}w^4 dt + \frac{9}{16}\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-5/2}w^5 dt \\ &\quad + \frac{1}{8}\int_{t_0}^{t} t^{-3}w^6 dt + \frac{3}{16}\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-5/2}w dt - \frac{3}{8}\int_{t_0}^{t} t^{-3}w^2 dt - \frac{1}{2}\sqrt{\frac{2}{3}}\alpha t^{-1/2}w \\ &\quad - \frac{1}{4}\alpha t^{-1}w^2 + \frac{1}{8}\sqrt{\frac{2}{3}}t^{-3/2}w - \frac{3}{16}t^{-2}w^2 - \frac{\beta t^{-1}}{8u^2} - \frac{\beta}{8}\sqrt{\frac{2}{3}}\int_{t_0}^{t} \frac{t^{-2}}{u^3}dt + C_3. \end{aligned} \tag{2.14}$$

Noticing that

$$\begin{aligned}
-\int_{t_0}^{t} t^{-3/2}(w^3+3\alpha w)\,dt &= \frac{-1}{6\lambda^2+2}\int_{t_0}^{t} t^{-3/2}(\lambda|w|+w)^3\,dt-\frac{3}{2}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|+w)\,dt\\
&\quad+\frac{1}{6\lambda^2+2}\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)^3\,dt+\frac{3}{2}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)\,dt\\
&\le-\frac{(\lambda-1)^3}{6\lambda^2+2}\int_{t_0}^{t} t^{-3/2}w^3\,dt+\frac{1}{6\lambda^2+2}\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)^3\,dt\\
&\quad+\frac{3}{2}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)\,dt,\\
-\sqrt{\frac{2}{3}}<t^{-1/2}w<\sqrt{\frac{4}{3}+\frac{2}{3}\sqrt{1+3\alpha t^{-1}}}&-\sqrt{\frac{2}{3}}<\frac{7}{9}\sqrt{\frac{2}{3}},\quad \text{for large } t,
\end{aligned}\tag{2.15}$$

we can obtain

$$\frac{1}{3}w^2+\frac{1}{6}\sqrt{\frac{2}{3}}t^{-1/2}w^3-\frac{3}{8}t^{-1}w^4-\frac{3}{8}\sqrt{\frac{2}{3}}t^{-3/2}w^5-\frac{1}{16}t^{-2}w^6\ge\frac{1}{10}w^2.\tag{2.16}$$

Since $t^{-1/2}w$ is bounded, we can claim that both $\int_{t_0}^{\infty} t^{-5/2}w\,dt$ and $\int_{t_0}^{\infty} t^{-3}w^2\,dt$ converge. Hence, for $\lambda>1$, the following inequality holds:

$$\begin{aligned}
&-\frac{1}{12}\sqrt{\frac{2}{3}}\frac{\lambda-1}{6\lambda^2+2}\int_{t_0}^{t} t^{-3/2}|w|^3\,dt+\frac{3}{8}\int_{t_0}^{t} t^{-2}w^4\,dt+\frac{9}{16}\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-5/2}w^5\,dt+\frac{1}{8}\int_{t_0}^{t} t^{-3}w^6\,dt\\
&\quad\le 0,\quad \text{for some large } t_0.
\end{aligned}\tag{2.17}$$

Substituting the above inequalities into (2.14) yields

$$\begin{aligned}
w'^2+\frac{1}{10}w^2 &\le \frac{1}{24(3\lambda^2+1)}\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)^3\,dt\\
&\quad+\frac{1}{12}\sqrt{\frac{3}{2}}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)\,dt+C_2.
\end{aligned}\tag{2.18}$$

Hence,

$$(\lambda|w|-w)^2\le\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)^3\,dt+(\lambda^2+1)\sqrt{\frac{2}{3}}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)\,dt+C_2.\tag{2.19}$$

Let

$$I=\sqrt{\frac{2}{3}}\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)^3\,dt+(\lambda^2+1)\sqrt{\frac{2}{3}}\alpha\int_{t_0}^{t} t^{-3/2}(\lambda|w|-w)\,dt+C_2.\tag{2.20}$$

Then,

$$I' = \sqrt{\frac{2}{3}}t^{-3/2}(\lambda|w| - w)\left[(\lambda|w| - w)^2 + \frac{3}{2}(\lambda^2+1)\alpha\right] \le \sqrt{\frac{2}{3}}t^{-3/2}\sqrt{I}\left[I + \frac{3}{2}(\lambda^2+1)\alpha\right]. \tag{2.21}$$

