ON SENSIBLE FUZZY IDEALS OF BCK-ALGEBRAS WITH RESPECT TO A *t*-CONORM

M. AKRAM AND JIANMING ZHAN

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We introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a t-conorm and investigate some of their properties. We give the conditions for a sensible fuzzy subalgebra with respect to a t-conorm to be a sensible fuzzy ideal with respect to a t-conorm. Some properties of the direct product and S-product of fuzzy ideals of BCK-algebras with respect to a t-conorm are also discussed.

1. Introduction

Imai and Iséki [3] introduced the class of logical algebras: BCK-algebras. This notion is originated from two different ways: one of the motivations is based on set theory, another motivation is from classical and nonclassical propositional calculus.

The notion of fuzzy sets was first introduced by Zadeh [8]. On the other hand, Schweizer and Sklar [5, 6] introduced the notions of triangular norm (t-norm) and triangular conorm (t-conorm). Triangular norm (t-norm) and triangular conorm (t-conorm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the t-norm generalizes the conjunctive (AND) operator and the t-conorm generalizes the disjunctive (OR) operator. In application, t-norm T and t-conorm S are two functions that map the unit square into the unit interval. Jun and Kim [4] introduced the notion of imaginable fuzzy ideals of BCK-algebras with respect to a t-norm. Cho et al. [1] have recently introduced the notion of sensible fuzzy subalgebras of BCK-algebras with respect to s-norm and studied some of their properties. In this paper, we introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a t-conorm and investigate some of their properties. We give conditions for a sensible fuzzy subalgebra with respect to a t-conorm to be a sensible fuzzy ideal with respect to a t-conorm. Some properties of the direct product and S-product of fuzzy ideals of BCK-algebras with respect to a t-conorm are also obtained.

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2. Preliminaries

In this section, we review some definitions and results that will be used in the sequel.

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCK-algebra if it satisfies the following conditions:

(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y=0$,
(3) $x*x=0$,
(4) $x*y=0$, $y*x=0 \Rightarrow x=y$,
(5) $0*x=0$

for all $x,y,z \in X$. We can define a partial ordering relation "$\leq$" on X by letting $x \leq y$ if and only if $x*y=0$. Let S be a nonempty subset of a BCK-algebra X, then S is called a *subalgebra* of X if $x*y \in S$ for all $x,y \in S$. A mapping $f: X \to Y$ of BCK-algebras is a *homomorphism* if $f(x*y)=f(x)*f(y)$ for all $x,y \in X$. A nonempty subset A of a BCK-algebra X is called an *ideal* of X if, for all $x,y \in X$, it satisfies (I1) $0 \in A$, (I2) $x*y$, $y \in A \Rightarrow x \in A$. A mapping $\mu: X \to [0,1]$, where X is an arbitrary nonempty set, is called a *fuzzy set* in X. For any fuzzy set μ in X and any $\alpha \in [0,1]$, we define the set $L(\mu;\alpha)=\{x \in X \mid \mu(x) \leq \alpha\}$, which is called *lower level cut* of μ.

Definition 2.1 [2]. A fuzzy set μ in a BCK-algebra X is called *an antifuzzy ideal* of X if
(AF1) $\mu(0) \leq \mu(x)$ for all $x \in X$;
(AF2) $\mu(x) \leq \max(\mu(x*y),\mu(y))$ for all $x,y \in X$.

Definition 2.2 [7]. A triangular conorm (t-conorm S) is a mapping $S:[0,1]\times[0,1] \to [0,1]$ that satisfies the following conditions:
(S1) $S(x,0)=x$,
(S2) $S(x,y)=S(y,x)$,
(S3) $S(x,S(y,z))=S(S(x,y),z)$,
(S4) $S(x,y) \leq S(x,z)$ whenever $y \leq z$

for all $x,y,z \in [0,1]$.

Replacing 0 by 1 in condition S_1, we obtain the concept of t-norm T.

Definition 2.3. Given a t-norm T and a t-conorm S, T and S are *dual (with respect to the negation ′)* if and only if $(T(x,y))'=S(x',y')$.

