FINITE SYMMETRIC TRILINEAR INTEGRAL TRANSFORM OF DISTRIBUTIONS. PART II

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The finite symmetric trilinear integral transform is extended to distributions by using quite different technique than Zemanian (1968) and Dube (1976) and an inversion formula is established using Parseval's identity. The operational calculus generated is applied to find the temperature inside an equilateral prism of semi-infinite length.

1. Introduction

Sen [6] with the help of trilinear coordinates has solved different types of boundary value problems relating to boundaries in the form of an equilateral triangle.

An equilateral triangular region T is described by the set

$$\{x = (x_1, x_2, x_3) \,/\, 0 < x_i < p,\ x_1 + x_2 + x_3 = p,\ x_i \in \mathbb{R},\ i = 1,2,3\}, \tag{1.1}$$

where x_1, x_2, and x_3 are the trilinear coordinates of a point and p is the height of an equilateral triangle.

Sen [6] has also expressed two-dimensional Laplace operator in trilinear coordinates as

$$\begin{aligned} \nabla_1^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &\equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_1 \partial x_3} \equiv \square \quad \text{(say)}. \end{aligned} \tag{1.2}$$

Later Patil [4] has developed the symmetric integral transform of function of trilinear coordinates which is defined on T as

$$S(f)(n) = F(n) = \int_0^p \int_0^p \int_0^p f(x_1, x_2, x_3)\, \psi_n(x_1, x_2, x_3)\, dx_1 dx_2 dx_3, \tag{1.3}$$

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where $\psi_n(x_1,x_2,x_3) = \sin\lambda_n x_1 + \sin\lambda_n x_2 + \sin\lambda_n x_3$ are eigenfunctions corresponding to the eigenvalues $\lambda_n = 2n\pi/p$, $n = 1,2,3,\ldots$, in an eigenvalue problem

$$\Box\psi_n + \lambda_n^2\psi_n = 0 \tag{1.4}$$

subjected to the Dirichlet type of boundary conditions

$$\psi_n(x) = 0 \quad \text{at } x_1 = 0,\ x_2 = 0,\ x_3 = 0. \tag{1.5}$$

If $f(x_1,x_2,x_3)$ is continuous, has piecewise continuous first- and second-order partial derivatives on T, and satisfies the above Dirichlet type of boundary conditions, then inverse transform of (1.6) is given by

$$f(x_1,x_2,x_3) = \sum_{n=1}^{\infty} F(n)c_n\psi_n(x_1,x_2,x_3), \tag{1.6}$$

where $1/c_n = \int_0^p\int_0^p\int_0^p \psi_n^2 dx_1dx_2dx_3 = 3p^3/2$ (see [4, page 129]).

In this paper we extend the finite symmetric trilinear integral transform to distributions analogous to the method employed in [1] which is quite different than Zemanian [8] and Dube [2] and establish an inversion theorem by using Parseval's identity as in [3]. At the end we find the temperature inside an equilateral prism of semi-infinite length.

2. The testing function space A

Let A denote the set of all infinitely differentiable complex-valued functions ϕ defined on T which satisfy the following two conditions:

(i) $\Box^m\phi$ satisfy Dirichlet type of boundary conditions on T for each $m = 0,1,2,\ldots$;

(ii)

$$\alpha_m(\phi) = \sup_{x\in T}|\Box^m\phi(x)| < \infty \quad \text{for each } m = 0,1,2,\ldots. \tag{2.1}$$

We note that A is nonempty and for each $n \in \mathbb{N}$, eigenfunction $\psi_n(x)$ is in A.

A is a linear space. The topology of A is that generated by the countable multinorm $\{\alpha_m\}_{m=0}^{\infty}$.

Theorem 2.1. *A is complete and therefore a Fréchet space.*

The proof of this theorem is similar to the proof of [7, Theorem 3.1].