Thus,

$$\tan^{-1}\sqrt{\frac{2I}{3(\lambda^2+1)\alpha}} \le \tan^{-1}\sqrt{\frac{2C_2}{3(\lambda^2+1)\alpha}} + \sqrt{\frac{4}{9(\lambda^2+1)\alpha}}\left(t_0^{-1/2} - t^{-1/2}\right). \tag{2.22}$$

We can take t_0 large enough such that the right side of (2.22) is less than $\pi/2$, and therefore, I is bounded and we complete the proof of the lemma. □

Combining all the lemmas in this section, we can conclude that the solution y of (PIV) either approaches $-(2/3)x$ as x approaches infinity or blows up at a finite point, and $y' = O(x)$ when y approaches $-(2/3)x$.

3. Monodromic data and proof of Theorem 1.1(1)

Now, based on the conclusion of the previous section, we seek a solution of the form

$$y = -\frac{2}{3}x + U(x), \qquad y' = \frac{2\sqrt{3}}{3}xV(x) \tag{3.1}$$

with $U(x) = O(1)$ and $V(x) = O(1)$ as $x \to \infty$.

Equation (PIV) can be obtained as the compatibility condition of the following linear systems [4]:

$$\frac{dY}{dz} = \begin{pmatrix} z + x + \dfrac{\theta_0 - v}{z} & u - \dfrac{yu}{2z} \\ \dfrac{2v - 2\theta_0 - 2\theta_\infty}{u} + \dfrac{2v(v-2\theta_0)}{yuz} & -z - x - \dfrac{\theta_0 - v}{z} \end{pmatrix} Y(z,x), \tag{3.2}$$

$$\frac{dY}{dx} = \begin{pmatrix} z & u \\ \dfrac{2(v - \theta_0 - \theta_\infty)}{u} & -z \end{pmatrix} Y(z,x). \tag{3.3}$$

Indeed, $d^2Y/dzdx = d^2Y/dxdz$ implies the following equations:

$$\frac{dy}{dx} = -4v + y^2 + 2xy + 4\theta_0, \tag{3.4}$$

$$\frac{du}{dx} = -u(y + 2x), \tag{3.5}$$

$$\frac{dv}{dx} = -\frac{2v^2}{y} + \left(\frac{4\theta_0}{y} - y\right)v + (\theta_0 + \theta_\infty)y, \tag{3.6}$$

where $\alpha = 2\theta_\infty - 1$ and $\beta = -8\theta_0^2$. It is important to point out that (PIV) is hiding in (3.4) and (3.6). Fokas et al. have proved the following result [4].

Proposition 3.1. *Let Y_j, $j = 1,\dots,4$ be a solution of (3.2) analytic in the neighborhood of infinity such that* $\det Y_j = 1$ *and $Y_j \sim Y_\infty$ as $|z| \to \infty$ in S_j, Y_∞ is the formal solution matrix of (3.2) in the neighborhood of infinity, and the sectors S_j are given by*

$$S_j : -\frac{\pi}{4} + (j-1)\frac{\pi}{2} \le \arg z < \frac{\pi}{4} + (j-1)\frac{\pi}{2}. \tag{3.7}$$

Then,

(1) *$Y_j \sim \widehat{Y}_\infty(z) z^{\operatorname{diag}(-\theta_\infty,\theta_\infty)} e^{\operatorname{diag}((z^2/2)+zx,-(z^2/2)-zx)}$ as $|z| \to \infty$ in S_j, where $\widehat{Y}_\infty$ is holomorphic at $z = \infty$, and $\widehat{Y}_\infty(z) \sim I + O(z^{-1})$,*

(2) *$Y_2(z) = Y_1(z)G_1$, $Y_3(z) = Y_2(z)G_2$, $Y_4(z) = Y_3(z)G_3$, $Y_1(z) = Y_4(ze^{2i\pi})G_4M_\infty$, where*

$$G_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \qquad G_3 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \qquad G_4 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \\ M_\infty = \operatorname{diag}\left(e^{2\pi i\theta_\infty}, e^{-2\pi i\theta_\infty}\right), \tag{3.8}$$

and the entries p, q, r, and s are independent of ξ,

(3) *all of the monodromy data of the system can be expressed in terms of two of the four entries p, q, r, and s.*

We need to find the monodromy data p and q corresponding to the asymptotic representations in (3.1). Since the values p and q are independent of the independent variable x, we will be able to solve the equation of the monodromic data and find the asymptotic expressions of U and V. To apply the uniform asymptotics method to system (3.2), we need to use a tricky transformation. It is interesting to notice that it is almost impossible to "integrate" the system (3.2) without the following tricky transformation and the magic number a.

