SOME EXISTENCE AND UNIQUENESS RESULTS FOR FIRST-ORDER BOUNDARY VALUE PROBLEMS FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN FRÉCHET SPACES

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A recent nonlinear alternative for contraction maps in Fréchet spaces due to Frigon and Granas is used to investigate the existence and uniqueness of solutions to first-order boundary value problems for impulsive functional differential equations with infinite delay. An example to illustrate the results is included.

1. Introduction

This paper is devoted to the study of the existence and uniqueness of solutions of first-order boundary value problems for impulsive functional and neutral impulsive functional differential equations with infinite delay. In particular, in Section 3, we will consider the class of first-order boundary value problem for functional differential equations with impulsive effects,

$$y'(t) = f(t, y_t) \quad \text{a.e. } t \in J := [0,\infty),\ t \neq t_k,\ k = 1,2,\dots, \tag{1.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = t_k,\ k = 1,2,\dots, \tag{1.2}$$

$$Ay_0 - x_\infty = \phi(t), \quad t \in (-\infty, 0], \tag{1.3}$$

where $f : J \times \mathcal{B} \to \mathbb{R}^n$ and $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,2,\dots$, are given functions, $\lim_{t\to\infty} y(t) = x_\infty$, $A \neq 1$, and $\phi \in \mathcal{B}$, which is called the *phase space*, will be defined later. Section 4 is devoted to the impulsive neutral functional differential equation with boundary conditions,

$$\begin{gathered} \frac{d}{dt}[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J,\ t \neq t_k,\ k = 1,2,\dots, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = t_k,\ k = 1,2,\dots, \\ Ay_0 - x_\infty = \phi(t), \quad t \in (-\infty, 0], \end{gathered} \tag{1.4}$$

where f, I_k, A, x_∞, and $\mathcal{B}$ are as in the problem (1.1)–(1.3) and $g : J \times \mathcal{B} \to \mathbb{R}^n$ is a given function.

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The notion of the phase space $\mathcal{B}$ plays an important role in the study of both the qualitative and quantitative theory of functional differential equations. A usual choice is a seminormed space satisfying suitable axioms as was introduced by Hale and Kato [18] (also see Kappel and Schappacher [21] and Schumacher [28]). For a detailed discussion on this topic, we refer the reader to Hino et al. [20]. For the case where the impulses are absent (i.e., $I_k = 0$, $k = 1,2,\ldots$), an extensive theory has been developed for the problem (1.1), (1.3), and we refer the reader to Corduneanu and Lakshmikantham [8], Hale and Kato [18], Hino et al. [20], Lakshmikantham et al. [23], and Shin [29].

Impulsive differential equations have become increasingly important in recent years as mathematical models of real-world processes and phenomena studied in control theory, physics, chemistry, population dynamics, biotechnology, and economics. There has been a significant development in impulse theory and this has been especially true in the area of impulsive differential equations with fixed moments; see, for example, the monographs of Baĭnov and Simeonov [3], Lakshmikantham et al. [22], and Samoĭlenko and Perestyuk [27], as well as the papers of Agur et al. [2], Ballinger and Liu [4], Benchohra et al. [5, 6], Franco et al. [9], and the references contained therein.

Boundary value problems on infinite intervals appear in many problems of practical interest, for example, in linear elasticity, nonlinear fluid flow, and foundation engineering (e.g., see, [1, 13, 16, 24–26]). Recently, fixed point arguments using such approaches as the Banach contraction principle, fixed point index theory, and monotone iterative technique have been applied to first- and second-order impulsive differential equations. We mention here the survey papers by Guo [12, 14, 15], Guo and Liu [17], Yan and Liu [30], and the references therein. Very recently, a nonlinear alternative due to Frigon and Granas [10] was applied to impulsive functional differential equation with variable times [11]; see the paper by Benchohra et al. [7] for first-order equations and Henderson and Ouahab [19] for second- and higher-order problems. Our goal here is to give existence and uniqueness results for the problems (1.1)–(1.3) and (1.4) above by using this nonlinear alternative for contraction maps.