Proposition 2.4. *Conjunctive (AND) operator is a t-norm T and disjunctive (OR) operator is its dual t-conorm S.*

Proposition 2.5 [5]. *For a t-conorm T, the following statement holds:*

$$S(x,y) \geq \max(x,y), \quad \forall x,y \in [0,1]. \tag{2.1}$$

Definition 2.6. Let S be a t-conorm. A fuzzy set μ in X is called sensible with respect to S if $\operatorname{Im}\mu \subseteq \Delta_S$, where $\Delta_S=\{\alpha \in [0,1] \mid S(\alpha,\alpha)=\alpha\}$.

3. Fuzzy ideals with respect to a t-conorm

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. Let S be a t-conorm. A fuzzy set $\mu : X \to [0,1]$ is called a *fuzzy ideal* of X with respect to S if

(SF1) $\mu(0) \leq \mu(x)$,

(SF2) $\mu(x) \leq S(\mu(x * y), \mu(y))$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, a, b, 1\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(x) = 0$ if $x \in \{0, a\}$ and $\mu(x) = 1$ for all $x \notin \{0, a\}$ and let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x, y) = \min(x + y, 1)$ which is a t-conorm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that μ is a sensible fuzzy ideal of X with respect to S_m.

Proposition 3.3. *Let S be a t-conorm. Then every sensible fuzzy ideal of X with respect to S is an antifuzzy ideal of X.*

Proof. The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [4]. □

The converse of Proposition 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.1$, $\mu(1) = \mu(2) = \mu(3) = 0.4$ and $\mu(4) = 0.7$ is an antifuzzy ideal of X. Let $\gamma \in (0,1)$ and define the binary operation S_γ on (0,1) as follows:

$$S_\gamma(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & \text{if } \min\{\alpha, \beta\} = 0, \\ 1 & \min\{\alpha, \beta\} > 0,\ \alpha + \beta \geq 1 + \gamma, \\ \gamma & \text{otherwise} \end{cases} \tag{3.1}$$

for all $\alpha, \beta \in [0,1]$. Then S_γ is a t-conorm. Thus $S_\gamma(\mu(0), \mu(0)) = S_\gamma(0.1, 0.1) = \gamma \neq \mu(0)$ whenever $\gamma < 0.8$. Hence μ is not a sensible fuzzy ideal of X with respect to S_γ.

Theorem 3.5. *Let S be a t-conorm and μ a nonempty fuzzy set of X. Then μ is fuzzy ideal of X with respect to S if and only if each nonempty level subset $L(\mu; \alpha)$ of μ is an ideal of X.*

Proof. Suppose that μ is a fuzzy ideal of X with respect to S. Since $L(\mu,\alpha)$ is nonempty, there exists $x \in L(\mu,\alpha)$. Now, from (SF1), $\mu(0) \le \mu(x) \le \alpha$, we have $0 \in L(\mu,\alpha)$. Let $x, y \in X$ be such that $x * y \in L(\mu,\alpha)$ and $y \in L(\mu,\alpha)$. Then we have $\mu(x) \le S(\mu(x*y),\mu(y)) \le S(\alpha,\alpha) = \alpha$, and so $x \in L(\mu,\alpha)$. This shows that the level set $L(\mu,\alpha)$ is an ideal of X.

Conversely, assume that every nonempty level subset $L(\mu;\alpha)$ of μ is an ideal of X. Then it can be easily checked that μ satisfies (SF1). If there exist $x, y \in X$ such that $\mu(x) > S(\mu(x*y),\mu(y))$, then by taking $t_0 := (1/2)\{\mu(x) + S(\mu(x*y),\mu(y))\}$, we have $x * y \in L(\mu;t_0)$ and $y \in L(\mu;t_0)$. Since μ is an ideal of X, $x \in L(\mu;t_0)$, we have $\mu(x) \le t_0$, a contradiction. Hence μ is a fuzzy ideal of X with respect to S. □

Definition 3.6. Let X be a BCK-algebra and a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a BCK-algebra X. Then the union $\bigvee_{i\in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) \mid i \in I\} \tag{3.2}$$

for each $x \in X$.