For every $\phi \in A$, the finite symmetric trilinear integral transform

$$S(\phi)(n) = \int_0^p\int_0^p\int_0^p \phi(x)\psi_n(x)dx_1dx_2dx_3 \tag{2.2}$$

exists and by (1.6), one has

$$\phi(x) = \sum_{n=1}^{\infty} c_n S(\phi)(n)\psi_n(x). \tag{2.3}$$

We call the sequence $(S(\phi)(n))_{n\in\mathbb{N}}$ as the finite symmetric trilinear integral transform $S(\phi)$ of ϕ. Therefore

$$S(\phi) = \big(S(\phi)(n)\big)_{n\in\mathbb{N}}. \tag{2.4}$$

The expression (2.3) can be seen as an inversion formula for the said transform.

Proposition 2.2. *The map $\phi \to S(\phi)$ is a continuous linear transformation from A into l^∞.*

Proof. It can be proved by making use of the property of integrals and

$$\big|\big|S(\phi)\big|\big|_{l^\infty} = \sup_n \big|S(\phi)(n)\big| \le 3p^3\alpha_0(\phi). \tag{2.5}$$

□

Let $L_2(T)$ denote the set of complex-valued functions ϕ defined on T such that

$$\|\phi\| = \left[\int_0^p \int_0^p \int_0^p |\phi(x)|^2 dx_1 dx_2 dx_3\right]^{1/2} < \infty. \tag{2.6}$$

An inner product in $L_2(T)$ is defined by

$$(\phi,\psi) = \int_0^p \int_0^p \int_0^p \phi(x)\overline{\psi(x)} dx_1 dx_2 dx_3, \quad \phi,\psi \in L_2(T), \tag{2.7}$$

where $\overline{\psi(x)}$ denotes the complex conjugate of $\psi(x)$.

We prove the following results which we need in subsequent sections.

Proposition 2.3. *If $\phi \in A$, then*

$$S(\Box^m\phi)(n) = (-1)^m\lambda_n^{2m}S(\phi)(n) \quad \text{for every } n \in \mathbb{N},\ m = 0,1,2,\ldots. \tag{2.8}$$

Proof. Integrating by parts and using boundary conditions, we get

$$S(\Box\phi)(n) = \int_0^p \int_0^p \int_0^p \phi(x)\Box\psi_n(x) dx_1 dx_2 dx_3 = (-1)\lambda_n^2 S(\phi)(n) \quad \text{for every } n \in \mathbb{N}. \tag{2.9}$$

Using (2.9), it is quite simple to see that

$$S(\Box^m\phi)(n) = (-1)^m\lambda_n^{2m}S(\phi)(n) \quad \text{for every } n \in \mathbb{N},\ m = 0,1,2,\ldots. \tag{2.10}$$

□

Proposition 2.4. *Let $\phi \in A$. Then the series*

$$\sum_{n=1}^{\infty} c_n \big|S(\Box^m\phi)(n)\big|^2 \tag{2.11}$$

converges and Bessel's inequality

$$\sum_{n=1}^{\infty} c_n \big|S(\Box^m\phi)(n)\big|^2 \le \big|\big|\Box^m\phi\big|\big|^2 < \infty \tag{2.12}$$

holds for each $m = 0,1,2,3,\ldots$.

Proof. Expanding the inner product and using the fact that $\{\psi_n\}$ is an orthogonal set, we get

$$0 \le \left|\left|\Box^m\phi - \sum_{k=1}^{n} c_k S(\Box^m\phi)(k)\psi_k\right|\right|^2 = ||\Box^m\phi||^2 - \sum_{k=1}^{n} c_k \left|\, S(\Box^m\phi)(k)\,\right|^2 \tag{2.13}$$

and so, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} c_k \left|\, S(\Box^m\phi)(k)\,\right|^2 \le ||\Box^m\phi||^2. \tag{2.14}$$

It is clear that the series (2.11) converges and taking the limit as $n \to \infty$ we get (2.12). □

Proposition 2.5. *If $\phi \in A$, then the series*

$$\sum_{n=1}^{\infty} (-1)^m c_n \lambda_n^{2m} S(\phi)(n)\psi_n(x), \quad m = 0,1,2,\ldots, \tag{2.15}$$

converges absolutely and uniformly over T.