Let

$$\widetilde{Y} = \begin{pmatrix} 1 & \dfrac{a}{\sqrt{2v}} \\ -\dfrac{1}{a}\sqrt{2v} & 1 \end{pmatrix} u^{-(1/2)\sigma_3} Y, \tag{3.9}$$

where the constant a will be determined later as needed. Then,

$$\frac{d\widetilde{Y}}{dz} = \frac{1}{2} \begin{pmatrix} \dfrac{1}{a}\sqrt{2v}B + \dfrac{a}{\sqrt{2v}}C & -\dfrac{2a}{\sqrt{2v}}A + B - \dfrac{a^2}{2v}C \\ -\dfrac{2}{a}\sqrt{2v}A + C - \dfrac{2v}{a^2}B & -\dfrac{1}{a}\sqrt{2v}B - \dfrac{aC}{\sqrt{2v}} \end{pmatrix} \widetilde{Y}. \tag{3.10}$$

Let $\phi = (B - (a^2C/2\nu) - (2a/\sqrt{2\nu})A)^{-1/2}\widetilde{Y}^{(1)}$, $z = x\eta$, and $\xi = ix^2$. Then,

$$\begin{aligned}\frac{d^2\phi}{d\eta^2} = x^2\Bigg\{ & z^2 + 2xz + x^2 + \frac{2x(\theta_0 - \nu)}{z} + \frac{\theta_0^2}{z^2} - 2\theta_\infty + \frac{2\nu(\nu - 2\theta_0)}{yz} \\ & - \frac{y(\nu - \theta_0 - \theta_\infty)}{z} + \frac{\sqrt{2\nu}y}{4az^2} - \frac{a\nu(\nu - 2\theta_0)}{\sqrt{2\nu}yz^2} \\ & - \frac{y/2 + a^2(\nu - 2\theta_0)/y - 2a/\sqrt{2\nu}(z^2 - \theta_0 + \nu)}{z[z - a^2z - y/2 - a^2\nu/y - 2a/\sqrt{2\nu}(z^2 + xz - \nu)]} \cdot \frac{\sqrt{2\nu}}{2a}\left(1 - \frac{y}{2z} + a^2 + \frac{a^2\nu}{yz}\right) \\ & + \frac{3[y/2 + a^2(\nu - 2\theta_0)/y - (2a/\sqrt{2\nu})(z^2 - \theta_0 + \nu)]^2}{4z^2[z - a^2z - y/2 - a^2\nu/y - (2a/\sqrt{2\nu})(z^2 + xz - \nu)]^2} \\ & - \frac{-y - 2a^2(\nu - 2\theta_0)/y - 4a(\theta_0 - \nu)/\sqrt{2\nu}}{z^2[z - a^2z - y/2 - a^2(\nu - 2\theta_0)/y - (2a/\sqrt{2\nu})(z^2 + xz + \theta_0 - \nu)]}\Bigg\}\phi. \end{aligned} \tag{3.11}$$

We denote $G = y/2 + a^2\nu/y - a\sqrt{2\nu}$, $H = -y/2 + a^2\nu/y$, and select $a = (i + \sqrt{3})/2$ such that $G - (2a/\sqrt{2\nu})z^2$ is a multiple of $3\eta + 1$. Hence,

$$\begin{aligned}\frac{d^2\phi}{d\eta^2} &= -\xi^2\Bigg\{\eta^2 + 2\eta + 1 + \frac{4}{27\eta} - \frac{2i\theta_\infty}{\xi} - \frac{i(U^2 + V^2 + (8/3)\theta_\infty)}{4\xi\eta} \\ &\quad + \frac{\sqrt{3}}{9\xi\eta^2} + \frac{i(\eta^2 - (\sqrt{2\nu}/2ax^2)G)[(1 + a^2)x\eta + H]}{\xi\eta^2[-(2a/\sqrt{2\nu})x^2\eta^2 + (1 - a^2 - (2ax/\sqrt{2\nu}))x\eta - G]} + O(\xi^{-3/2})\Bigg\}\phi \\ &= -\xi^2F(\eta,\xi). \end{aligned} \tag{3.12}$$

This equation has three turning points

$$\begin{aligned}\eta_{1,2} &= -\frac{1}{3} \pm \xi^{-1/2}T_0 + O(\xi^{-1/2}), \\ \eta_3 &= -\frac{4}{3} + O(\xi^{-1}), \end{aligned} \tag{3.13}$$

where η_2 takes the positive sign and $T_0^2 = i(U^2 + V^2)/4 + \sqrt{3}/3 = \mathrm{id}^2/4 + \sqrt{3}/3$ and the Stokes curve of this equation is defined by $\arg(\xi\eta^2) = k\pi$ or $\arg(\eta) = k\pi/2 - (1/2)\arg(\xi)$.

