CLOSED CONFORMAL VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Received 1 March 2006; Revised 24 July 2006; Accepted 8 August 2006

We give here a geometric proof of the existence of certain local coordinates on a pseudo-Riemannian manifold admitting a closed conformal vector field.

1. Introduction

A vector field V on a pseudo-Riemannian manifold (M,g) is called *conformal* if

$$\mathcal{L}_V g = 2\lambda g \tag{1.1}$$

for a scalar field λ, where $\mathcal{L}$ denotes the Lie derivative on M. It is easy to see that if V is locally a gradient field, then (1.1) is equivalent to

$$\nabla_X V = \lambda X \quad \text{for every vector field } X. \tag{1.2}$$

Here ∇ denotes the Levi-Civita connection of g. We call vector fields satisfying (1.2) *closed conformal vector fields.* They appear in the work of Fialkow [3] about conformal geodesics, in the works of Yano [7–11] about concircular geometry in Riemannian manifolds, and in the works of Tashiro [6], Kerbrat [4], Kühnel and Rademacher [5], and many other authors.

If V is lightlike on (M,g), then from (1.2), we get

$$Xg(V,V) = 2g(\nabla_X V, V) = 2\lambda g(X,V) = 0 \tag{1.3}$$

for every vector field X. Thus $\lambda \equiv 0$ and V is parallel. About lightlike parallel vector fields, we have the following theorem.

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 36545, Pages 1–8
DOI 10.1155/IJMMS/2006/36545

THEOREM 1.1 (Brinkmann [2]). *If (M,g) admits a lightlike parallel vector field V, then there are local coordinates $u^1,u^2,\dots,u^n$ $(n := \dim M > 2)$ such that $V = \partial/\partial u^1$ and*

$$(g_{ij}) = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right), \tag{1.4}$$

where $\alpha,\beta \in \{3,\dots,n\}$ and $\partial g_{\alpha\beta}/\partial u^1 = 0$.

Brinkmann's proof is purely analytical. We will give, in the next section, geometric tools which will allow us to generalize Brinkmann's theorem.

2. Geometric constructions

Let (M,g) be a connected pseudo-Riemannian manifold of dimension n and signature $(k,n-k)$ with $0 < k < n$. Given a vector field W on M, we denote by $W^\flat$ the one-form defined by $W^\flat(X) = g(W,X)$. Then W is locally a gradient field if and only if $dW^\flat = 0$. In the following, a vector field W satisfying $\nabla_W W = 0$ will be called *geodesic*.

LEMMA 2.1. *If W is a geodesic vector field, then $dW^\flat$ is invariant under the flow of W.*

Proof. Let $(\nabla W^\flat)(X,Y) = (\nabla_X W^\flat)(Y) = g(\nabla_X W, Y)$. Then, from the fact that W is geodesic, it follows that

$$\begin{aligned}(\mathcal{L}_W \nabla W^\flat)(X,Y) &= Wg(\nabla_X W, Y) - g(\nabla_{[W,X]} W, Y) - g(\nabla_X W, [W,Y]) \\ &= g(R(W,X)W, Y) + g(\nabla_X W, \nabla_Y W),\end{aligned} \tag{2.1}$$

where R denotes the Riemannian curvature tensor,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.2}$$

Since $g(R(W,X)W,Y)$ is symmetric with respect to X, Y, from

$$dW^\flat(X,Y) = (\nabla W^\flat)(X,Y) - (\nabla W^\flat)(Y,X), \tag{2.3}$$

we get $(\mathcal{L}_W dW^\flat)(X,Y) = (\mathcal{L}_W \nabla W^\flat)(X,Y) - (\mathcal{L}_W \nabla W^\flat)(Y,X) = 0$. □

LEMMA 2.2. *If W is a lightlike geodesic vector field, then $dW^\flat(X,W) = 0$.*

Proof. We have the following.

$$\left.\begin{array}{l} W \text{ lightlike } \Rightarrow (\nabla W^\flat)(X,W) = g(\nabla_X W, W) = 0 \\ W \text{ geodesic } \Rightarrow (\nabla W^\flat)(W,X) = g(\nabla_W W, X) = 0 \end{array}\right\} \Rightarrow dW^\flat(X,W) = 0. \qquad \square$$