Theorem 3.7. *If $\{\mu_i \mid i \in I\}$ is a family of fuzzy ideals of a BCK-algebra X with respect to S, then $\bigvee_{i\in I} \mu(x_i)$ is a fuzzy ideal of X with respect to S.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy ideals of X with respect to S. It is easy to see that $\mu_i(0) \le \mu_i(x)$ for all $x \in X$. For $x, y \in X$, we have

$$\begin{aligned}\left(\bigvee_{i\in I} \mu_i\right)(x) &= \sup\{\mu_i(x) \mid i \in I\} \le \sup\{S(\mu_i(x*y),\mu_i(y)) \mid i \in I\} \\ &= S(\sup\{\mu_i(x*y) \mid i \in I\}, \sup\{\mu_i(y) \mid i \in I\}) \\ &= S\left(\bigvee_{i\in I} \mu_i(x*y), \bigvee_{i\in I} \mu_i(y)\right).\end{aligned} \tag{3.3}$$

Hence $\bigvee_{i\in I}$ is a fuzzy ideal of X with respect to S. □

Proposition 3.8. *Every sensible fuzzy ideal of X with respect to S is order preserving.*

Proposition 3.9. *Let μ be a sensible fuzzy ideal of X with respect to S. If the inequality $x * y \le z$ holds in X, then $\mu(x) \le S(\mu(y),\mu(z))$ for all $x, y, z \in X$.*

Definition 3.10 [1]. A fuzzy set μ is called a *fuzzy subalgebra* of X with respect to a t-conorm S if $\mu(x*y) \le S(\mu(x),\mu(y))$ for all $x, y \in X$.

Theorem 3.11. *Let S be a t-conorm. Then every sensible fuzzy ideal of X with respect to S is a sensible fuzzy subalgebra of X with respect to S.*

Proof. Straightforward. □

The converse of Theorem 3.11 is not true in general as seen in the following example.

Example 3.12. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	b
c	c	c	c	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = \mu(b) = \mu(c) = 0$ and $\mu(c) = 1$ and let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x, y) = \min\{x + y, 1\}$ which is a t-conorm for all x, $y \in [0,1]$. By routine computation, we can easily check that μ is a sensible fuzzy subalgebra of X with respect to S_m. But μ is not a sensible fuzzy ideal of X with respect to S_m because $\mu(a) = 1 \geq 0 = S_m(\mu(a * b), \mu(b))$.

Remark 3.13. In Example 3.12, we observe that a sensible fuzzy subalgebra with respect to S is not a sensible fuzzy ideal with respect to S. So, a question arises: under what condition(s) a sensible fuzzy subalgebra with respect to S is a sensible fuzzy ideal with respect to S? We answer this question in the following theorems without proofs.

THEOREM 3.14. *Let S be a t-conorm. A sensible fuzzy subalgebra μ of X with respect to S is a sensible fuzzy ideal of X with respect to S if and only if for all $x, y, z \in X$, the inequality $x * y \leq z$ implies that $\mu(x) \leq S(\mu(y), \mu(z))$.*

THEOREM 3.15. *Let S be a t-conorm and let X be a BCK-algebra in which the equality $x = (x * y) * y$ holds for all distinct elements x and y of X. Then every sensible fuzzy subalgebra of X with respect to S is a sensible fuzzy ideal of X with respect to S.*

Definition 3.16. Let $f : X \to Y$ be a mapping, where X and Y are nonempty sets, and μ is fuzzy set of Y. The preimage of μ under f written μ^f is a fuzzy set of X defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

THEOREM 3.17. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If μ is a fuzzy ideal of Y with respect to S, then μ^f is a fuzzy ideal of X with respect to S.*

Proof. For any $x \in X$, we have $\mu^f(x) = \mu(f(x)) \geq \mu(\acute{0}) = \mu(f(0)) = \mu^f(0)$. Let $x, y \in X$. Then we have

$$\begin{aligned} S(\mu^f(x * y), \mu^f(y)) &= S(\mu(f(x * y)), \mu(f(y))) \\ &= S(\mu(f(x) * f(y)), \mu(f(y))) \\ &\leq \mu(f(x)) = \mu^f(x). \end{aligned} \tag{3.4}$$