To transform (3.12) to an equation related to the parabolic cylinder equation, we define a constant $\hat{\alpha}$ and a new variable ζ by

$$\begin{aligned}\frac{1}{2}\pi i\hat{\alpha}^2 = \int_{-\hat{\alpha}}^{\hat{\alpha}} (\zeta^2 - \hat{\alpha}^2)^{1/2}d\zeta &= \int_{\eta_1}^{\eta_2} F^{1/2}(\eta,\xi)d\eta, \\ \int_{\hat{\alpha}}^{\zeta} (\zeta^2 - \hat{\alpha}^2)^{1/2}d\zeta &= \int_{\eta_2}^{\eta} F^{1/2}(\eta,\xi)d\eta. \end{aligned} \tag{3.14}$$

We need to find the asymptotic expression of ζ first. Let $\eta^* = -1/3 + \xi^{-1/2}T$ where T is a large parameter with $T \gg \xi^{1/4}$ determined as needed later and

$$\int_{\eta_2}^{\eta} F^{1/2}(\eta,\xi)d\eta = \int_{\eta_2}^{\eta^*} F^{1/2}(\eta,\xi)d\eta + \int_{\eta^*}^{\eta} F^{1/2}(\eta,\xi)d\eta = I_1 + I_2. \tag{3.15}$$

Then,

$$\begin{aligned}
I_1 &= -\frac{i\sqrt{3}}{2\xi}\left(T^2 - \frac{T_0^2}{2} - T_0^2\log(2T) + T_0^2\log T_0\right) + O\left(\frac{T_0^2}{\xi T^2}\right),\\
I_2 &= \int_{\eta^*}^{\eta} F^{1/2}(\eta,\xi)d\eta\\
&= \left\{\frac{\sqrt{3\eta}}{18}(3\eta+4)^{3/2} - \frac{i\theta_\infty}{\xi}\log\frac{\sqrt{3\eta+4}+\sqrt{3\eta}}{\sqrt{3\eta+4}-\sqrt{3\eta}} + \frac{i\sqrt{3}T_0^2}{2\xi}\log\frac{\sqrt{3\eta+4}+3i\sqrt{\eta}}{\sqrt{3\eta+4}-3i\sqrt{\eta}}\right.\\
&\quad - \frac{1}{4\xi}\sqrt{\frac{3\eta+4}{\eta}} + \frac{i}{2\xi}\log\frac{\sqrt{(3R_1+4)\eta}-\sqrt{(3\eta+4)R_1}}{\sqrt{(3R_1+4)\eta}+\sqrt{(3\eta+4)R_1}}\\
&\quad \left.+ \frac{i}{2\xi}\log\frac{\sqrt{(3R_2+4)\eta}-\sqrt{(3\eta+4)R_2}}{\sqrt{(3R_2+4)\eta}+\sqrt{(3\eta+4)R_2}} + O(\xi^{-3/2})\right\}\Bigg|_{\eta^*}^{\eta}\\
&= \frac{1}{2}\eta^2 + \eta + \frac{1}{3} - \frac{i\theta_\infty}{\xi}\log(3\eta) - \frac{\pi\sqrt{3}T_0^2}{3\xi} - \frac{\sqrt{3}}{4\xi} + \frac{\pi}{2\xi} - \frac{\sqrt{3}i}{6} + \frac{\sqrt{3}i}{2\xi}T^2\\
&\quad - \frac{\pi\theta_\infty}{3\xi} - \frac{i\sqrt{3}T_0^2}{2\xi}\log\left(\xi^{-1/2}T\right) + \frac{i}{\xi}\log 2 - \frac{i}{2\xi}\log(U+iV) + O(\eta^{-1}) + O(\xi^{-3/2}),\\
\widehat{\alpha}^2 &= -\frac{i\sqrt{3}T_0^2}{\xi} + o(\xi^{-1}),
\end{aligned} \tag{3.16}$$

