ONE-PILE MISÈRE NIM FOR THREE OR MORE PLAYERS

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The analysis of the classical game of Nim relies on binary representation of numbers as shown in the books of Berlekamp and Gardener. There is considerable interest in generalizations and modifications of the game. We will consider one-pile misère Nim for more than two players. In the case of three or more players, the impartial game theory results rarely apply. In this note, we analyze the game in a variety of cases where alliances are formed among the players.

The traditional game of Nim is an impartial game for two players that plays a central role in combinatorial game theory. The analysis for several variants of the game relies on binary representation (see [1, 2]). There is great interest in generalizations and modifications of the game. We will consider a one-pile misère Nim version for more than two players.

The rules for one-pile Nim are as follows. Players alternate turns by taking up to m counters from the pile of N counters. The player who takes the last counter wins. In the misère version, the player who takes the last counter loses. For the two-player version, the analysis of the game can also be described using modular calculations. In one-pile misère Nim for two players having maximum move m, K is a losing position for the first player if and only if $K \bmod (m+1) = 1$.

We generalize the one-pile misère Nim for more than two players. In the case of three or more players, the impartial game theory results do not always apply. Moreover, the result of the game cannot be completely determined without considering alliances.

For example, here is the game tree for three-player one-pile misère Nim starting with 5 counters, where each player can take 1 or 2 counters at their turn. Notice that we have omitted the branches that would lead to a player's immediate loss as shown in the following diagram:

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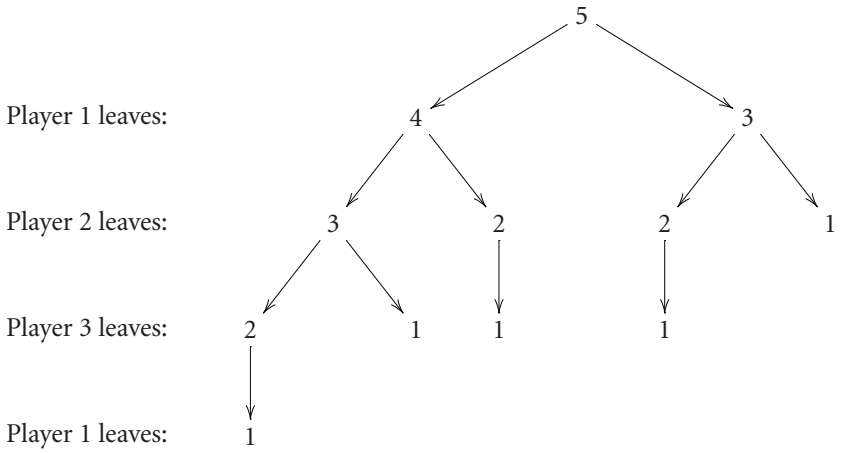

In this game, player 2 is safe if he plays wisely and he also determines who will lose the game. Hence player 1 and player 3 should each attempt to persuade player 2 to cooperate with them. This leads us to considering the game with known alliances. The game of misère Nim with alliances resembles the early phases of the "Survivor" game.

Definition 1. Consider the game for three or more players that follows the rules of misère Nim for each player. Suppose that the players form two alliances and that each player is in exactly one alliance. Also assume that each player will support his alliance's interests. This game will be called *Survivor Nim*.

We first consider several games with an even number of players.

Proposition 2. *Consider Survivor Nim for $2n$ players, where the even-numbered players form one alliance and the odd-numbered players form another alliance. Suppose each player is allowed to remove $1,\dots,m$ counters on his turn. This game is isomorphic to the two-player one-pile misère Nim, where each player removes $1,\dots,m$ counters at his turn.*

Proof. Suppose $P_1,P_3,P_5,\dots,P_{2n-1}$ form one alliance while $P_2,P_4,P_6,\dots,P_{2n}$ form another alliance. By applying the calculations for determining the advantage in the two-player game, each player in the alliance with an advantage will be able to make a move to maintain the advantage for his alliance. Hence we can look at this game as a two-player game. □

In the games, where the alliances formed involve two or more consecutive players, the end of the game differs from genuine two-player misère Nim. In misère Nim, the player who makes the last possible move loses. In Survivor Nim, if there remain fewer counters than an alliance's combined minimum move, the player or players in that alliance must take these (if there are more counters than each player's minimum move) and lose the game.