2. Preliminaries

In this short section, we introduce notations and definitions that are used throughout the remainder of this paper. We let $C([0,b],\mathbb{R}^n)$ denote the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \le t \le b\}, \tag{2.1}$$

and we let $L^1([0,\infty),\mathbb{R}^n)$ be the Banach space of measurable functions $y : [0,\infty) \to \mathbb{R}^n$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^\infty \|y(t)\|dt \quad \forall y \in L^1([0,\infty),\mathbb{R}^n). \tag{2.2}$$

For more details on the following notions, we refer the reader to Frigon and Granas [10]. Let X be a Fréchet space with a family of seminorms $\{\|\cdot\|_n,\ n \in \mathbb{N}\}$. If $Y \subset X$, we say

that Y is *bounded* if for every $n \in \mathbb{N}$ there exists $M_n > 0$ such that

$$\|y\|_n \le M_n \quad \forall y \in Y. \tag{2.3}$$

We associate to X a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: for every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We let $X^n = (X/\sim_n, \|\cdot\|)$ denote the quotient space that is the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: for every $x \in X$, let $[x]_n$ denote the equivalence class of x of subsets X^n, and we define $Y^n = \{[x]_n : x \in Y\}$. We let $\overline{Y}^n$, $\operatorname{int}_n(Y^n)$, and $\partial_n Y^n$, respectively, denote the closure, the interior, and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We will assume that the family of seminorms $\{\|\cdot\|_n\}$ satisfies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X. \tag{2.4}$$

Next, we define what we mean by a contraction.

Definition 2.1. A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

$$\|f(x) - f(y)\|_n \le k_n \|x - y\|_n \quad \forall x, y \in X. \tag{2.5}$$

The following nonlinear alternative will be used to prove our main results.

Theorem 2.2 (nonlinear alternative [10]). *Let X be a Fréchet space, let $Y \subset X$ be a closed subset in X, and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(C1) *N has a unique fixed point;*

(C2) *there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.*

We will also need the following definition.

Definition 2.3. The map $f : [0,\infty) \times \mathcal{B} \to \mathbb{R}^n$ is said to be L^1-Carathéodory if

(i) $t \mapsto f(t,x)$ is measurable for each $x \in \mathcal{B}$;

(ii) $x \mapsto f(t,x)$ is continuous for almost all $t \in [0,\infty)$;

(iii) for each $q > 0$ there exists $h_q \in L^1([0,\infty), \mathbb{R}_+)$ such that

$$\|f(t,x)\| \le h_q(t) \quad \forall \|x\|_{\mathcal{B}} \le q \text{ and for almost all } t \in [0,\infty). \tag{2.6}$$

3. Functional differential equations

In order to define the phase space and a solution of the problem (1.1)–(1.3), we consider the space

$$\begin{aligned} \mathrm{PC} = \{ & y : (-\infty,\infty) \longrightarrow \mathbb{R}^n \mid y(t_k^-) \text{ and } y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), \\ & Ay(t) - x_\infty = \phi(t) \ \text{ for } t \le 0,\ y_k \in C(J_k, \mathbb{R}^n),\ k = 1,2,\dots\}, \end{aligned} \tag{3.1}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 1,2,\dots$. In the remainder of this paper, we will assume that $\mathcal{B}$ satisfies the following properties.

(A-1) If $y:(-\infty,\infty)\to\mathbb{R}^n$ and $y_0\in\mathcal{B}$, then for every t in $[0,\infty)$, we have
(i) y_t is in $\mathcal{B}$,
(ii) $\|y_t\|_{\mathcal{B}}\le K(t)\sup\{|y(s)|:0\le s\le t\}+M(t)\|y_0\|_{\mathcal{B}}$, and
(iii) $|y(t)|\le H\|y_t\|_{\mathcal{B}}$,
where $H\ge 0$ is a constant, $K:[0,\infty)\to[0,\infty)$ is continuous, $M:[0,\infty)\to[0,\infty)$ is locally bounded, and H, K, and M are independent of y.

(A-2) For the function y in (A-1), y_t is a continuous function on $[0,\infty)\setminus\{t_1,t_2,\dots\}$.

(A-3) The space $\mathcal{B}$ is complete.

Now set

$$B_*=\{y:(-\infty,\infty)\longrightarrow\mathbb{R}^n:y\in \mathrm{PC}\cap\mathcal{B}\},$$
$$B_k=\left\{y\in B_*:\sup_{t\in J_k^*}|y(t)|<\infty\right\},\quad \text{where } J_k^*=(-\infty,t_k]. \tag{3.2}$$

Let $\|\cdot\|_k$ be the seminorm in B_k defined by

$$\|y\|_k=\|y_0\|_{\mathcal{B}}+\sup\{|y(s)|:0\le s\le t_k\},\quad y\in B_k. \tag{3.3}$$

We will next define what we mean by a solution of (1.1)–(1.3).

Definition 3.1. A function y is said to be a solution of the problem (1.1)–(1.3) if $y\in B_*$ and y satisfies (1.1)–(1.3).

To prove our existence results for the problem (1.1)–(1.3), we first establish the following lemma.