Proof. We have

$$\left|(-1)^m c_n \lambda_n^{2m} S(\phi)(n)\psi_n(x)\right| \le 3c_n\lambda_n^{2m}\left|\, S(\phi)(n)\,\right| \quad (x \in T,\ n = 1,2,\ldots). \tag{2.16}$$

By using Proposition 2.3, Cauchy-Schwarz inequality, and (2.12), we get

$$\sum_{n=1}^{k} 3c_n\lambda_n^{2m}\left|\, S(\phi)(n)\,\right| \le 3c_0||\mathbb{I}^{m+1}\phi|| \quad \text{for every } k \in \mathbb{N}, \text{ where } c_0 = \left[\sum_{n=1}^{\infty}\frac{c_n}{\lambda_n^4}\right]^{1/2}. \tag{2.17}$$

In view of $c_n = 2/3p^3$ (see [4, page 129]), $\lambda_n = 2n\pi/p$, and $\sum_{n=1}^{\infty}(1/n^4) = \pi^4/90$ (see [3, page 112]), we have $c_0 = (1/6)\sqrt{p/60}$.

It is obvious that $\sum_{n=1}^{\infty} 3c_n\lambda_n^{2m}|S(\phi)(n)|$ is convergent. By Weierstrass M test, the series (2.15) converges absolutely and uniformly over T. □

Corollary 2.6. *If $\phi \in A$, then*

$$\mathbb{I}^m\phi(x) = \sum_{n=1}^{\infty}(-1)^m c_n\lambda_n^{2m}S(\phi)(n)\psi_n(x), \quad m = 0,1,2,\ldots, \tag{2.18}$$

and the series converges uniformly over T.

Proof. From (1.6), we have

$$\begin{aligned}\phi(x) &= \lim_{n\to\infty}\sum_{k=1}^{n} c_k S(\phi)(k)\psi_k(x),\\ \mathbb{I}^m\phi(x) &= \lim_{n\to\infty}\mathbb{I}^m\left(\sum_{k=1}^{n} c_k S(\phi)(k)\psi_k(x)\right) = \lim_{n\to\infty}\sum_{k=1}^{n} c_k(-1)^m\lambda_k^{2m}S(\phi)(k)\psi_k(x).\end{aligned} \tag{2.19}$$

In view of (2.19) and in view of Proposition 2.5, it is inferred that the series $\sum_{n=1}^{\infty}(-1)^m c_n\lambda_n^{2m}S(\phi)\cdot(n)\psi_n(x)$ converges to $\mathbb{I}^m\phi$ uniformly on T for each $m=0,1,2,\ldots$. □

The following is an immediate consequence of Corollary 2.6.

Corollary 2.7. *For all $\phi\in A$, $\phi_n\to\phi$ in A, where $\phi_n(x)=\sum_{k=1}^{n}c_kS(\phi)(k)\psi_k(x)$.*

Theorem 2.8. *For every $\phi\in A$, Parseval's identity holds, that is,*

$$\sum_{n=1}^{\infty}c_n\left|S(\phi)(n)\right|^2=\|\phi\|^2. \tag{2.20}$$

Equivalently

$$(S(\phi_1),S(\phi_2))=\sum_{n=1}^{\infty}c_nS(\phi_1)(n)\overline{S(\phi_2)(n)}=(\phi_1,\phi_2). \tag{2.21}$$

Proof. We have from Proposition 2.4 that

$$||\phi-t_n||^2=\|\phi\|^2-\sum_{k=1}^{n}c_k\left|S(\phi)(k)\right|^2,\quad \text{where } t_n(x)=\sum_{k=1}^{n}c_kS(\phi)(k)\psi_k(x). \tag{2.22}$$

But

$$||\phi-t_n||^2=\int_0^p\int_0^p\int_0^p\left|\phi(x)-t_n(x)\right|^2dx_1dx_2dx_3\le\left[\alpha_0(\phi-t_n)\right]^2p^3. \tag{2.23}$$

Hence

$$0\le\|\phi\|^2-\sum_{k=1}^{n}c_k\left|S(\phi)(k)\right|^2\le\left[\alpha_0(\phi-t_n)\right]^2p^3. \tag{2.24}$$

Taking the limit as $n\to\infty$ and using Corollary 2.7, we get (2.20).