For example, consider a Survivor Nim game with 7 players, where each player removes 1 or 2 counters at their turn. Let the first four players form alliance 1 and the last three players form alliance 2. Suppose that at some point in the game, alliance 2 reduced the

pile to three counters with their turns. In Survivor Nim, alliance 1 removes 3 counters and loses. In misère Nim, if we consider alliance 1 as a single player, he cannot move, since he needs at least 4 counters, hence alliance 2 loses the game by making the last possible move. Because we will have several cases where the only discrepancies in comparing two games happen for a "few small values," we make the following definition. In our note, we follow the notation introduced in [1].

Definition 3. Let G_1 be a misère Nim game with K_1 counters and let G_2 be a Survivor Nim game with K_2 counters. Let j be a positive integer. Suppose the positive and zero positions in the misère Nim game with K_1 counters are the same as the positive and zero positions in the Survivor Nim game with $K_2 - j$ counters. Then say that the game G_2 is reducible to the game G_1.

PROPOSITION 4. *Consider the Survivor Nim for $2n$ players, where each player removes $1,\dots,m$ counters on his turn. If the first n players form one alliance and the second n players form another alliance, then the game is reducible to two-player misère Nim, where each player can take $n,\dots,mn$ counters.*

Proof. On his turn, each player in the winning alliance will use the strategy to support his alliance. Hence, we can look at this game as a two-player game, where we combine all the possible moves for players in a given alliance. Hence, this game is reducible to a two-player game, where each player removes $n,\dots,nm$ counters. Note that the zero positions for this Survivor game are shifted by n from the zero positions of misère Nim game with the same number of counters. □

COROLLARY 5. *The zero positions (those such that the player who starts must lose) for the Survivor Nim game in Proposition 4 satisfy the condition $N \bmod ((m+1)n) \in \{0,\dots,n\}$.*

Proof. Using the Mex rule [1, page 56] for this game, we can calculate the Nim values $\mathcal{G}$ for this game with N counters by $\mathcal{G}(N) = \operatorname{mex}(\mathcal{G}(N-n), \mathcal{G}(N-n-1), \dots, \mathcal{G}(N-mn))$. The Nim sequence for this game is given in the table below:

$$
\begin{gathered}
\frac{N}{G(N)} \left| \frac{1}{0} \right| \frac{2}{0} \left| \frac{3}{0} \right| \cdots \left| \frac{n-1}{0} \right| \frac{n}{0} \left| \frac{n+1}{1} \right| \cdots \left| \frac{2n}{1} \right| \frac{2n+1}{2} \Big| \cdots \\
\left| \frac{3n-1}{2} \right| \frac{3n}{2} \left| \cdots \right| \frac{(m+1)n-1}{m} \left| \frac{(m+1)n}{0} \right| \dots .
\end{gathered}
\tag{1}
$$

Note that this sequence is periodic and the zeros occur when $N \bmod ((m+1)n) \in \{0,\dots,n\}$. □

Combinations of unequal alliances will require a more detailed analysis. If we consider an odd number of players, we need to develop new methods of proof as these games do not follow the classical combinatorial game theory.

For example, consider a Survivor Nim game with 7 players, where each player removes 1 or 2 counters on his turn. Suppose the first four players form alliance 1, and the last three players form alliance 2. Notice that if alliance 2 reduced the pile to 1, 2, 3, or 4 counters on his turn, then alliance 1 loses. On the other hand, if alliance 1 reduced the

pile to 1, 2, or 3 counters on their turn, then alliance 2 loses. If alliance 1 reduces the pile to 4 (or 5, 6, 7, 8, 9, and 10) counters at his turn, then alliance 1 loses. Similarly, if alliance 2 reduces the pile to 5, 6, 7, 8, 9, 10, or 11 counters at his turn, then alliance 2 loses. We see from here that it is a Partizan game, that is, players have different options on a given position. The following definition introduces useful terminology we will be using for the rest of the note.

Definition 6. A collection of consecutive losing positions for one alliance is called a *round*.

Definition 7. Suppose one has a game, where k players take turns. The set of turns, where one starts with player one, and all k players have their turn, is called a *cycle*.