Lemma 3.2. *Let $f:[0,\infty)\to\mathbb{R}^n$ be a continuous function with $\int_0^\infty f(s)ds<\infty$. Then y is a solution of the impulsive integral equation*

$$y(t)=\begin{cases}\dfrac{\phi(0)}{A(A-1)}+\dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(y_s)ds+\sum_{k=1}^\infty I_k(y(t_k^-))\right]+\dfrac{\phi(t)}{A}, & t\in(-\infty,0],\\[2ex] \dfrac{\phi(0)}{A-1}+\dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(y_s)ds+\sum_{k=1}^\infty I_k(y(t_k^-))\right]+\displaystyle\int_0^t f(y_s)ds & \\ +\displaystyle\sum_{0<t_k<t} I_k(y(t_k^-)), & t\in[0,\infty),\end{cases} \tag{3.4}$$

if and only if y is a solution of the impulsive boundary value problem

$$y'(t)=f(y_t),\quad t\in[0,\infty),\ t\ne t_k,\ k=1,2,\dots, \tag{3.5}$$
$$y(t_k^+)-y(t_k^-)=I_k(y(t_k^-)),\quad k=1,2,\dots, \tag{3.6}$$
$$Ay_0-y_\infty=\phi(t),\quad t\in(-\infty,0], \tag{3.7}$$

where $\lim_{t\to\infty}y(t)=y_\infty$.

Proof. Let y be a solution of the impulsive integral equation (3.4). Then for $t \in [0,\infty)$ and $t \neq t_k$, $k = 1,2,\ldots$, we have

$$y(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)). \tag{3.8}$$

Thus, $y'(t) = f(y_t)$ for $t \in [0,\infty)$ and $t \neq t_k$, $k = 1,2,\ldots$. From the definition of y, we see that

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)) \quad \text{for } k = 1,2,\ldots. \tag{3.9}$$

Finally, to see that $Ay_0 - y_\infty = \phi(t)$ for $t \in (-\infty,0]$, note that

$$\begin{aligned} y_\infty &= \frac{\phi(0)}{A-1} + \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right], \\ y_0 &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \frac{\phi(t)}{A}. \end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned} Ay_0 - y_\infty = {} & \frac{\phi(0)}{A-1} + \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \phi(t) - \frac{\phi(0)}{A-1} \\ & - \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] = \phi(t). \end{aligned} \tag{3.11}$$

Now let y be a solution of the problem (3.5)–(3.7). Then,

$$y'(t) = f(y_t) \quad \text{for } t \in [0,t_1], \tag{3.12}$$

and an integration from 0 to $t \in (0,t_1]$ yields

$$y(t) - y(0) = \int_0^t y'(s)ds = \int_0^t f(y_s)ds \tag{3.13}$$

or

$$y(t) = y(0) + \int_0^t f(y_s)ds \tag{3.14}$$

for $t \in [0,t_1]$. If $t \in (t_1,t_2]$, then

$$y(t_1^-) - y(0) + y(t) - y(t_1^+) = \int_0^{t_1} y'(s)ds + \int_{t_1}^t y'(s)ds = \int_0^t f(y_s)ds, \tag{3.15}$$

so

$$y(t) = y(0) + \int_0^t f(y_s)\,ds + I_1(y(t_1^-)). \tag{3.16}$$

Continuing this procedure, we obtain

$$y(t) = y(0) + \int_0^t f(y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)) \tag{3.17}$$

for $t \in [0,\infty)$. Since $\lim_{t\to\infty} y(t) = y_\infty$, we have

$$y_\infty = y(0) + \int_0^\infty f(y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-)). \tag{3.18}$$

From (3.7), we have $y_\infty = Ay(0) - \phi(0)$, and so

$$y(0) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right]. \tag{3.19}$$

Substituting (3.19) into (3.17), we obtain

$$y(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \int_0^t f(y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)) \tag{3.20}$$

for $t \in [0,\infty)$.

From (3.7), (3.18), and (3.19), we have

$$\begin{aligned} y(t) &= \frac{\phi(t)}{A} + \frac{1}{A}\left[y(0) + \int_0^\infty f(y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] \\ &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \frac{\phi(t)}{A} \end{aligned} \tag{3.21}$$

for $t \in (-\infty,0]$. This completes the proof of the lemma. □

We are now ready to prove our existence and uniqueness result for the problem (1.1)–(1.3).