By using polarization identity, we get (2.21). □

3. The space of rapidly decreasing sequences

Let B be the set of all complex sequences $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\sum_{n=1}^{\infty}c_n\lambda_n^{2m}\left|a_n\right|<\infty\quad\forall m=0,1,2,\ldots. \tag{3.1}$$

B is a linear space and

$$\beta_m\left((a_n)_{n\in\mathbb{N}}\right)=\sum_{n=1}^{\infty}c_n\lambda_n^{2m}\left|a_n\right|,\quad m=0,1,2,\ldots, \tag{3.2}$$

defines a countable multinorm on B. B is complete and therefore a Fréchet space.

Theorem 3.1. *For each continuous linear functional h defined on B, there exist a positive constant C and a nonnegative integer r such that for every $(a_n)_{n\in\mathbb{N}} \in B$,*

$$|\langle h,(a_n)_{n\in\mathbb{N}}\rangle| \le C\max_{0\le k\le r}\beta_k((a_n)_{n\in\mathbb{N}}). \tag{3.3}$$

The proof is similar to the proof of [8, Theorem 1.8.1].

Theorem 3.2. *The finite symmetric trilinear integral transform S is a homeomorphism from A onto the space B.*

Proof. If $\phi\in A$, then, by Proposition 2.5, $\sum_{n=1}^{\infty} c_n\lambda_n^{2m}|S(\phi)(n)|<\infty$ for every $m=0,1,2,\ldots$.

It is simple to prove that the mapping $S:A\to B$ defined by $S(\phi)=(S(\phi)(n))_{n\in\mathbb{N}}$ is one-one and onto.

Hence $S^{-1}:B\to A$ exists and is given by

$$S^{-1}(a)(x)=S^{-1}\Big((a_n)_{n\in\mathbb{N}}\Big)(x)=\sum_{n=1}^{\infty}c_n a_n\psi_n(x),\quad x\in T,\ \text{for each } a=(a_n)_{n\in\mathbb{N}}\in B. \tag{3.4}$$

Assume that $(\phi_k)_{k\in\mathbb{N}}\to\phi$ in A as $k\to\infty$. By Proposition 2.3,

$$\begin{aligned}
&\lambda_n^{2m}|S(\phi_k)(n)-S(\phi)(n)|\\
&\quad=|S(\mathbb{I}^m(\phi_k-\phi))(n)|\\
&\quad\le 3p^3\alpha_m(\phi_k-\phi)\longrightarrow 0\quad \text{as } k\longrightarrow\infty,\ \forall n\in\mathbb{N},\ m=0,1,2,\ldots,\\
&\beta_m(S(\phi_k)-S(\phi))\\
&\quad=\sum_{n=1}^{\infty}c_n\lambda_n^{2m}|S(\phi_k)(n)-S(\phi)(n)|\longrightarrow 0\quad \text{as } k\longrightarrow\infty,\ \text{for each } m=0,1,2,\ldots.
\end{aligned} \tag{3.5}$$

This proves S is continuous. To prove S^{-1} is continuous, we proceed as follows.

Let $a^k\to a$ in B as $k\to\infty$, where $a^k=(a_n^{(k)})_{n\in\mathbb{N}}$, $a=(a_n)_{n\in\mathbb{N}}$. Using (3.4),

$$\begin{aligned}
S^{-1}(a^k-a)(x)&=\sum_{n=1}^{\infty}c_n(a_n^{(k)}-a_n)\psi_n(x),\quad x\in T,\\
\alpha_m(S^{-1}(a^k-a))&=\sup_{x\in T}\left|\sum_{n=1}^{\infty}c_n(a_n^{(k)}-a_n)(-1)^m\lambda_n^{2m}\psi_n(x)\right|\quad \text{(by Corollary 2.6)}\\
&\le 3\sum_{n=1}^{\infty}c_n\lambda_n^{2m}|a_n^{(k)}-a_n|\longrightarrow 0\quad \text{as } k\longrightarrow\infty,\ \text{for each } m=0,1,2,\ldots.
\end{aligned} \tag{3.6}$$

This completes the proof. □

4. Distribution space

In this section we will introduce the space of distributions and study its basic properties.