A nontangent vector field $\widetilde{W}$ on a pseudo-Riemannian hypersurface $\widetilde{M}$ can be extended to a geodesic vector field W in a neighbourhood of $\widetilde{M}$ in the following way. Let $c(s,p)$ be the geodesic starting at $p = c(0,p) \in \widetilde{M}$ with $\dot{c}(0,p) = \widetilde{W}(p)$ and $W(c(s,p)) := \dot{c}(s,p)$. Then, taking into account the fact that $\widetilde{W}$ is transversal (i.e. nontangent) to $\widetilde{M}$, we conclude that W is a geodesic vector field on a neighbourhood of $\widetilde{M}$ extending $\widetilde{W}$. Moreover, if $\widetilde{W}$ is lightlike, then so is W. Denoting with $\widetilde{W}^\top$, $\widetilde{W}^\perp$ the tangent and normal component of $\widetilde{W}$, for vector fields X, Y on $\widetilde{M}$ tangent to $\widetilde{M}$, we have the following lemma.

Lemma 2.3. $dW^\flat(X,Y) = d(\widetilde{W}^\top)^\flat(X,Y)$.

Proof. The statement follows from $g(\nabla_X \widetilde{W}^\perp, Y) - g(\nabla_Y \widetilde{W}^\perp, X) = -g(\widetilde{W}^\perp, [X,Y]) = 0$. $\square$

The following remark will be used in the proof of the next proposition.

Remark 2.4. Let V be a vector field and let φ be a function on M. At a point $p_0 \in M$, the gradient of the solutions of $Vf = \varphi$ span an affine hyperplane H of $T_{p_0}M$. Let $v := V(p_0)$, then $H = \{x \in T_{p_0}M \mid g(x,v) = \varphi(p_0)\}$ and

(a) if $\varphi(p_0) \neq 0$, then H contains lightlike, spacelike, and timelike vectors,

(b) if $\varphi(p_0) = 0$, then H contains only lightlike vectors and the zero vector if and only if $n = 2$ and v is lightlike.

Proposition 2.5. *If V is a closed conformal vector field on (M,g), then in a neighbourhood of a point p_0 where $V(p_0) \neq 0$, there is a lightlike geodesic gradient field W such that $g(V,W) = 1$.*

Proof. We divide the proof into two cases.
Case 1. $n > 2$ or $n = 2$ and $V(p_0)$ is nonlightlike.

Let u be a solution of $Vu = 0$ with $g(p_0)(\nabla u, \nabla u) \neq 0$ (here ∇u denotes the gradient of u). According to Remark 2.4(b), such a solution exists. Let $\mathcal{U}$ be an open neighbourhood of p_0 on which $g(\nabla u, \nabla u) \neq 0$, and let $\widetilde{M}$ be the pseudo-Riemannian hypersurface $u^{-1}(u(p_0)) \cap \mathcal{U}$. Then ∇u is a normal vector field on $\widetilde{M}$ and, from $Vu = 0$, we have that $\tilde{V} := V|_{\widetilde{M}}$ is a tangent vector field on $\widetilde{M}$. Let $\tilde{f} : \widetilde{M} \to \mathbb{R}$ be a solution of $\tilde{V}\tilde{f} = 1$ such that $g(p_0)(\nabla \tilde{f}, \nabla \tilde{f})$ and $g(p_0)(\nabla u, \nabla u)$ have opposite sign (see Remark 2.4(a)). Without loss of generality, we assume that $g(\nabla \tilde{f}, \nabla \tilde{f}) \neq 0$ on $\widetilde{M}$. Setting $\widetilde{W} := \nabla \tilde{f} + h\nabla u$, where $h^2 := -g(\nabla \tilde{f}, \nabla \tilde{f})/g(\nabla u, \nabla u) > 0$, we get

$$g(\widetilde{W}, \widetilde{W}) = g(\nabla \tilde{f}, \nabla \tilde{f}) + h^2 g(\nabla u, \nabla u) = 0, \qquad g(\tilde{V}, \widetilde{W}) = \tilde{V}\tilde{f} = 1. \tag{2.4}$$

Let now W be the geodesic vector field extending $\widetilde{W}$ in a neighbourhood of $\widetilde{M}$. Then W is lightlike. From $Wg(V,W) = g(\nabla_W V, W) + g(V, \nabla_W W) = 0$ and $g(\tilde{V}, \widetilde{W}) = 1$, we conclude that $g(V,W) = 1$. It remains to show that W is locally a gradient.