Theorem 3.3. *Let $f : J \times \mathcal{B} \to \mathbb{R}^n$ be an L^1-Carathéodory function and assume that*

(H1) *there exist constants $d_k > 0$, $k = 1,2,\ldots$, such that*

$$\|I_k(x) - I_k(\overline{x})\| \le d_k\|x - \overline{x}\| \quad \text{for each } k = 1,2,\ldots,\ \forall x,\overline{x} \in \mathbb{R}^n; \tag{3.22}$$

(H2) *there exists a function $l \in L^1([0,\infty),\mathbb{R}^n)$ such that*

$$\|f(t,x) - f(t,\overline{x})\| \le l(t)\|x - \overline{x}\|_{\mathcal{B}} \quad \text{for each } t \in [0,\infty),\ \forall x,\overline{x} \in \mathcal{B}; \tag{3.23}$$

(H3) *there exist a function* $p \in L^1([0,\infty),\mathbb{R}_+)$ *and positive constants* c_k, $k = 1,2,\ldots$, *such that*

$$\begin{gathered} \|f(t,u)\| \le p(t) \quad \text{for } (t,u) \in [0,\infty)\times\mathcal{B}, \\ \|I_k(z)\| \le c_k \quad \forall z \in \mathbb{R}^n, \\ \sum_{k=1}^{\infty} c_k < \infty. \end{gathered} \tag{3.24}$$

If $\sum_{k=1}^{\infty} d_k < 1$, *then the problem (1.1)–(1.3) has a unique solution.*

Proof. We transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : B_* \to B_*$ defined by

$$N(y)(t) = \begin{cases} \dfrac{\phi(0)}{A(A-1)} + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \dfrac{\phi(t)}{A}, & t \in (-\infty,0], \\ \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] & \\ + \displaystyle\int_0^{t} f(s,y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in [0,\infty). \end{cases} \tag{3.25}$$

It should be clear from Lemma 3.2 that the fixed points of N are solutions of the problem (1.1)–(1.3). Let $x : (-\infty,\infty) \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,x_s)\,ds + \sum_{k=1}^{\infty} I_k(x(t_k^-))\right] & \text{if } t \in [0,\infty), \\ \dfrac{\phi(0)}{A(A-1)} + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,x_s)\,ds + \sum_{k=1}^{\infty} I_k(x(t_k^-))\right] + \dfrac{\phi(t)}{A} & \text{if } t \in (-\infty,0]. \end{cases} \tag{3.26}$$

Then,

$$x_0 = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^{\infty} f(s,x_s)\,ds + \sum_{k=1}^{\infty} I_k(x(t_k^-))\right]. \tag{3.27}$$

For each $z \in C([0,\infty),\mathbb{R}^n)$ with $z(0) = 0$, denote by $\bar{z}$ the function given by

$$\bar{z}(t) = \begin{cases} z(t) & \text{if } t \in [0,\infty), \\ 0 & \text{if } t \in (-\infty,0]. \end{cases} \tag{3.28}$$

If y satisfies the integral equation

$$y(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^{\infty} f(s,y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] + \int_0^{t} f(s,y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \tag{3.29}$$

we can decompose y into $y(t) = \bar{z}(t) + x(t)$, $0 \le t < \infty$, which implies that $y_t = \bar{z}_t + x_t$ for $0 \le t < \infty$, and z satisfies

$$z(t) = \int_0^t f(s, \bar{z}_s + x_s)\,ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k^-) + x(t_k^-)). \tag{3.30}$$

Let

$$B_*^k = \{z \in B_k : z_0 = 0\}. \tag{3.31}$$

For any $z \in B_*^k$, we have

$$\|z\|_k = \|z_0\|_{\mathcal{B}} + \sup\{\|z(s)\| : 0 \le s \le t_k\} = \sup\{\|z(s)\| : 0 \le s \le t_k\}. \tag{3.32}$$

Thus $(B_*^k, \|\cdot\|_k)$ is a Banach space. If we set

$$C_0 = \{z \in B_* : z_0 = 0\} \tag{3.33}$$

with the Bielecki-type norm on B_*^k defined by

$$\|z\|_{B_*^k} = \max\left\{\|z(t)\| e^{-\tau \hat{l}(t)} : t \in [0, t_k]\right\}, \tag{3.34}$$

where $\hat{l}(t) = \int_0^t \tilde{l}(s)\,ds$, $\tilde{l}(t) = K_k l(t)$, $K_k = \sup\{|K(t)| : t \in [0, t_k]\}$, $k = 1, 2, \ldots$, and $\tau > 0$ is a constant, then C_0 is a Fréchet space with the family of seminorms $\|\cdot\|_{B_*^k}$.

Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \begin{cases} 0, & t \le 0, \\ \displaystyle\int_0^t f(s, \bar{z}_s + x_s)\,ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k^-) + x(t_k^-)), & t \in [0, \infty). \end{cases} \tag{3.35}$$

Clearly, the operator N having a fixed point is equivalent to the operator P having one, and so we turn our attention to proving that P does have a fixed point. We will use the nonlinear alternative given in Theorem 2.2 to show this.