Hence μ^f is a fuzzy ideal of X with respect to S. □

THEOREM 3.18. *Let $f : X \to Y$ be an epimorphism of BCK-algebras. If μ^f is a fuzzy ideal of X with respect to S, then μ is a fuzzy ideal of Y with respect to S.*

Proof. Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) = \mu^f(x) \geq \mu^f(0) = \mu(f(0)) = \mu(\acute{0})$, where $\acute{0} = f(0)$. Let $x, y \in Y$. Then there exist $a, b \in X$ such that

$f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned}\mu(x) &= \mu(f(a)) = \mu(f(a)) = \mu^f(a) \\ &\leq S(\mu^f(a * b), \mu^f(b)) = S(\mu(f(a * b)), \mu(f(b))) \\ &= S(\mu(f(a) * f(b)), \mu(f(b))) = S(\mu(x * y), \mu(y)).\end{aligned} \tag{3.5}$$

Hence μ is a fuzzy ideal of Y with respect to S. □

Definition 3.19. Let f be a mapping defined on X. If ν is a fuzzy set in $f(X)$, then the fuzzy set $\mu = \nu \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called *preimage* of ν under f.

Theorem 3.20. *Let S be a t-conorm and let $f : X \to Y$ be an epimorphism of BCK-algebras, ν sensible fuzzy ideal of Y with respect to S and μ, the preimage of ν under f. Then μ is a sensible fuzzy ideal of X with respect to S.*

Proof. The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [4]. □

Theorem 3.21. *Let μ be a fuzzy set in X and* $\operatorname{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$, *where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{A_n \mid n = 0, 1, \dots, k\}$ be a family of ideals of X with respect to a t-conorm S such that*

(i) $A_0 \subset A_1 \subset \cdots \subset A_k = X$,

(ii) $\mu(A^*) = \alpha_n$, *where* $A_n^* = A_n \setminus A_{n-1}$, $A_{-1} = \emptyset$ *for* $n = 0, 1, \dots, k$.

Then μ is a fuzzy ideal of X with respect to S.

Proof. Since $0 \in A_0$, we have $\mu(0) = \alpha_0 \leq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then we discuss the following cases: if $x * y \in A_n^*$ and $y \in A_n^*$, then $x \in A_n$ because A_n is an ideal of X. Thus

$$\mu(x) \leq \alpha_n = S(\mu(x * y), \mu(y)). \tag{3.6}$$

If $x * y \notin A_n^*$ and $y \notin A_n^*$, then the following four cases arise:

(1) $x * y \in X \setminus A_n$ and $y \in X \setminus A_n$,

(2) $x * y \in A_{n-1}$ and $y \in A_{n-1}$,

(3) $x * y \in X \setminus A_n$ and $y \in A_{n-1}$,

(4) $x * y \in A_{n-1}$ and $y \in X \setminus A_n$.

But, in either case, we know that

$$\mu(x) \leq S(\mu(x * y), \mu(y)). \tag{3.7}$$

If $x * y \in A_n^*$ and $y \notin A_n^*$, then either $y \in A_{n-1}$ or $y \in X \setminus A_n$. It follows that either $x \in A_n$ or $x \in X \setminus A_n$. Thus

$$\mu(x) \leq S(\mu(x * y), \mu(y)). \tag{3.8}$$

If $x * y \notin A_n^*$ and $y \in A_n^*$, then by similar process, we have

$$\mu(x) \leq S(\mu(x * y), \mu(y)). \tag{3.9}$$

This completes the proof. □

Definition 3.22 [9]. A BCK-algebra X is said to satisfy the ascending (resp., descending) chain condition (ACC (resp., DCC)) if for every ascending (resp., descending) sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ (resp., $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$) of ideals of X there exists a natural number n such that $A_n = A_k$ for all $n \geqslant k$. If X satisfies DCC, X is an Artin BCK-algebras.