where $R_{1,2} = (\sqrt{2\nu}/4ax^2)[(1-a^2-2ax/\sqrt{2\nu})x \pm \sqrt{(1-a^2-2ax/\sqrt{2\nu})^2x^2 + (8ax^2/\sqrt{2\nu})G}]$ with R_1 taking the + sign and R_2 taking the − sign. Moreover,

$$\begin{aligned}
&\frac{1}{2}\zeta^2 - \frac{\widehat{\alpha}^2}{2}\log(2\zeta) - \frac{\widehat{\alpha}^2}{4} + \frac{\alpha^2}{2}\log\alpha + O\left(\frac{\widehat{\alpha}^4}{\zeta^2}\right)\\
&\quad = I_1 + I_2\\
&\quad = -\frac{i\sqrt{3}}{2\xi}\left(T^2 - \frac{T_0^2}{2} - T_0^2\log(2T) + T_0^2\log T_0\right) + O\left(\frac{T_0^2}{\xi T^2}\right)\\
&\qquad + \frac{1}{2}\eta^2 + \eta + \frac{1}{3} + \frac{3M_1}{2\xi}\log(-3\eta) - \frac{\pi M_1\sqrt{3}}{3\xi} + \frac{\pi(3M_2+i)}{12\sqrt{3}\xi} - \frac{\sqrt{3}i}{6} + \frac{\sqrt{3}i}{2}\xi^{-1}T^2\\
&\qquad + \frac{\pi i M_1}{\xi} - \frac{\sqrt{3}M_1 i}{\xi}\log\left(\xi^{-1/4}T^{1/2}\right) + \frac{i(3M_2+i)}{2\sqrt{3}\xi}\log\left(\frac{\sqrt{7}i}{2}\xi^{-1/2}T\right).
\end{aligned} \tag{3.17}$$

Now, we get

$$\begin{aligned}\frac{1}{2}\zeta^2+\frac{\sqrt{3}iT_0^2}{2\xi}\log\zeta=\frac{1}{2}\eta^2+\eta+\frac{1}{3}-\frac{\sqrt{3}i}{6}-\frac{i\theta_\infty}{\xi}\log\eta-\frac{i}{2\xi}\log(U+iV)\\ +\frac{b}{\xi}+O\left(\frac{1}{\xi\eta}\right)+O(\xi^{-3/2}),\end{aligned}\tag{3.18}$$

where $b=((i\sqrt{3}/8)T_0^2-i\theta_\infty)\log 3-5\pi\sqrt{3}T_0^2/24-\sqrt{3}/4+\pi/2-\pi\theta_\infty/3+i\log 2$.

For the function $F(\eta,\xi)$ to be a polynomial on η, [2, Theorem 1] is proved. Actually, its proof is still valid for our case although our function $F(\eta,\xi)$ is rational on η. Hence, we have the following theorem.

Theorem 3.2. *Given any solution ϕ of (3.12), there exist constant c_1 and c_2 such that, uniformly for η on the Stokes curve, as $\xi\to\infty$,*

$$\left(\frac{\zeta^2-\hat{\alpha}^2}{F(\eta,\xi)}\right)^{-1/4}\phi(\eta,\xi)=\left\{(c_1+o(1))D_\nu\left(e^{\pi i/4}\sqrt{2\xi}\zeta\right)+(c_2+o(1))D_{-\nu-1}\left(e^{-\pi i/4}\sqrt{2\xi}\zeta\right)\right\},\tag{3.19}$$

where $\nu=-1/2+(1/2)i\xi\hat{\alpha}^2$ and $D_\nu(\tau)$, $D_{-\nu-1}(\tau)$ are the solutions of the parabolic cylinder equation.

By Proposition 3.1,

$$\begin{aligned}\tilde{Y}_1^{(11)}&\sim u^{-1/2}z^{-\theta_\infty}e^{(1/2)z^2+\xi z}\quad\text{as } z\longrightarrow\infty \text{ along } \arg(z)=-\frac{\pi}{4},\\ \tilde{Y}_1^{(12)}&\sim\frac{\mathrm{au}^{1/2}}{\sqrt{2\nu}}z^{\theta_\infty}e^{-(1/2)z^2-zx}\quad\text{as } z\longrightarrow\infty \text{ along } \arg(z)=-\frac{\pi}{4},\\ \tilde{Y}_2^{(11)}=\tilde{Y}_1^{(11)}+p\tilde{Y}_1^{(12)}&\sim u^{-1/2}z^{-\theta_\infty}e^{(1/2)z^2+zx}\quad\text{as } z\longrightarrow\infty \text{ along } \arg(z)=\frac{\pi}{4}.\end{aligned}\tag{3.20}$$