Let z be a possible solution of the problem $z = \lambda P(z)$ for some $0 < \lambda < 1$. Then, for each $t \in [0, \infty)$,

$$z(t) = \lambda\left[\int_0^t f(s, \bar{z}_s + x_s)\,ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k^-) + x(t_k^-))\right], \tag{3.36}$$

so by (H3), we have

$$\|z(t)\| < \int_0^t p(s)\,ds + \sum_{i=1}^{i=k} I_i(\bar{z}(t_i) + x(t_i)) \mid \le \int_0^{t_k} p(s)\,ds + \sum_{i=1}^{i=k} c_i := M_k^*. \tag{3.37}$$

Hence,

$$\sup\{\|z(t)\| : t \in [0, t_k]\} < M_k^*. \tag{3.38}$$

Let

$$Y = \{z \in C_0 \mid \sup\{\|z\|_{B_*^k} : 0 \le t \le t_k\} \le M_k^* \ \forall k = 1,2,\dots\}. \tag{3.39}$$

Clearly, Y is a closed subset of C_0. We will show that $P : Y \to B_*^k$ is a contraction map. To see this, consider $z, z^* \in Y$; then for each $t \in [0,t_k]$ and $k = 1,2,\dots$, we have

$$\begin{aligned}
\|P(z)(t) - P(z^*)(t)\| &\le \int_0^t \|f(s,\bar{z}_s + x_s) - f(s,\bar{z}_s^* + x_s)\| ds \\
&\quad + \sum_{i=1}^{i=k} \|I_i(\bar{z}(t_i^-) + x(t_i^-)) - I_i(\bar{z}^*(t_i^-) + x(t_i^-))\| \\
&\le \int_0^t l(s)\|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}} ds + \sum_{i=1}^{i=k} d_i \|z(t_i^-) - z^*(t_i^-)\| \\
&\le \int_0^t l(s) K_k \sup_{s\in[0,t]} \|z(s) - z^*(s)\| ds + \sum_{i=1}^{i=k} d_i \|z(t_i^-) - z^*(t_i^-)\| \\
&\le \int_0^t \widetilde{l}(s) e^{\tau \hat{l}(s)} e^{-\tau \hat{l}(s)} \sup_{s\in[0,t]} \|z(s) - z^*(s)\| ds \\
&\quad + \sum_{i=1}^{i=k} d_i e^{\tau \hat{l}(t)} e^{-\tau \hat{l}(t)} \sup_{s\in[0,t_k]} \|z(s) - z^*(s)\| \\
&= \int_0^t \widetilde{l}(s) e^{\tau \hat{l}(s)} ds \|z - z^*\|_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau \hat{l}(t)} \|z - z^*\|_{B_*^k} \\
&= \frac{1}{\tau} \int_0^t \left(e^{\tau \hat{l}(s)}\right)' ds \|z - z^*\|_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau \hat{l}(t)} \|z - z^*\|_{B_*^k} \\
&\le \frac{1}{\tau} e^{\tau \hat{l}(t)} ds \|z - z^*\|_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau \hat{l}(t)} \|z - z^*\|_{B_*^k}.
\end{aligned} \tag{3.40}$$

Thus,

$$e^{-\tau \hat{l}(t)} \|P(z)(t) - P(z^*)(t)\| \le \frac{1}{\tau} \|z - z^*\|_{B_*^k} + \sum_{i=1}^{i=k} d_i \|z - z^*\|_{B_*^k} = \left(\frac{1}{\tau} + \sum_{i=1}^{i=k} d_i\right) \|z - z^*\|_{B_*^k}. \tag{3.41}$$

Therefore,

$$\|P(z) - P(z^*)\|_{B_*^k} \le \left(\frac{1}{\tau} + \sum_{i=1}^{i=k} d_i\right) \|z - z_*\|_{B_*^k}, \tag{3.42}$$

which shows that if τ is large enough, then P is a contraction. From the choice of Y, there is no $z \in \partial Y^n$ such that $z = \lambda P(z)$ for some $\lambda \in (0,1)$. As a consequence of Theorem 2.2,

we conclude that P has a unique fixed point. In turn, this implies that the operator N has a unique fixed point that is a solution to (1.1)–(1.3). This completes the proof of the theorem. □

4. Neutral functional differential equations

This section is concerned with the existence of solutions to the boundary value problem for first-order neutral functional differential equations with infinite delay and impulses given in (1.4). Our main result is as follows.