Definition 4.1. A linear functional U on a Fréchet space A, $U : A \to C$, is called a distribution if there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ in A such that

$$\langle U, \phi \rangle = \lim_{n\to\infty} \int_0^p \int_0^p \int_0^p \chi_n(x)\phi(x)dx_1dx_2dx_3 \quad \text{exists for each } \phi \in A. \tag{4.1}$$

The set of all distributions is a complex linear space and it will be denoted by A'.

Let A_0 denote the set of all functions $f(x)$ which are continuous, have piecewise continuous first- and second-order partial derivatives on T, and satisfy the Dirichlet type of boundary conditions on T.

Proposition 4.2. *Let $f \in A_0$. Then the formula*

$$\langle U_f, \phi \rangle = \int_0^p \int_0^p \int_0^p f(x)\phi(x)dx_1dx_2dx_3, \quad \phi \in A, \tag{4.2}$$

defines a distribution U_f on A. A_0 can be embedded in A'.

Proof. U_f is clearly linear. Define $\chi_n(x) = \sum_{k=1}^{n} c_k S(f)(k)\psi_k(x)$, $\chi_n \in A$, for all values of n.

Moreover, $\chi_n \to f$ uniformly on T follows from Corollary 2.6. Therefore

$$\langle U_f, \phi \rangle = \lim_{n\to\infty} \int_0^p \int_0^p \int_0^p \chi_n(x)\phi(x)dx_1dx_2dx_3 \tag{4.3}$$

which proves that (4.2) defines a distribution U_f on A.

It is clear that the map $f \to U_f$ is linear, one-to-one, and continuous.

Finally, if $U_{f_n} \to U_f$ in image of A_0, then

$$\int_0^p \int_0^p \int_0^p (f_n - f)(x)\phi(x)dx_1dx_2dx_3 = \langle U_{f_n} - U_f, \phi \rangle \longrightarrow 0 \tag{4.4}$$

as $n \to \infty$ for all $\phi \in A$.

This implies $f_n \to f$ as $n \to \infty$. Thus A_0 can be embedded in A'. □

There are distributions that do not have the form (4.2) with $f \in A_0$.

Example 4.3. Dirac function δ_x centered at $x \in T$ is given by

$$\langle \delta_x, \phi \rangle = \phi(x), \quad \phi \in A. \tag{4.5}$$

It is easy to prove that δ_x is linear.

Take $\chi_n(y) = \sum_{k=1}^{n} c_k \psi_k(x)\psi_k(y)$, $x, y \in T$, and x is fixed. Then χ_n is in A for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_0^p \int_0^p \int_0^p \chi_n(y)\phi(y)dy_1dy_2dy_3 = \lim_{n\to\infty} \sum_{k=1}^{n} c_k S(\phi)(k)\psi_k(x) = \phi(x) = \langle \delta_x, \phi \rangle. \quad (4.6)$$

So $\delta_x \in A'$.

Define $d(\phi,\psi) = \sum_{m=0}^{\infty}(1/2^m)(\alpha_m(\phi-\psi)/(1+\alpha_m(\phi-\psi)))$. Then d is a compatible translation invariant metric on A [5, page 27]. Furthermore (A,d) is a complete metric space.

Theorem 4.4. *Every distribution is a continuous linear functional on A.*

The proof is similar to the proof of [3, Theorem 3.143].