Theorem 3.23. *Let S be a t-conorm. If μ is a fuzzy ideal of X, with respect to S, having finite image, then X is an Artin BCK-algebra.*

Proof. Suppose that there exists a strictly descending chain $A_0 \supset A_1 \supset A_2 \supset \cdots$ of fuzzy ideals of X which does not terminate at finite step. Define a fuzzy set μ in X by

$$\mu(x) := \begin{cases} \dfrac{1}{n+1} & \text{if } x \in A_n \setminus A_{n+1},\ n = 0, 1, 2, \ldots, \\ 0 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n, \end{cases} \tag{3.10}$$

where $A_0 = X$. We prove that μ is a fuzzy ideal of X with respect to S. Clearly, $\mu(0) \leq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x * y \in A_n \setminus A_{n+1}$ and $y \in A_k \setminus A_{k+1}$ for $n = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $y \in A_n$, and so $x \in A_n$ because A_n is a fuzzy ideal of X. Hence

$$\mu(x) \leq \frac{1}{n+1} = S(\mu(x * y), \mu(y)). \tag{3.11}$$

If $x * y$, $y \in \bigcap_{n=0}^{\infty} A_n$, then $x \in \bigcap_{n=0}^{\infty} A_n$. Thus

$$\mu(x) = 0 = S(\mu(x * y), \mu(y)). \tag{3.12}$$

If $x * y \notin \bigcap_{n=0}^{\infty} A_n$ and $y \in \bigcap_{n=0}^{\infty} A_n$, then there exists $k \in \mathbb{N}$ such that $x * y \in A_k \setminus A_{k+1}$. It follows that $x \in A_k$ so that

$$\mu(x) \leq \frac{1}{k+1} = S(\mu(x * y), \mu(y)). \tag{3.13}$$

Finally, suppose that $x * y \in \bigcap_{n=0}^{\infty} A_n$ and $y \notin \bigcap_{n=0}^{\infty} A_n$. Then $y \in A_r \setminus A_{r+1}$ for some $r \in \mathbb{N}$. Hence $x \in A_r$, and so

$$\mu(x) \leq \frac{1}{r+1} = S(\mu(x * y), \mu(y)). \tag{3.14}$$

Consequently, we conclude that μ is a fuzzy ideal of X with respect to S and μ has infinite number of different values. This is a contradiction, and the proof is complete. □

Theorem 3.24. *Let S be a t-conorm. The following statements are equivalent:*

(i) *every ascending chain of ideals of X with respect to S terminates at finite step,*

(ii) *the set of values of any fuzzy ideal with respect to S is a well-ordered subset of [0, 1].*

Proof. Let μ be a fuzzy ideal of X with respect to S. Suppose that the set of values of μ is not a well-ordered subset of $[0,1]$. Then there exists a strictly increasing sequence $\{\alpha_n\}$ such that $\mu(x) = \alpha_n$. Let $G_n := \{x \in X \mid \mu(x) \leq \alpha\}$. Then

$$G_1 \subset G_2 \subset G_3 \subset \cdots \tag{3.15}$$

is a strictly ascending chain of ideals of X which is not terminating. This is a contradiction.

Conversely, suppose that there exists a strictly ascending chain

$$G_1 \subset G_2 \subset G_3 \subset \cdots \tag{*}$$

of ideals of X with respect to S which does not terminate at finite step. Define a fuzzy set μ in X by

$$\mu(x) := \begin{cases} \frac{1}{k}, & \text{where } k = \max\{n \in \mathbb{N} \mid x \in G_n\}, \\ 1 & \text{if } x \in G_n, \end{cases} \tag{3.16}$$

where $G = \bigcup_{n\in\mathbb{N}} G_n$. Since $0 \in G_n$ for all $n = 0,1,\ldots$, therefore, $\mu(0) \leq \mu(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y$, $y \in G_n \backslash G_{n-1}$ for $n = 2,3,\ldots$, then $x \in G_n$. Thus, we obtain

$$\mu(x) \leq \frac{1}{n} = S(\mu(x*y),\mu(y)). \tag{3.17}$$

Assume that $x * y \in G_n$ and $y \in G_n \backslash G_m$ for all $m < n$. Since μ is an ideal of X, therefore, $x \in G_n$. Thus