Theorem 4.1. *Let $f : J \times \mathcal{B} \to \mathbb{R}^n$ be an L^1-Carathéodory function. In addition to (H1)–(H3), assume that $M(0) < 1$ and*

(B) *there exist constants $c_1^*, c_2^* \geq 0$, and $\bar{d}_k > 0$, $k = 1,2,\ldots$, such that*

$$\begin{gathered} \|g(t,u)\| \leq c_1^* \|u\|_{\mathcal{B}} + c_2^*, \quad t \in [0,\infty),\ u \in \mathcal{B},\ c_1^* K_k < 1, \\ \|g(t,u) - g(t,\overline{u})\| \leq \bar{d}_k \|u - \overline{u}\|_{\mathcal{B}} \quad \text{for } t \in [0,t_k], \end{gathered} \tag{4.1}$$

where

$$K_k = \sup\{|K(t)| : t \in [0,t_k]\}, \quad k = 1,2,\ldots. \tag{4.2}$$

If $\sum_{k=1}^{\infty}[d_k + \bar{d}_k K_k] < 1$, then the problem (1.4) has a unique solution.

Proof. We proceed similarly to what we did in the proof of Theorem 3.3 by defining the operators $N_1 : B_* \to B_*$ and $P_1 : C_0 \to C_0$ by

$$N_1(y)(t) = \begin{cases} g(0,\phi(0)) - g(t,\phi(t)) + \dfrac{\phi(0)}{A(A-1)} + \dfrac{\phi(t)}{A} \\ + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right], & t \in (-\infty,0], \\ g(0,\phi) - g(t,y_s) + \dfrac{1}{A-1}\left[\displaystyle\int_0^{\infty} f(s,y_s)\,ds + \sum_{k=1}^{\infty} I_k(y(t_k^-))\right] \\ + \dfrac{\phi(0)}{A-1} + \displaystyle\int_0^t f(s,y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in [0,\infty), \end{cases}$$

$$(P_1 z)(t) = \begin{cases} 0, & t \leq 0, \\ g(0,\phi) - g(t,\overline{z}_t + x_t) + \displaystyle\int_0^t f(s,\overline{z}_s + x_s)\,ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k^-) + x(t_k^-)), & t \in [0,\infty), \end{cases} \tag{4.3}$$

where $\overline{z}_t$ and x_t are defined similar to the way they are in the proof of Theorem 3.3. In order to apply the nonlinear alternative, Theorem 2.2, we first obtain a priori estimates for the solutions of the integral equation

$$z(t) = \lambda\left[g(0,\phi) - g(t,\overline{z}_t + x_t) + \int_0^t f(s,\overline{z}_s + x_s)\,ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k^-) + x(t_k^-))\right] \tag{4.4}$$

for any $\lambda \in (0,1)$. We have

$$\begin{aligned}
||z(t)|| &< ||g(0,\phi(0))|| + ||g(t,\bar{z}_t + x_t)|| + \int_0^t p(s)ds + \sum_{0<t_k<t} ||I_k(\bar{z}(t_k^-) + x(t_k^-))|| \\
&\leq ||g(0,\phi)|| + c_1^* ||\bar{z}_t + x_t||_{\mathcal{B}} + c_2^* + \int_0^t p(s)ds + \sum_{i=1}^{i=k} c_i \\
&\leq ||g(0,\phi(0))|| + c_1^* K(t) \sup_{s\in[0,t]} |z(s)| + c_1^* K(t) \sup_{s\in[0,t]} |x(s)| + c_1^* M(t) ||x_0||_{\mathcal{B}} \\
&\quad + c_2^* + \int_0^t p(s)ds + \sum_{i=1}^{k} c_i \leq ||g(0,\phi(0))|| + c_1^* K_k \sup_{s\in[0,t]} |z(s)| \\
&\quad + K_k c_1^* \|x\|_\infty + M_k c_1^* ||x_0||_{\mathcal{B}} + c_2^* + \int_0^t p(s)ds + \sum_{i=1}^{k} c_i,
\end{aligned} \tag{4.5}$$

where

$$M_k = \sup\{M(t) : t \in [0,t_k]\}. \tag{4.6}$$

Hence,

$$\|z\|_\infty < ||g(0,\phi)|| + c_1^* K_k \|z\|_\infty + K_k c_1^* \|x\|_\infty + M_k c_1^* ||x_0||_{\mathcal{B}} + c_2^* + \int_0^\infty p(s)ds + \sum_{i=1}^{k} c_i. \tag{4.7}$$