For vector fields X, Y on $\widetilde{M}$ (not necessarily tangent to $\widetilde{M}$), we can write

$$X = X^{\top} + \alpha\widetilde{W}, \qquad Y = Y^{\top} + \beta\widetilde{W}, \tag{2.5}$$

where α and β are certain functions on $\widetilde{M}$ and $X^{\top}$, $Y^{\top}$ are tangent to $\widetilde{M}$. Using Lemma 2.2, we get

$$0 = dW^{\flat}(X, W) = dW^{\flat}(X^{\top} + \alpha W, W) = dW^{\flat}(X^{\top}, W). \tag{2.6}$$

In the same way, we get $dW^{\flat}(W, Y^{\top}) = 0$, and therefore $dW^{\flat}(X,Y) = dW^{\flat}(X^{\top}, Y^{\top})$. Now Lemma 2.3 and $\widetilde{W}^{\top} = \nabla\widetilde{f}$ imply that $dW^{\flat}(X,Y) = 0$ on $\widetilde{M}$. Using Lemma 2.1, we conclude that $dW^{\flat} = 0$.

Case 2. $n = 2$ and $V(p_0)$ is lightlike.

According to Remark 2.4(b), we cannot proceed as in Case 1 since the gradient at p_0 of a solution of $Vu = 0$ is a lightlike vector. Remarking that along an integral curve α of V through p_0 V is lightlike, we set $\widetilde{M} := Im\alpha$. Let now $\widetilde{W}$ be a lightlike vector field along α such that V and $\widetilde{W}$ are linearly independent. Then, since g is nondegenerate, $g(V,V)g(\widetilde{W},\widetilde{W}) - g(V,\widetilde{W})^2 = -g(V,\widetilde{W})^2 \neq 0$. Therefore we can assume that $g(V,\widetilde{W}) = 1$. Since $\widetilde{W}$ is not tangent to α, we can extend it to a geodesic vector field W on a neighbourhood $\mathcal{U}$ of p_0. Then $Wg(W,W) = 0$ which, together with $\widetilde{W}$ lightlike, implies W lightlike, and $Wg(V,W) = g(\nabla_W V, W) = 0$ which, together with $g(V,\widetilde{W}) = 1$, implies $g(V,W) = 1$. Since every vector field on $\mathcal{U}$ can be written as a linear combination of V and W, we have $g(\nabla_X W, Y) - g(\nabla_Y W, X) = 0$ for every vector field X, Y on $\mathcal{U}$ if and only if $g(\nabla_V W, W) - g(\nabla_W W, V) = 0$.

Thus W being lightlike and geodesic implies that W is a gradient vector field.

It remains to show that V is lightlike along an integral curve α through $p_0 := \alpha(0)$. This follows from $(d/dt)g(V,V) = 2g(\nabla_V V, V) = 2\lambda g(V,V)$, since its general solution is $g(\alpha(t))(V,V) = g(p_0)(V,V)e^{2\int_0^t \lambda(u)du}$. □

For example, let $M = \mathbb{R}^n_k$ be the pseudo-Euclidian space of dimension n and signature $(k, n-k)$ with $0 < k < n$, that is, $\langle x,x\rangle = -(x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2)$. The position vector field $V(x) = \sum_{i=1}^n x_i(\partial/\partial x_i)|_x$ satisfies $\nabla_X V = X$, and therefore it is a closed conformal vector field. We will construct, following the proof of Proposition 2.5, a lightlike geodesic gradient field W with $\langle V, W\rangle = 1$ in a neighbourhood of a point $x_0 \neq 0$ ($V(x) = 0$ if and only if $x = 0$). We take for simplicity $x_0 = (1, 0, \dots, 0)$, then $u(x_1, \dots, x_n) := x_n/x_1$ is a solution of $Vu = 0$ with $\langle \nabla u, \nabla u\rangle|_{x_0} = 1$. The hypersuface $\widetilde{M} := u^{-1}(u(x_0)) = u^{-1}(0)$ is the hyperplane $x_n = 0$. Let $\widetilde{V} := V|_{\widetilde{M}}$, then $\widetilde{f}(x_1, \dots, x_{n-1}) := \ln x_1$ is a solution of $\widetilde{V}\widetilde{f} = 1$ with $\langle \nabla\widetilde{f}, \nabla\widetilde{f}\rangle|_{x_0} = -1$. Defining for every $x \in \widetilde{M}$ that

$$\widetilde{W}(x) := \nabla\widetilde{f}(x) + \nabla u(x) = \frac{1}{x_1}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n}\right)\bigg|_x, \tag{2.7}$$

it is easy to see that

$$W(x) := \frac{1}{x_1 + x_n}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n}\right)\bigg|_x \tag{2.8}$$

is a geodesic vector field on M extending $\widetilde{W}$. Moreover W is lightlike, $\langle V,W\rangle = 1$, and $W = \nabla \ln|x_1 + x_n|$. It is clear that W is not unique and not everywhere defined. More generally, for an arbitrary point $x_0 \neq 0$, we have, for instance, that

$$W = \nabla \ln |\langle a,x\rangle|, \quad \text{where } a \text{ is a lightlike vector in } \mathbb{R}^n_k \text{ with } \langle a,x_0\rangle \neq 0, \tag{2.9}$$

is a lightlike geodesic gradient field satisfying $\langle V,W\rangle = 1$.