$$\mu(x) \leq \frac{1}{n} \leq \frac{1}{m+1} \leq \mu(y), \tag{3.18}$$

and hence

$$\mu(x) \leq S(\mu(x*y),\mu(y)). \tag{3.19}$$

Similarly, for the case $x * y \in G_n \backslash G_m$ and $y \in G_n$, we have

$$\mu(x) \leq S(\mu(x*y),\mu(y)). \tag{3.20}$$

Hence μ is an ideal of X with respect to t-conorm S. Since the chain (*) is not terminating, μ has strictly descending sequence of values. This contradicts that the value of any set of fuzzy ideal with respect to S is well ordered. This ends the proof. □

Lemma 3.25. *Let T be a t-norm. Then t-conorm S can be defined as*

$$S(x,y) = 1 - T(1-x, 1-y). \tag{3.21}$$

Proof. Straightforward. □

Theorem 3.26. *A fuzzy set μ of a BCK-algebra X is a T-fuzzy ideal of X if and only if its complement μ^c is an S-fuzzy ideal of X.*

Proof. Let μ be a T-fuzzy ideal of X. For $x, y \in X$, we have

$$\begin{aligned}\mu^c(0) &= 1-\mu(0) \le 1-\mu(x) = \mu^c(x),\\ \mu^c(x) &= 1-\mu(x) \le 1 - T\mu((x*y),\mu(y))\\ &= 1 - T1 - \mu^c((x*y),1-\mu^c(y))\\ &= S(\mu^c(x*y),\mu^c(y)).\end{aligned} \tag{3.22}$$

Hence μ^c is an S-fuzzy ideal of X.

The converse is proved similarly. □

4. S-product and direct product with respect to a t-conorm

In this section, we discuss properties of S-product and direct product of fuzzy ideals of a BCK-algebra with respect to a t-conorm.

Definition 4.1. Let S be a t-conorm and let λ and μ be two fuzzy sets in X. Then the S-product of λ and μ is denoted by $[\lambda\cdot\mu]_S$ and defined by $[\lambda\cdot\mu]_S(x) = S(\lambda(x),\mu(x))$, for all $x \in X$.

Theorem 4.2. *Let λ and μ be two fuzzy ideals of X with respect to S. If a t-conorm S^* dominates S, that is, if $S^*(S(\alpha,\gamma),S(\beta,\delta)) \le S(S^*(\alpha,\beta),S^*(\gamma,\delta))$ for all α, $\beta,\gamma,\delta \in [0,1]$, then S^*-product $[\lambda\cdot\mu]_S*$ is a fuzzy ideal of X with respect to S.*

Proof. For any $x \in X$, we have

$$[\lambda\cdot\mu]_S*(0) = S^*(\lambda(0),\mu(0)) \le S^*(\lambda(x),\mu(x)) = [\lambda\cdot\mu]_{S*}(x). \tag{4.1}$$

Let $x, y \in X$. Then

$$\begin{aligned}[\lambda\cdot\mu]_S*(x) &= S^*(\lambda(x),\mu(x))\\ &\le S^*(S(\lambda(x*y),\lambda(y)),S(\mu(x*y),\mu(y)))\\ &\le S(S^*(\lambda(x*y),\mu(x*y)),S^*(\lambda(y),\mu(y)))\\ &= S([\lambda\cdot\mu]_S*(x*y),[\lambda\cdot\mu]_S*(y)).\end{aligned} \tag{4.2}$$

Hence $[\lambda\cdot\mu]_S*$ is a fuzzy ideal of X with respect to S. □

Theorem 4.3. *Let S and S^* be t-conorms in which S^* dominates S. Let $f : X \to Y$ be an epimorphism of BCK-algebras. If λ and μ are fuzzy ideals of Y with respect to S, then $f^{-1}([\lambda\cdot\mu]_S*) = [f^{-1}(\lambda), f^{-1}(\mu)]_S*$.*