Now,

$$||x_0||_{\mathcal{B}} \leq K(0)|x(0)| + M(0)||x_0||_{\mathcal{B}}, \tag{4.8}$$

so

$$||x_0||_{\mathcal{B}} \leq \frac{K(0)}{1-M(0)} |x(0)|, \tag{4.9}$$

and since

$$\|x\|_\infty := \sup_{t\in[0,t_k]} |x(t)| \leq \frac{|\phi(0)|}{A-1} + \frac{1}{A-1}\left[\|p\|_{L^1} + \sum_{k=1}^{\infty} c_k\right] := \ell, \tag{4.10}$$

we then have

$$\|z\|_\infty < \frac{1}{1-c_1^* K_k}\left[||g(0,\phi)|| + c_2^* + K_k c_1^* \ell + M_k c_1^* \frac{K(0)}{1-M(0)}\ell + \|p\|_{L^1} + \sum_{i=1}^{i=k} c_i\right] := \overline{M}_k. \tag{4.11}$$

Now set

$$Y_1 = \{z \in C_0 : \sup\{||z(t)|| : 0 \le t \le t_k\} \le \overline{M}_k \ \forall k = 1,2,\dots\}. \tag{4.12}$$

Clearly, Y_1 is a closed subset of C_0. To show that P_1 is a contraction, let $z, z^* \in Y_1$. Then, for each $t \in [0,t_k]$ and $k = 1,2,\dots$, we have

$$\begin{aligned}
&||P_1(z)(t) - P_1(z^*)(t)|| \\
&\quad \le ||g(t,\bar{z}_t + x_t) - g(t,\bar{z}_t^* + x_t)|| + \int_0^t ||f(s,\bar{z}_s + x_s) - f(s,\bar{z}_s^* + x_s)||ds \\
&\qquad + \sum_{i=1}^{i=k} ||I_i(\bar{z}(t_i^-) + x(t_i^-)) - I_i(\bar{z}^*(t_i^-) + x(t_i^-))|| \le \bar{d}_k ||\bar{z}_t - z_t^*||_{\mathcal{B}} \\
&\qquad + \int_0^t l(s)||\bar{z}_s - \bar{z}_s^*||_{\mathcal{B}} ds + \sum_{i=1}^{i=k} d_i ||z(t_i^-) - z^*(t_i^-)|| \le \bar{d}_k K_k \sup_{t\in[0,t_k]} ||z(t) - z^*(t)|| \\
&\qquad + \int_0^t l(s) K_k \sup_{\tau\in[0,s]} ||z(\tau) - z^*(\tau)||ds + \sum_{i=1}^{i=k} d_i ||z(t_i^-) - z^*(t_i^-)|| \\
&\quad \le \bar{d}_k K_k e^{\tau\hat{l}(t)} e^{-\tau\hat{l}(t)} \sup_{t\in[0,t_k]} ||z(t) - z^*(t)|| + \int_0^t \tilde{l}(s) e^{\tau\hat{l}(s)} e^{-\tau\hat{l}(s)} \sup_{\tau\in[0,s]} ||z(\tau) - z^*(\tau)||ds \\
&\qquad + \sum_{i=1}^{i=k} d_i e^{\tau\hat{l}(t)} e^{-\tau\hat{l}(t)} \sup_{s\in[0,t_k]} ||z(s) - z^*(s)|| \le \bar{d}_k K_k e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k} \\
&\qquad + \int_0^t \tilde{l}(s) e^{\tau\hat{l}(s)} ds ||z - z^*||_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k} = \bar{d}_k K_k e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k} \\
&\qquad + \frac{1}{\tau} \int_0^t (e^{\tau\hat{l}(s)})' ds ||z - z^*||_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k} \le \bar{d}_k K_k e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k} \\
&\qquad + \frac{1}{\tau} e^{\tau\hat{l}(t)} ds ||z - z^*||_{B_*^k} + \sum_{i=1}^{i=k} d_i e^{\tau\hat{l}(t)} ||z - z^*||_{B_*^k}.
\end{aligned} \tag{4.13}$$

Thus,

$$e^{-\tau\hat{l}(t)} ||P_1(z)(t) - P_1(z^*)(t)|| \le \left(\frac{1}{\tau} + \sum_{i=1}^{i=k} d_i + \bar{d}_k K_k\right) ||z - z^*||_{B_*^k}. \tag{4.14}$$

Therefore,

$$||P_1(z) - P_1(z^*)||_{B_*^k} \le \left(\frac{1}{\tau} + \sum_{i=1}^{i=k} d_i + \bar{d}_k K_k\right) ||z - z_*||_{B_*^k}, \tag{4.15}$$

and again if τ is large enough, P_1 is a contraction. From the choice of Y_1, there is no $z \in \partial Y_1^n$ such that $z = \lambda P_1(z)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative (Theorem 2.2), we see that P_1 has a unique fixed point which again leads to the existence of a unique solution to (1.4). □