Proposition 4.5. *A' is the dual of A, that is, A' is precisely the collection of all continuous linear functionals from A into $\mathbb{C}$.*

Proof. Let $f : A \to \mathbb{C}$ be a continuous linear functional. For each $\phi \in A$, by Corollary 2.7,

$$E_n(\phi) = \sum_{k=1}^{n} c_k S(\phi)(k)\psi_k \longrightarrow \phi \quad \text{in } A. \quad (4.7)$$

We have

$$\langle f, \phi \rangle = \lim_{n\to\infty} \langle f, E_n(\phi) \rangle = \lim_{n\to\infty} \int_0^p \int_0^p \int_0^p \phi(x)\chi_n(x)dx_1dx_2dx_3. \quad (4.8)$$

Thus the condition in the definition of distribution is satisfied with

$$\chi_n(x) = \sum_{k=1}^{n} c_k \langle f, \psi_k \rangle \psi_k(x). \quad (4.9)$$

□

Proposition 4.6. *A is a subspace of A' and the topology of A is stronger than that induced on it by A'.*

Proof. Let $f \in A$. By Proposition 4.2,

$$\langle f, \phi \rangle = \int_0^p \int_0^p \int_0^p f(x)\phi(x)dx_1dx_2dx_3 \quad (4.10)$$

defines a distribution U_f. Therefore $A \subset A'$.

Further

$$\xi_\phi(f) = |\langle f, \phi \rangle| \le \alpha_0(f)\alpha_0(\phi)p^3 \quad f \in A,\ \phi \in A, \quad (4.11)$$

implies the second statement. □

Proposition 4.7. *For each $f \in A'$, there exist a nonnegative integer r and a positive constant C such that*

$$\left|\langle f,\phi\rangle\right| \leq C\max_{0\leq m\leq r}\alpha_m(\phi). \tag{4.12}$$

Here C and r depend on f but not on ϕ.

The proof is similar to proof of [8, Theorem 1.8.1].

5. Generalized finite symmetric trilinear integral transform

The generalized finite symmetric trilinear integral transform S' of $f \in A'$ is defined by

$$\langle S'(f),(a_n)_{n\in\mathbb{N}}\rangle = \left\langle f(x),\sum_{n=1}^{\infty} c_n a_n \psi_n(x)\right\rangle, \quad (a_n)_{n\in\mathbb{N}} \in B. \tag{5.1}$$

If $\phi(x) = \sum_{n=1}^{\infty} c_n a_n \psi_n(x)$, then, by Theorem 3.2, $(a_n)_{n\in\mathbb{N}} = (S(\phi)(n))_{n\in\mathbb{N}} = S(\phi)$ and (5.1) can be written as

$$\langle S'(f),S(\phi)\rangle = \langle f,\phi\rangle. \tag{5.2}$$

Theorem 5.1. *S' is a homeomorphism from B' onto the space A'.*

Proof. This result can be seen as a consequence of Theorem 3.2 and [8, Theorem 1.10.2]. □

Proposition 5.2. *The finite symmetric trilinear integral transform S is a special case of the generalized transform S'. That is, $S'f = Sf$ for every $f \in A$.*

Proof. By using Proposition 4.6, (5.1), Theorems 3.2, and 2.8, it is simple to prove

$$\langle S'f,(a_n)_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{\infty} c_n S(f)(n)a_n = \langle (S(f)(n))_{n\in\mathbb{N}},(a_n)_{n\in\mathbb{N}}\rangle, \quad (a_n)_{n\in\mathbb{N}} \in B. \tag{5.3}$$

□

Motivated by the above result, we define the generalized integral transform $S'f$ of $f \in A'$ as

$$S'(f) = \left(\langle f(x),\psi_n(x)\rangle\right)_{n\in\mathbb{N}} \tag{5.4}$$

and we set

$$S'(f)(n) = \langle f(x),\psi_n(x)\rangle, \quad \psi_n \in A,\ n \in \mathbb{N}. \tag{5.5}$$

We now state and prove an inversion theorem for the elements of A' that can be seen as an inversion formula for the S'-transformation.

Theorem 5.3. *Let $f \in A'$. Then*

$$f = \lim_{n\to\infty} \sum_{k=1}^{n} c_k \langle f, \psi_k \rangle \psi_k, \tag{5.6}$$

where the limit is taken in the sense of A'.