Proof. For any $x \in X$, we have

$$\begin{aligned}f^{-1}([\lambda\cdot\mu]_S*)(x) &= [\lambda\cdot\mu]_S*(f(x)) = S^*(\lambda(f(x)),\mu(f(x)))\\ &= S^*([f^{-1}(\lambda)](x),[f^{-1}(\mu)](x)) = [f^{-1}(\lambda),f^{-1}(\mu)]_S*(x).\end{aligned} \tag{4.3}$$

□

Theorem 4.4. *Let S be a t-conorm. Let X_1 and X_2 be BCK-algebras and let $X = X_1 \times X_2$ be the direct product BCK-algebra of X_1 and X_2. Let λ be a fuzzy ideal of a BCK-algebra X_1 with* *respect to S and let μ be a fuzzy ideal of a BCK-algebra X_2 with respect to S. Then $\nu = \lambda \times \mu$ is a fuzzy ideal of $X = X_1 \times X_2$ with respect to S defined by*

$$\nu(x_1,x_2) = (\lambda \times \mu)(x_1,x_2) = S(\lambda(x_1),\mu(x_2)). \tag{4.4}$$

Proof. For any $(x,y) \in X_1 \times X_2 = X$, we have

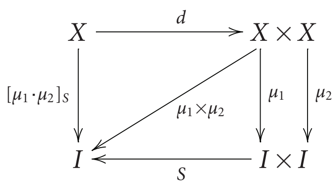
$$\begin{aligned} \nu(0,0) &= (\lambda \times \mu)(0,0) = S(\lambda(0),\mu(0)) \\ &\leq S(\lambda(x),\mu(y)) = (\lambda \times \mu)(x,y) = \nu(x,y). \end{aligned} \tag{4.5}$$

Let $x = (x_1,x_2)$ and $y = (y_1,y_2) \in X_1 \times X_2 = X$. Then we have

$$\begin{aligned} \nu(x) &= (\lambda \times \mu)(x) = (\lambda \times \mu)(x_1,x_2) = S(\lambda(x_1),\mu(x_2)) \\ &\leq S(S(\lambda(x_1 * y_1),\lambda(y_1)),S(\mu(x_2 * y_2),\mu(y_2))) \\ &= S(S(\lambda(x_1 * y_1),\mu(x_2 * y_2)),S(\lambda(y_1),\mu(y_2))) \\ &= S((\lambda \times \mu)(x_1 * y_1,x_2 * y_2),(\lambda \times \mu)(y_1,y_2)) \\ &= S((\lambda \times \mu)((x_1,x_2) * (y_1,y_2)),(\lambda \times \mu)(y_1,y_2)) \\ &= S((\lambda \times \mu)(x * y),(\lambda \times \mu)(y)) = S(\nu(x * y),\nu(y)). \end{aligned} \tag{4.6}$$

Hence ν is a fuzzy ideal of X with respect to S. □

The relationship between fuzzy ideals $\mu_1 \times \mu_2$ and $[\mu_1 \cdot \mu_2]_S$ with respect to S can be viewed via the following diagram:



(4.7)

where $I = [0,1]$ and $d : X \to X \times X$ is defined by $d(x) = (x,x)$. It is easy to see that $[\mu_1 \cdot \mu_2]_S$ is the preimage of $\mu_1 \times \mu_2$ under d.

Converse of Theorem 4.4 may not be true as seen in the following example.

Example 4.5. Let X be a BCK-algebra and let $s,t \in [0,1]$. Define fuzzy sets μ_1 and μ_2 in X by $\mu_1(x) = 1$ and

$$\mu_2(x) = \begin{cases} 1 & \text{if } x = 0, \\ t & \text{otherwise} \end{cases} \tag{4.8}$$

for all $x \in X$, respectively.

If $x = 0$, then $\mu_2(x) = 1$, and thus

$$(\mu_1 \times \mu_2)(x,x) = S(\mu_1(x),\mu_2(x)) = S(1,1) = 1. \tag{4.9}$$

If $x \neq 0$, then $\mu_2(x) = t$, and thus

$$(\mu_1 \times \mu_2)(x,x) = S(\mu_1(x),\mu_2(x)) = S(1,t) = 1. \tag{4.10}$$

That is, $\mu_1 \times \mu_2$ is a constant function and so $\mu_1 \times \mu_2$ is a fuzzy ideal of $X_1 \times X_2$. Now μ_1 is a fuzzy ideal of X, but μ_2 is not a fuzzy ideal of X since for $x \neq 0$, we have $\mu_2(0) = 1 > t = \mu_2(x)$.