Using the asymptotic expressions for the parabolic cylinder function [2]

$$D_\nu(z)\sim\begin{cases}z^\nu e^{-(1/4)z^2} & \text{if } |\arg z|<\frac{3}{4}\pi,\\ z^\nu e^{-(1/4)z^2}-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\pm\pi i\nu}z^{-\nu-1}e^{(1/4)z^2} & \text{on } \arg z=\pm\frac{3}{4}\pi,\\ e^{-2\pi i\nu}z^\nu e^{-(1/4)z^2}-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\pi i\nu}z^{-\nu-1}e^{(1/4)z^2} & \text{on } \arg z=\frac{5}{4}\pi,\\ e^{-2\pi i\nu}z^\nu e^{-(1/4)z^2} & \text{if } \frac{5}{4}\pi<\arg z<\frac{11}{4}\pi,\end{cases}\tag{3.21}$$

and (3.20), we have to choose

$$\begin{aligned}\tilde{Y}_1^{(11)}&=c_1\eta^{1/2}\left(\frac{\zeta^2-\hat{\alpha}^2}{F(\eta,\xi)}\right)^{1/4}D_\nu\left(e^{\pi i/4}\sqrt{2\xi\zeta}\right),\\ \tilde{Y}_1^{(12)}&=c_2\eta^{1/2}\left(\frac{\zeta^2-\hat{\alpha}^2}{F(\eta,\xi)}\right)^{1/4}D_{-\nu-1}\left(e^{-\pi i/4}\sqrt{2\xi\zeta}\right)\end{aligned}\tag{3.22}$$

with

$$c_1 = u^{-1/2}x^{-\theta_\infty} \times 2^{1/4-(\sqrt{3}/4)T_0^2}\xi^{1/4-(\sqrt{3}/4)T_0^2}e^{\pi i/8-(\pi i\sqrt{3}/8)T_0^2+\xi i/3+\sqrt{3}\xi/6+(1/2)\log(U+iV)+\mathrm{bi}+O(\eta^{-1}\xi)+O(\xi^{-1/2})},$$
$$c_2 = \frac{\mathrm{au}^{1/2}}{\sqrt{2\nu}}x^{\theta_\infty} \times 2^{1/4+(\sqrt{3}/4)T_0^2}\xi^{1/4+(\sqrt{3}/4)T_0^2}e^{-\pi i/8-(\pi i\sqrt{3}/8)T_0^2-\xi i/3-\sqrt{3}\xi/6-(1/2)\log(U+iV)-\mathrm{bi}+O(\eta^{-1}\xi)+O(\xi^{-1/2})}. \tag{3.23}$$

Thus, as $z \to \infty$ along $\arg z = \pi/4$,

$$\begin{aligned}\overline{Y}_2^{(11)} &= \widetilde{Y}_1^{(11)} + p\widetilde{Y}_1^{(12)} \\ &\sim c_1\zeta^{1/2}\left[\left(e^{\pi i/4}\sqrt{2\xi}\zeta\right)^{\nu}e^{-(1/4)(e^{\pi i/4}\sqrt{2\xi}\zeta)^2} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\pi i\nu}\left(e^{\pi i/4}\sqrt{2\xi}\zeta\right)^{-\nu-1}e^{(1/4)(e^{\pi i/4}\sqrt{2\xi}\zeta)^2}\right] \\ &\quad + pc_2\zeta^{1/2}\left(e^{-\pi i/4}\sqrt{2\xi}\zeta\right)^{-\nu-1}e^{-(1/4)(e^{-\pi i/4}\sqrt{2\xi}\zeta)^2}.\end{aligned} \tag{3.24}$$

Comparing (3.20) with (3.24), we obtain the entry p in Proposition 3.1:

$$\begin{aligned}p &= \frac{c_1\sqrt{2\pi}}{c_2\Gamma(-(\sqrt{3}/2)M)}e^{(\pi\sqrt{3}i/4)T_0^2-(3\pi i/4)} \\ &= \frac{2\sqrt{\pi\nu}(U+iV)}{\mathrm{au}\,\Gamma(-(\sqrt{3}/2)M)} \\ &\quad\times x^{-2\theta_\infty}2^{-(\sqrt{3}/2)T_0^2}\xi^{-(\sqrt{3}/2)T_0^2}e^{-\pi i/2+(\pi i\sqrt{3}/4)T_0^2+2\xi i/3+\sqrt{3}\xi/3+2\,\mathrm{bi}+O(\xi\eta^{-1})+O(\xi^{-(1/2)})}.\end{aligned} \tag{3.25}$$