Finally we remark that a nontrivial conformal vector field (a vector field V is nontrivial if there is a point $p \in M$ with $V(p) \neq 0$) has isolated zeros (see [4]). This is in general not true if the conformal vector field is not closed (see, e.g., an example in [1]).

3. Local coordinates

Let V and W be vector fields as in Proposition 2.5 and let $E_1 = V - g(V,V)W$, $E_2 = W$. It is easy to see that

(i) E_1, E_2 are linearly independent;

(ii) the distribution $\mathfrak{D}$ spanned by E_1, E_2 is integrable and the metric g is nondegenerate on $\mathfrak{D}$;

(iii) the distribution $\mathfrak{D}^\perp$ spanned by the vector fields orthogonal to E_1, E_2 is integrable and g is nondegenerate on $\mathfrak{D}^\perp$;

(iv) $[E_1, E_2] = 0$.

We can now state the following theorem.

Theorem 3.1. *If (M,g) admits a closed conformal vector field V, then in a neighbourhood of a point p_0 where $V(p_0) \neq 0$, there are local coordinates $u^1, u^2, \dots, u^n$ such that $V = \partial/\partial u^1 + a(\partial/\partial u^2)$, for some function $a = a(u^2)$, and*

$$(g_{ij}) = \left(\begin{array}{cc|ccc} -a & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right), \tag{3.1}$$

where $\alpha, \beta \in \{3,\dots,n\}$, $\det(g_{\alpha\beta}) \neq 0$, and $\partial g_{\alpha\beta}/\partial u^1 + a(\partial g_{\alpha\beta}/\partial u^2) = a' g_{\alpha\beta}$ ($a' := da/du^2$).

Proof. From Frobenius theorem, we know that there are local coordinates $u^1, u^2, \dots, u^n$ such that

$$\frac{\partial}{\partial u^1} = E_1, \qquad \frac{\partial}{\partial u^2} = E_2, \qquad g_{1\alpha} = g_{2\alpha} = 0, \quad \alpha = 3,\dots,n. \tag{3.2}$$

Hence $g_{11}=g(E_1,E_1)=g(V,V)-2g(V,V)g(V,W)=-g(V,V)$, $g_{12}=g(V,W)=1$, $g_{22}=g(W,W)=0$ and, setting $E_i=\partial/\partial u^i$, $i=1,\dots,n$, we have that

$$\begin{aligned}\frac{\partial g_{\alpha\beta}}{\partial u^1}+a\frac{\partial g_{\alpha\beta}}{\partial u^2}&=g(\nabla_{E_1}E_\alpha+g(V,V)\nabla_{E_2}E_\alpha,E_\beta)\\&\quad+g(E_\alpha,\nabla_{E_1}E_\beta+g(V,V)\nabla_{E_2}E_\beta)\\&=g(\nabla_{E_\alpha}E_1+g(V,V)\nabla_{E_\alpha}E_2,E_\beta)\\&\quad+g(E_\alpha,\nabla_{E_\beta}E_1+g(V,V)\nabla_{E_\beta}E_2)\\&=g(\nabla_{E_\alpha}(E_1+g(V,V)E_2),E_\beta)\\&\quad+g(E_\alpha,\nabla_{E_\beta}(E_1+g(V,V)E_2))\\&=g(\nabla_{E_\alpha}V,E_\beta)+g(E_\alpha,\nabla_{E_\beta}V,)=2\lambda g_{\alpha\beta},\end{aligned}\tag{3.3}$$

where $a=g(V,V)$. From $Xg(V,V)=2\lambda g(X,V)$ and $g(E_1,V)=g(E_3,V)=\cdots=g(E_n,V)=0$, we conclude that $a=a(u^2)$. Furthermore

$$a'=Wg(V,V)=2\lambda\tag{3.4}$$

and $a=0$ if and only if V is lightlike (cf. with Brinkmann's theorem). □

On the other hand, we have the following proposition.