Now we generalize the product of two fuzzy ideals with respect to S to the product of n fuzzy ideals with respect to S. We first need to generalize the domain of t-conorm S to $\prod_{i=1}^{n}[0,1]$ as follows.

Definition 4.6. The function $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by

$$S_n(\alpha_1,\alpha_2,\ldots,\alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1,\alpha_2,\ldots,\alpha_i,\alpha_{i+1},\ldots,\alpha_n)) \tag{4.11}$$

for all $1 \leq i \leq n$, $n \geq 2$, $S_2 = S$, and $S_1 =$ identity.

Lemma 4.7. *For a t-conorm S and every $\alpha_i,\beta_i \in [0,1]$, where $1 \leq i \leq n$, $n \geq 2$,*

$$S_n(S(\alpha_1,\beta_1),S(\alpha_2,\beta_2),\ldots,S(\alpha_n,\beta_n)) = S(S_n(\alpha_1,\alpha_2,\ldots,\alpha_n),S_n(\beta_1,\beta_2,\ldots,\beta_n)). \tag{4.12}$$

Theorem 4.8. *Let S be a t-conorm and let $X = \prod_{i=0}^{n} X_i$ be the direct product of BCK-algebras. If μ_i is a fuzzy ideal of X_i with respect to S, where $1 \leq i \leq n$, then $\mu = \prod_{i=1}^{n}\mu_i$ defined by*

$$\mu(x) = \left(\prod_{i=1}^{n}\mu_i\right)(x_1,x_2,\ldots,x_n) = S_n(\mu_1(x_1),\mu_2(x_2),\ldots,\mu_n(x_n)) \tag{4.13}$$

for all $x = (x_1,x_2,\ldots,x_n) \in X$ is a fuzzy ideal of X with respect to S.

Proof. Clearly, $\mu(0) \leq \mu(x)$ for all $x = (x_1,x_2,\ldots,x_n) \in X = \prod_{i=1}^{n} X_i$.

Let $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots,y_n)$ be the elements of $X = \prod_{i=1}^{n} X_i$. Then

$$\begin{aligned}
\mu(x) &= \left(\prod_{i=1}^{n}\mu_i\right)(x_1,x_2,\ldots,x_n) = S_n(\mu_1(x_1),\mu_2(x_2),\ldots,\mu_n(x_n)) \\
&\leq S_n(S(\mu_1(x_1 * y_1),\mu(y_1)),S(\mu_2(x_2 * y_2),\mu(y_2)),\ldots,S(\mu_n(x_n * y_n),\mu(y_n))) \\
&= S(S_n(\mu_1(x_1 * y_1),\mu_2(x_2 * y_2),\ldots,\mu_n(x_n * y_n)),S_n(\mu(y_1),\mu(y_2),\ldots,\mu(y_n))) \\
&= S\left(\left(\prod_{i=1}^{n}\mu_i\right)(x_1 * y_1,x_2 * y_2,\ldots,x_n * y_n),\left(\prod_{i=1}^{n}\mu_i\right)(y_1,y_2,\ldots,y_n)\right) \\
&= S(\mu(x * y),\mu(y)).
\end{aligned} \tag{4.14}$$

Hence $\mu = \prod_{i=1}^{n}\mu_i$ is a fuzzy ideals of X with respect to S. □

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M. Akram: Punjab University College of Information Technology, University of The Punjab,
Allama Iqbal Campus (Old Campus), P. O. Box 54000, Lahore, Pakistan
E-mail address: m.akram@pucit.edu.pk

Jianming Zhan: Department of Mathematics, Hubei Institute for Nationalities, Enshi,
Hubei Province 445000, China
E-mail address: zhanjianming@hotmail.com