Theorem 8. *Consider the Survivor Nim for $2n+1$ players, where each player removes 1 or 2 counters on his turn. If the first $n+1$ players form one alliance and the last n players form another alliance, then the game is reducible to a two-player misère game, where the first player takes $n+1,\dots,2(n+1)$ counters and the second player takes $n,\dots,2n$ counters. Moreover, the largest game that the first alliance loses is with $3n^2+2n+1$ counters.*

Proof. Suppose the first $n+1$ players form alliance 1 and the last n players form alliance 2. Obviously, alliance 1 loses the games with up to $n+1$ counters. Denote the largest number of counters that alliance 1 loses in his first round of losses by $N_{11}=n+1$. Alliance 2 loses a round of games starting with $N=n+2$ up through $N-2(n+1)=n$ counters, that is, $n+2\le N\le 3n+2$. This follows because alliance 1 can take up to $2(n+1)$ counters at his turn and leaves at most n. Let us call $N_{12}=3n+2$ the largest number of counters for which alliance 2 loses in his first round of losses. Similarly, alliance 1 loses the games when $N=3n+3$ until $N-(n+1)-2n=n+1$, so $3n+3\le N\le 4n+2$. This is because no matter what alliance 1 does, alliance 2 can always leave $n+1$ counters or less after his turn. Let us call it the second round of losing positions for alliance 1 and denote the largest number by N_{21}. Moreover, alliance 2 loses for the next positions when $N=4n+3$ up to $N-2(n+1)-n=3n+2$ implying that $4n+3\le N\le 6n+4$. Let us call it the second round for alliance 2 and denote the largest number by N_{22}. To summarize these observations, we have

$$\begin{aligned} N_{11}-(n+1)&=0,\\ N_{12}-2(n+1)-n&=n,\\ N_{21}-(n+1)-2n&=N_{11},\\ N_{22}-2(n+1)-n&=N_{12}. \end{aligned} \tag{2}$$

We use induction to prove that

$$\begin{aligned} N_{j1}&=j(n+1)+(j-1)2n, \quad 0<j\le n+1,\\ N_{j2}&=j2(n+1)+jn, \quad 0<j\le n. \end{aligned} \tag{3}$$

To verify the induction step, note that if $N_{j1}=j(n+1)+(j-1)2n$ and $N_{j2}=2j(n+1)+jn$, $j>0$, then in the games with $N_{j2}+1,\dots,N_{(j+1)1}$ counters, no matter how many counters alliance 1 takes, there will be still N_{j1} or less counters left after alliance 2 turn. Hence, the largest number of counters that alliance 1 loses in the $j+1$ cycle is if they

first take $n+1$ and alliance 2 takes $2n$ and there are N_{j1} counters left. Therefore $N_{(j+1)1} = N_{j1} + (n+1) + 2n$, $0 < j \leq n+1$. One can show similarly that $N_{(j+1)2} = 2(j+1)(n+1) + (j+1)n$, $0 < j \leq n$.

This process continues and the formulas hold as long as alliance 1 gets two consecutive losses in a row during the previous cycle. If they only have one isolated losing position, then on the next round, they can always choose a position that they can win. To justify this, observe that in any game, the possible moves can be grouped as follows:

$$\begin{array}{lccc} \text{alliance 1:} & n+1 & \cdots & 2(n+1) \\ & \downarrow & & \downarrow \\ \text{alliance 2:} & n \cdots 2n & \cdots & n \cdots 2n \end{array}$$

Therefore following any moves for alliance 1, alliance 2 can leave either $N - (n+1+2n) = N - (3n+1)$ or $N - (2n+2+n) = N - (3n+2)$ counters. Alliance 1 will lose the whole game if alliance 1 must lose in both of these situations. If not, alliance 2 will lose as alliance 1 can choose his move so that all the moves for alliance 2 are losing. In general, the number of positions that alliance 1 loses in a row in each round is decreasing by 1, as this number can be computed from

$$N_{j1} - N_{(j-1)2} = j(n+1) + (j-1)2n - (j-1)n - (j-1)2(n+1) = n - j + 2. \tag{4}$$

Hence, $n - j + 2 = 2$, that is, $n = j$ is the largest cycle where alliance 1 loses at least two in a row, which proves our upper bound for j. Moreover, the largest game that alliance 1 loses is with $N_{(n+1)1} = n(n+1) + (n-1)(2n) = 3n^2 + 2n + 1$ counters. □

Corollary 9. *If alliances are partitioned as in Theorem 8, the safe games for the first alliance are the games where the number of counters N satisfies*

$$3jn + j - 2n + 1 \leq N \leq 3jn + 2j, \quad 0 < j \leq n. \tag{5}$$

Furthermore, the first alliance wins all the games with more than $3n^2 + 2n + 1$ counters.