Proof. Let $F_n(x) = \sum_{k=1}^{n} c_k \langle f, \psi_k \rangle \psi_k$. Since $F_n \in A$ for every n, by Proposition 4.2,

$$\langle F_n, \psi_m \rangle = \begin{cases} \langle f, \psi_m \rangle & \text{if } m \le n, \\ 0 & \text{if } m > n. \end{cases} \tag{5.7}$$

By using Theorem 2.8, Parseval's identity, we have

$$\begin{aligned} \langle F_n, \phi \rangle &= \sum_{k=1}^{\infty} c_k \langle F_n, \psi_k \rangle S(\phi)(k) \\ &= \sum_{k=1}^{n} c_k \langle f, \psi_k \rangle S(\phi)(k) \quad \text{(by (5.7))} \\ &= \langle f, E_n(\phi) \rangle \quad \text{for every } \phi \in A,\ E_n(\phi) = \sum_{k=1}^{n} c_k S(\phi)(k) \psi_k. \end{aligned} \tag{5.8}$$

By Corollary 2.7, $E_n(\phi) \to \phi$ for all $\phi \in A$. Therefore

$$\lim_{n\to\infty} \langle F_n, \phi \rangle = \lim_{n\to\infty} \langle f, E_n(\phi) \rangle = \langle f, \phi \rangle, \quad \forall \phi \in A. \tag{5.9}$$

□

Theorem 5.4 (uniqueness theorem). *If $f, g \in A'$ are such that $S'(f)(n) = S'(g)(n)$ for every n, then $f = g$ in the sense of equality in A'.*

The following example illustrates the inversion theorem.

Example 5.5. Dirac function δ_x centered at $x \in T$ is given by

$$\langle \delta_x, \phi \rangle = \phi(x), \quad \phi \in A. \tag{5.10}$$

In Example 4.3, we have shown that δ_x is in A'. The finite symmetric trilinear integral transform of δ_x is given as

$$S'(\delta_x)(n) = \langle \delta_x(t), \psi_n(t) \rangle = \psi_n(x). \tag{5.11}$$

By virtue of Proposition 4.2, for all $\phi(t) \in A$,

$$\left\langle \sum_{m=0}^{\mathbb{N}} c_m \psi_m(x) \psi_m(t), \phi(t) \right\rangle = \sum_{m=0}^{\mathbb{N}} c_m S(\phi)(m) \psi_m(x) \longrightarrow \phi(x) \quad \text{as } \mathbb{N} \longrightarrow \infty. \tag{5.12}$$

But $\phi(x) = \langle \delta_x(t), \phi(t) \rangle$. Therefore $\delta_x(t) = \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} c_m \psi_m(x) \psi_m(t)$.

A trivial consequence of Theorem 5.3 is the following version of Parseval's identity.

Corollary 5.6. *Let $\phi \in A$ and $f \in A'$. Then*

$$\langle f, \phi \rangle = \sum_{n=1}^{\infty} c_n \langle f, \psi_n \rangle S(\phi)(n) = \langle S' f, S\phi \rangle. \tag{5.13}$$

6. Characterization of distributions in A' and their generalized finite symmetric trilinear integral transform S'

Proposition 6.1. *Let $H : B \to \mathbb{C}$. Then $H \in B'$ if and only if there exists a complex sequence $(b_n)_{n\in\mathbb{N}}$ such that*

$$|b_n| \leq C\lambda_n^{2k}, \quad n \in \mathbb{N}, \tag{6.1}$$

for some $C > 0$ and some $k \in \mathbb{N}$, for which

$$\langle H, (a_n)_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} c_n a_n b_n, \quad (a_n)_{n\in\mathbb{N}} \in B. \tag{6.2}$$

Proof. Assume that H takes the form (6.2) where $(b_n)_{n\in\mathbb{N}}$ satisfies (6.1). By using (3.2) it is simple to prove $H \in B'$. Conversely, let $H \in B'$. Then by Theorem 3.1, there exists $k \in \mathbb{N}$ such that

$$|\langle H, (a_n)_{n\in\mathbb{N}} \rangle| \leq C_{0\leq m\leq k}^{\max} \sum_{n=1}^{\infty} c_n \lambda_n^{2m} |a_n| \leq C \sum_{n=1}^{\infty} c_n \lambda_n^{2k} |a_n|, \quad (a_n)_{n\in\mathbb{N}} \in B. \tag{6.3}$$

We now define the mapping, $J : B \to J(B) \subset l_1$

$$(a_n)_{n\in\mathbb{N}} \longrightarrow (\lambda_n^{2k} a_n)_{n\in\mathbb{N}}, \tag{6.4}$$

J is one-one.