5. Example

In this section we give an example to illustrate the usefulness of our main results. Consider the problem

$$\begin{gathered} y'(t) = \frac{e^{-\gamma t}\,|\,y_t\,|}{(t+1)(t+2)(1+|\,y_t\,|)}, \quad \text{a.e. } t \in J := [0,\infty) - \{t_1, t_2, \dots\}, \\ y(t_k^+) - y(t_k^-) = b_k y(t_k^-), \quad k = 1,2,\dots, \\ y(t) = \phi(t), \quad t \in (-\infty, 0], \end{gathered} \tag{5.1}$$

where $\gamma > 0$ is a constant and $b_k > 0$ for $k = 1,2,\dots$. Let

$$\begin{aligned} \mathcal{D} = \{\psi : (-\infty,0] \longrightarrow \mathbb{R}^n \mid \psi &\text{ is continuous everywhere except for a countable number of} \\ &\text{points } \bar{t} \text{ at which } \psi(\bar{t}^-), \psi(\bar{t}^+)\ \exists, \psi(\bar{t}^-) = \psi(\bar{t})\}. \end{aligned} \tag{5.2}$$

Let

$$B_\gamma = \{y \in \mathcal{D} : \lim_{\theta \longrightarrow -\infty} e^{\gamma\theta} y(\theta) \text{ exists in } \mathbb{R}^n\} \tag{5.3}$$

with the norm in B_γ given by

$$\|y\|_\gamma = \sup_{-\infty<\theta\le 0} e^{\gamma\theta} \|y(\theta)\|. \tag{5.4}$$

Let $y : (-\infty,\infty) \to \mathbb{R}^n$ be such that $y_0 \in B_\gamma$; then

$$\lim_{\theta \longrightarrow -\infty} e^{\gamma\theta} y_t(\theta) = \lim_{\theta \longrightarrow -\infty} e^{\gamma\theta} y(t+\theta) = \lim_{\theta \longrightarrow -\infty} e^{\gamma(\theta-t)} y(\theta) = e^{-\gamma t} \lim_{\theta \longrightarrow -\infty} e^{\gamma\theta} y_0(\theta) < \infty. \tag{5.5}$$

Hence, $y_t \in B_\gamma$.

Next, we prove that $|y(t)| \le H\|y_t\|_{B_\gamma}$ and

$$\|y_t\| \le K(t) \sup\{\|y(s)\| : 0 \le s \le t\} + M(t)\|y_0\|_\gamma, \tag{5.6}$$

where $\|y_t(\theta)\| = \|y(t+\theta)\|$, $K(t) \equiv M(t) \equiv 1$, and $H = 1$. Now, if $\theta + t \le 0$, we have

$$\|y_t(\theta)\| \le \sup\{\|y(s)\| : -\infty < s \le 0\}. \tag{5.7}$$

For $t + \theta \ge 0$, we see that

$$\|y_t(\theta)\| \le \sup\{\|y(s)\| : 0 < s \le t\}. \tag{5.8}$$

Thus, for all $t+\theta \in \mathbb{R}$, we obtain

$$\|y_t(\theta)\| \le \sup\{\|y(s)\| : -\infty < s \le 0\} + \sup\{\|y(s)\| : 0 \le s \le t\}. \tag{5.9}$$

Then,

$$\|y_t\|_\gamma \le \|y_0\|_\gamma + \sup\{\|y(s)\| : 0 \le s \le t\}. \tag{5.10}$$

Now $(B_\gamma, \|\cdot\|)$ is a Banach space and B_γ will serve as the phase space for our problem. We see that the function f satisfies

$$\begin{gathered} |f(t,x)| = \frac{e^{-\gamma t}|x|}{(t+1)(t+2)(1+|x|)} \le \frac{1}{(t+1)(t+2)} = p(t), \\ \int_0^\infty p(t)dt = \int_0^\infty \frac{1}{(t+1)(t+2)}dt = \ln 2. \end{gathered} \tag{5.11}$$

Let $x, y \in B_\gamma$; then,

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{e^{-\gamma t}}{(t+1)(t+2)}\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| = \frac{e^{-\gamma t}\big||x|-|y|\big|}{(t+1)(t+2)(1+|x|)(1+|y|)} \\ &\le \frac{e^{-\gamma t}|x-y|}{(t+1)(t+2)(1+|x|)(1+|y|)} \le \frac{\|x-y\|_{B_\gamma}}{(t+1)(t+2)}. \end{aligned} \tag{5.12}$$

Hence, the hypotheses of Theorem 3.3 are satisfied, so if $\sum_{k=1}^\infty b_k < 1$, then the problem (5.1) has a unique solution.

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