Proof. The safe games for the first alliance are from $N_{11} + 1$ to N_{12}, from $N_{21} + 1$ to N_{22}, from $N_{31} + 1$ to N_{32}, and so forth. Using the formulas from Theorem 8, we obtain the safe positions from $N_{j1} + 1 = 3jn + j - 2n + 1$ to $N_{j2} = 3jn + 2j$, where $1 \leq j \leq n$ and all games with more than $N_{n1} = 3n^2 + 2n + 1$ counters. □

One can extend this theorem for any game with two alliances, where any group of $n+1$ consecutive players form one alliance and the other n players form the second alliance. We will show this in the sequel.

Theorem 10. *Consider the Survivor Nim for $2n+1$ players, where each player removes 1 or 2 counters at his turn. Suppose $1 \leq k \leq n$. If the players $1, \ldots, k$ and $n+k+2, \ldots, 2n+1$ form one alliance, that is called alliance 2, and $k+1, \ldots, n+k+2$ players form another alliance, that is called alliance 1, then the game is reducible to a two-player game, where the first player takes $n, \ldots, 2n$ counters and the second player takes $n+1, \ldots, 2(n+1)$ counters. Moreover, the largest game that alliance 1 loses is with $3kn + 3k + n + 1$ counters.*

Proof. Note that alliance 1 denotes the group with $n+1$ players (the players $k+1,\dots,n+k+1$) while alliance 2 has n players (the players $1,\dots,k$ and $n+k+2,\dots,2n+1$). Even though alliance 2 starts the game now, this convention fits our final analysis. The given game is a generalization of the game in the previous theorem, where we have shifted the first alliance position by k. Let us denote the largest value of losing positions in this round for alliance 2 by N_{02} as it is different from the complete cycles. Starting from the game with $k+1$ counters, alliance 1 will be losing up to the game with $N_{11}-2k-(n+1)=0$ counters. Thereafter, alliance 2 will be losing the next games up to $N_{12}=k+2(n+1)+n$ and so on. One can use induction to prove much like in Theorem 8 that

$$\begin{aligned} N_{j1} &= 2k+j(n+1)+(j-1)2n, \quad 0<j\le k+1,\\ N_{j2} &= k+j2(n+1)+jn, \quad 0\le j\le k+1. \end{aligned} \tag{6}$$

To justify the upper bound for j, note that the number of consecutive positions where alliance 1 loses in each cycle is decreasing by 1. This number can be computed from

$$N_{j1}-N_{(j-1)2}=2k+j(n+1)+(j-1)2n-k-(j-1)n-(j-1)2(n+1)=n+k-j+2, \tag{7}$$

for all values of j starting from 1 and up to the cycle where $N_{j1}-N_{(j-1)2}=1$.

Hence, $n+k-j+2=n+2$, that is, $k=j$ is the last cycle where alliance 1 loses $n+2$ in a row. The largest round alliance 1 loses is the round $k+1$, since on the next round alliance 1 has $n+2$ choices and can choose a winning one. This verifies our upper bound for j. Moreover, the largest game that alliance 1 loses is with $N_{(k+1)1}=2k+(k+1)(n+1)+(k)(2n)=3kn+3k+n+1$ counters. □

Corollary 11. *Suppose the alliances are partitioned as in Theorem 10, where alliance 1 has $n+1$ players, alliance 2 has n players, and alliance 2 starts the game. The safe positions for alliance 1 are the games with $N\le k$ counters, the games where the number of counters N satisfies*

$$3jn+2k+j-2n+1\le N\le 3jn+k+2j, \quad 1\le j\le k+1, \tag{8}$$

and all the games with $N>3kn+3k+n+1$ counters.