Similarly, by Proposition 3.1, we have

$$\begin{aligned}\widetilde{Y}_3^{(12)} &= q\widetilde{Y}_2^{(11)} + \widetilde{Y}_2^{(12)}, \\ \widetilde{Y}_2^{(12)} &\sim \frac{\mathrm{au}^{1/2}}{\sqrt{2\nu}}z^{\theta_\infty}e^{-(1/2)z^2-xz}, \\ \widetilde{Y}_2^{(11)} &\sim u^{-1/2}z^{-\theta_\infty}e^{(1/2)z^2+xz}.\end{aligned} \tag{3.26}$$

Thus, we have to choose

$$\begin{aligned}\widetilde{Y}_2^{(11)} &= c_3\eta^{1/2}\left(\frac{\zeta^2-\widehat{\alpha}^2}{F(\eta,\xi)}\right)^{1/4}D_\nu\left(e^{\pi i/4}\sqrt{2\xi}\zeta\right) + c_4\eta^{1/2}\left(\frac{\zeta^2-\widehat{\alpha}^2}{F(\eta,\xi)}\right)^{1/4}D_{-\nu-1}\left(e^{-\pi i/4}\sqrt{2\xi}\zeta\right), \\ \widetilde{Y}_2^{(12)} &= c_5\left(\frac{\zeta^2-\widehat{\alpha}^2}{F(\eta,\xi)}\right)^{1/4}D_{-\nu-1}\left(e^{-\pi i/4}\sqrt{2\xi}\zeta\right)\end{aligned} \tag{3.27}$$

with

$$c_3 = u^{-1/2}x^{-\theta_\infty}2^{1/4-(\sqrt{3}/4)T_0^2}\xi^{1/4-(\sqrt{3}/4)T_0^2}e^{\pi i/8-(\pi i\sqrt{3}/8)T_0^2+\xi i/3+\sqrt{3}\xi/6+(1/2)\log(U+iV)+\text{bi}},$$
$$c_4 = \frac{c_3\sqrt{2\pi}}{\Gamma(-(\sqrt{3}/2)M)}e^{-3\pi i/4+(\sqrt{3}\pi i/4)T_0^2}, \tag{3.28}$$
$$c_5 = \frac{\text{au}^{1/2}}{\sqrt{2v}}x^{\theta_\infty}2^{1/4+(\sqrt{3}/4)T_0^2}\xi^{1/4+(\sqrt{3}/4)T_0^2}e^{-\pi i/8-(\pi i\sqrt{3}/8)T_0^2-\xi i/3-\sqrt{3}\xi/6-(1/2)\log(U+iV)-\text{bi}}.$$

Therefore,

$$\begin{aligned} q &= \frac{c_5\sqrt{2\pi}}{c_3\Gamma(1+(\sqrt{3}/2)M)}e^{-5\pi i/4+(3\pi i/4)T_0^2} \\ &= \frac{\text{au}\sqrt{2\pi}(U+iV)^{-1}}{\sqrt{2v}\Gamma(1+(\sqrt{3}/2)M)}x^{2\theta_\infty}(2\xi)^{(\sqrt{3}/2)T_0^2}e^{-3\pi i/2+(3\pi i\sqrt{3}/4)T_0^2-2\xi i/3-(\sqrt{3}/3)\xi-2\text{bi}}, \end{aligned} \tag{3.29}$$

$$pq = 1 - e^{\pi i\sqrt{3}M}. \tag{3.30}$$

Since the monodromic data p and q are independent of x, we have proved that $M = i(U^2+V^2)/4 = \text{id}^2/4$ is a constant relative to x. Solving the equation of q, we get

$$U+iV = \frac{\text{au}\sqrt{2\pi}}{q\sqrt{2v}\Gamma(1+(\sqrt{3}/2)M)}x^{2\theta_\infty}e^{-3\pi i/2+(3\pi i\sqrt{3}/4)T_0^2-2\xi i/3-(\sqrt{3}/3)\xi+(\sqrt{3}/2)T_0^2\log(2\xi)-2\text{bi}}. \tag{3.31}$$