Proposition 3.2. *If on a neighbourhood $\mathcal{U}$ of a point $p_0\in M$, there are local coordinates as in Theorem 3.1, then $V=\partial/\partial u^1+a(\partial/\partial u^2)$ is a closed conformal vector field on $\mathcal{U}$.*

Proof. The statement follows from

$$\begin{aligned}g(\nabla_{E_i}V,E_j)&=g(\nabla_{E_i}E_1,E_j)+a'\delta_{2i}\delta_{1j}+ag(\nabla_{E_i}E_2,E_j)\\&=\frac{1}{2}\left(\frac{\partial g_{1j}}{\partial u^i}+\frac{\partial g_{ij}}{\partial u^1}-\frac{\partial g_{1i}}{\partial u^j}+a\frac{\partial g_{ij}}{\partial u^2}\right)+a'\delta_{2i}\delta_{1j}\\&=\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^1}+a\frac{\partial g_{ij}}{\partial u^2}\right)+\frac{1}{2}a'(\delta_{1i}\delta_{2j}+\delta_{2i}\delta_{1j}),\end{aligned}\tag{3.5}$$

where δ is the Kronecker delta. Namely, for every pair (i,j), we get $g(\nabla_{E_i}V,E_j)=(1/2)a'g_{ij}$. Moreover, V is lightlike if and only if $a=0$. □

Remark 3.3. If in Proposition 3.2 we assume that $a\neq 0$, then according to Fialkow results, see [3, formulas (12.9) and (12.10)], we must be able to prove that $(\mathcal{U},g)$ is locally isometric to a warped product with a one-dimensional base manifold. This can be seen in the following way: take local coordinates $\overline{u}^1,\dots,\overline{u}^n$ in $\mathcal{U}$ such that

$$\frac{\partial}{\partial\overline{u}^1}=\frac{1}{\sqrt{|a|}}\left(\frac{\partial}{\partial u^1}+a\frac{\partial}{\partial u^2}\right),\qquad \frac{\partial}{\partial\overline{u}^2}=\frac{\partial}{\partial u^1},\qquad \frac{\partial}{\partial\overline{u}^\alpha}=\frac{\partial}{\partial u^\alpha},\quad \alpha=3,\dots,n. \tag{3.6}$$

This is reached by the coordinate transformation

$$\overline{u}^1=\int\frac{\sqrt{|a|}}{a}du^2,\qquad \overline{u}^2=u^1-\int\frac{1}{a}du^2,\qquad \overline{u}^\alpha=u^\alpha,\quad \alpha=3,\dots,n. \tag{3.7}$$

Then it is easy to see that $a=a(\overline{u}^1)$ and that

$$(\overline{g}_{ij}):=\left(g\left(\frac{\partial}{\partial\bar{u}^i},\frac{\partial}{\partial\bar{u}^j}\right)\right)=\left(\begin{array}{c|cccc}\pm1&0&0&\cdots&0\\ \hline 0&-a&0&\cdots&0\\ 0&0&&&\\ \vdots&\vdots&&(g_{\alpha\beta})&\\ 0&0&&&\end{array}\right). \tag{3.8}$$

Furthermore, from $\partial g_{\alpha\beta}/\partial u^1+a(\partial g_{\alpha\beta}/\partial u^2)=a'g_{\alpha\beta}$, we get

$$\frac{\partial g_{\alpha\beta}}{\partial\overline{u}^1}=\frac{1}{\sqrt{|a|}}\left(\frac{\partial g_{\alpha\beta}}{\partial u^1}+a\frac{\partial g_{\alpha\beta}}{\partial u^2}\right)=\frac{1}{\sqrt{|a|}}\frac{da}{du^2}g_{\alpha\beta}=\frac{1}{a}\frac{da}{d\overline{u}^1}g_{\alpha\beta}, \tag{3.9}$$

and therefore $g_{\alpha\beta}=a\overline{g}_{\alpha\beta}$, where $\partial\overline{g}_{\alpha\beta}/\partial\overline{u}^1=0$. Thus $(\mathcal{U},g)$ is locally isometric to a warped product with a one-dimensional base manifold and warped factor a. In these local coordinates, the metric of the fiber manifold is given by

$$\left(\begin{array}{c|ccc}-1&0&\cdots&0\\ \hline 0&&&\\ \vdots&&(\overline{g}_{\alpha\beta})&\\ 0&&&\end{array}\right) \tag{3.10}$$

which means, in other words, that $\overline{u}^2,\dots,\overline{u}^n$ are Fermi coordinates on the fiber manifold.

Acknowledgment

The author wishes to thank Professor K. Voss of the Swiss Federal Institute of Technology in Zurich for helpful suggestions on the subject.

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