Proof. The safe positions for alliance 1 are the first k games and from $N_{11}+1$ to N_{12}, from $N_{21}+1$ to N_{22}, from $N_{31}+1$ to N_{32}, and so forth. Using the formulas from Theorem 10, we obtain that the safe positions are from $N_{j1}+1=3jn+2k+j-2n+1$ to $N_{j2}=3jn+k+2j$, where $1\le j\le k+1$ and all games are with more than $3kn+3k+n+1$ counters. □

Theorem 12. *Consider the Survivor Nim for $2n+1$ players, where each player removes 1 or 2 counters at his turn. Suppose $n<k\le 2n$. If the players $1,\dots,k-n$ and $k+1,\dots,2n+1$ form one alliance and the players $k-n+1,\dots,k$ form another alliance, then the game is reducible to a two-player game, where the first player takes $n+1,\dots,2(n+1)$ counters and the second player takes $n,\dots,2n$ counters. Moreover, the largest game that the first alliance loses is with $9n^2-3kn+5n+1$ counters.*

Proof. We will call the alliance with $n+1$ players alliance 1 (the players $1,\dots,k-n$ and $k+1,\dots,2n+1$) and the alliance with n players alliance 2 (the players $k-n+1,\dots,k$).

The first round of losses for alliance 1 are the games with $1,\dots,k-n$ counters, let us denote this largest value by N_{01} as it is different from the complete rounds. Similarly, to match our previous notation, let us denote the first round of losses for alliance 2 by N_{02}. Alliance 2 loses the games with $k-n+1,\dots,2(k-n)+n$ counters, hence $N_{02} = 2(n-k)+n$.

Next, alliance 1 will be losing up to the game with $N_{11} = (k-n)+2n+(n+1) = 0$ counters. Moreover, alliance 2 will be losing the next games up to $N_{12} = 2(k-n)+2(n+1)+2n$, and so forth. One can use induction to prove similarly as in the previous theorems that

$$\begin{aligned} N_{j1} &= (k-n)+2jn+j(n+1), \quad 0 < j \le 3n-k, \\ N_{j2} &= 2(k-n)+2j(n+1)+(j+1)n, \quad 0 \le j \le 3n-k. \end{aligned} \tag{9}$$

To justify the upper bound for j, the number of games that alliance 1 loses in a row in each round is decreasing by 1, as this number can be computed from

$$\begin{aligned} N_{j1} - N_{(j-1)2} &= (k-n)+j2n+j(n+1) - \big(2(k-n)+2j(n+1)+(j+1)n\big) \\ &= 3n-k-j+2, \end{aligned} \tag{10}$$

for all values of j starting from 1 and up to the cycle where this difference is 1. Hence, $3n-k-j+2 = 2$, that is, $3n-k = j$ is the last cycle where alliance 1 loses two in a row. This proves our upper bound for j. Moreover, the largest game that alliance 1 loses is with $N_{(3n-k+1)1} = (k-n)+2(3n-k+1)n+(3n-k+1)(n+1) = 9n^2-3kn+5n+1$ counters. □

Corollary 13. *If alliances are partitioned as in Theorem 12, the safe positions for alliance 1 are the games with*

$$3jn+k+j-n+1 \le N \le 3jn+2k+2j-n, \quad 0 \le j \le 3n-k, \tag{11}$$

counters and all the games with more than $9n^2-3kn+5n+1$ *counters.*

Proof. Using the formulas from Theorem 12, we obtain the safe positions from $N_{j1}+1$ to N_{j2} with

$$N_{j1}+1 = 3jn+k+j-n+1 \le N \le N_{j2} = 3jn+2k+2j-n, \quad 0 \le j \le 3n-k, \tag{12}$$

counters and all the games with more than $N_{(3n-k+1)1} = 9n^2-3kn+5n+1$ counters. □

As an example of summarizing these results for specific games, we note the following corollary.

Corollary 14. *Consider the Survivor Nim for* 3 *players, where each player removes* 1 *or* 2 *counters at his turn. If the game is played with more than 11 counters, then the alliance with two players wins the game.*

Proof. If player 1 and player 2 form an alliance against player 3, then by Corollary 9, the safe positions for the larger alliance are the games with 3, 4, 5, and more than 7 counters. If player 2 and player 3 form an alliance against player 1, then by Corollary 11, the safe positions for the larger alliance are the games with 1, 5, 6, and more than 11 counters. If player 1 and player 3 form an alliance against player 2, then by Corollary 13, the safe positions for the larger alliance are the games with 2, 3, and more than 9 counters. □

The last result shows that if the game is played with a large enough number of counters, then the larger alliance will always win. The question of who wins when one has an arbitrary partition of 2n+1 players into coalitions of $n+1$ and n players is unanswered and would be interesting to pursue.

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