Solving for the real and imaginary parts of this equation, we obtain the expected asymptotic expression

$$\begin{aligned} U &= \pm d\cos\left(\frac{\sqrt{3}}{3}x^2 - \frac{\sqrt{3}}{4}d^2\log x + \phi_0 + O(x^{-1})\right), \\ V &= \mp d\sin\left(\frac{\sqrt{3}}{3}x^2 - \frac{\sqrt{3}}{4}d^2\log x + \phi_0 + O(x^{-1})\right), \end{aligned} \tag{3.32}$$

where $\phi_0 = \arg q + \arg\Gamma(3/2+(i\sqrt{3}/8)d^2) - \pi/4 - (\sqrt{3}/8)d^2(\log 2 + (1/2)\log 3) - 2\pi\theta_\infty/3 +$ mod (2π).

4. Proof of Theorem 1.1(2)

To prove Theorem 1.1(2) which concerns the behaviour of the solution in the third quadrant, we can study the behaviour of the solution in the first quadrant by its symmetry. Now, we let $y = xu^2(t)$ and $t = x^2$. Then,

$$u'' + t^{-1}u' = \frac{1}{16}t^{-2}u + \frac{3}{16}u^5 + \frac{1}{2}u^3 + \frac{1}{4}u - \frac{\alpha}{4}t^{-2}u + \frac{\beta}{8t^2u^3}. \tag{4.1}$$

For the sake of contradiction, we assume that u does not blow up at any finite point. Then, there would be four possibilities:

(i) u is oscillating when t goes to infinity;

(ii) u approaches a finite positive limit l as t goes to infinity;

(iii) u goes to zero monotonously when t goes to infinity;
(iv) u goes to infinity monotonously as t goes to infinity.
Now, we proceed to eliminate all four possibilities.

Case 1. In this case, there would be a large value t_0 such that $u'(t_0) = 0$ and $1 - \alpha t^{-2} > 0$ for all $t \geq t_0$. Hence, integrating both sides of (4.1) yields

$$tu'(t) = \int_{t_0}^{t} \left(\frac{1}{16} t^{-1} u + \frac{3t}{16} u^5 + \frac{t}{2} u^3 + \frac{t}{4}(1 - \alpha t^{-2}) u + \frac{\beta}{8tu^3} \right) dt > 0. \tag{4.2}$$

This is clearly a contradiction.

Case 2. In this case, there would exist constants C and t_0 such that

$$u'' + t^{-1} u' > C \quad \text{for } t \geq t_0. \tag{4.3}$$

Multiplying both sides of this inequality by t and integrating from t_0, we get

$$tu' > \frac{1}{2} C t^2 + C_1 \quad \text{for } t \geq t_0. \tag{4.4}$$

Integrating again, we have

$$u > \frac{1}{4} C t^2 + C_1 \ln t + C_2. \tag{4.5}$$

This is clearly a contradiction again.

Case 3. In this case, there would be a positive constant C_3 such that

$$u'' > C_3 u. \tag{4.6}$$

Multiplying both sides of (4.6) by $2u'$ which is nonpositive and integrating it from $t_1 \geq t_0$, we have

$$u'^2(t) < C_3 u^2(t) + u'^2(t_1) - C_3 u^2(t_1). \tag{4.7}$$

If there is a constant $t_1 > t_0$ such that $u'^2(t_1) - C_3 u^2(t_1) < 0$, we can find a large t such that $u'^2(t) < 0$ and this is a contradiction. Thus, for all $t > t_0$, we have

$$u'(t) < -C_4 u(t). \tag{4.8}$$

Therefore, we have

$$u(t) < C_5 e^{-C_4 t}. \tag{4.9}$$

Now, we go back to (4.1) and get that, for large t,

$$u''(t) > \frac{C_6}{t^2 u^3} > C_7 t^{-2} e^{3C_4 t} > \frac{C_7}{t^2}. \tag{4.10}$$

And then,

$$u'(t) - u'(t_1) > \frac{C_7}{t_1} - \frac{C_7}{t}. \tag{4.11}$$

Because $u'(t) \le 0$, $u'(t_1) + C_7/t_1 \le 0$ for all $t_1 > t_0$. Hence, we have

$$u(t) \le -C_7 \ln t + C_8. \tag{4.12}$$

This is impossible.

Case 4. This case can be eliminated by using the argument used at the beginning of the proof of Lemma 2.2.

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