Moreover, the linear mapping $M : J(B) \subset l_1 \to \mathbb{C}$

$$(\lambda_n^{2k} a_n)_{n\in\mathbb{N}} \longrightarrow \langle H, (a_n)_{n\in\mathbb{N}} \rangle \tag{6.5}$$

is continuous by virtue of (6.3).

By Hahn-Banach theorem, M can be extended to l_1 as a member of l_1'. Then there exists $(d_n)_{n\in\mathbb{N}} \in l_\infty$ such that

$$M(\lambda_n^{2k} a_n)_{n\in\mathbb{N}} = \langle H, (a_n)_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} d_n a_n \lambda_n^{2k}, \quad (a_n)_{n\in\mathbb{N}} \in B. \tag{6.6}$$

Taking $b_n = d_n \lambda_n^{2k}/c_n$, where $c_n = 2/3p^3$ [4, page 129], $n \in \mathbb{N}$. Since every sequence in l_∞ is bounded sequence, we have

$$\langle H, (a_n)_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} c_n a_n b_n. \tag{6.7}$$

□

Theorem 6.2. *Let $(b_n)_{n\in\mathbb{N}}$ be a complex sequence. There exists $f \in A'$ such that $F(n) = S'(f)(n) = b_n$, $n \in \mathbb{N}$, if and only if there exist $C > 0$ and $k \in \mathbb{N}$ such that*

$$|b_n| \leq C\lambda_n^{2k}, \quad n \in \mathbb{N}. \tag{6.8}$$

Proof. The condition is necessary and follows from Proposition 4.7.

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence satisfying condition (6.8).

For any $\phi \in A$,

$$\lim_{k\to\infty} \left\langle \sum_{n=1}^{k} c_n b_n \psi_n(x), \phi(x) \right\rangle \equiv \sum_{n=1}^{\infty} c_n b_n S(\phi)(n). \tag{6.9}$$

From (6.8) and Theorem 3.2, it is clear that the series $\sum_{n=1}^{\infty} c_n b_n S(\phi)(n)$ is absolutely convergent.

Define $f : A \to \mathbb{C}$ by the formula

$$\langle f, \phi \rangle = \sum_{n=1}^{\infty} c_n b_n S(\phi)(n), \quad \phi \in A. \tag{6.10}$$

Then f is a linear functional on A and

$$\langle f, \phi \rangle = \lim_{n\to\infty} \int_0^p \int_0^p \int_0^p \phi(x)\chi_n(x)dx_1dx_2dx_3, \tag{6.11}$$

where $\chi_n(x) = \sum_{k=1}^{n} c_k b_k \psi_k(x) \in A$ for every $n \in \mathbb{N}$.

By using orthogonal relations, we have $S'(f)(n) = b_n$, $n \in \mathbb{N}$.

Thus the condition is sufficient. □

7. Operational calculus

Integrating by parts and using boundary conditions, one can easily prove that if $f \in A_0$, then

$$\langle \mathbb{I} f, \phi \rangle = \langle f, \mathbb{I}\phi \rangle \quad \text{for every } \phi \in A. \tag{7.1}$$

It allows us to define that for any $f \in A'$,

$$\langle \mathbb{I} f, \phi \rangle = \langle f, \mathbb{I}\phi \rangle, \quad \phi \in A. \tag{7.2}$$

It is clear that $\mathbb{I} f \in A'$.

It can also be seen inductively that for any integer m,

$$\langle \mathbb{I}^m f, \phi \rangle = \langle f, \mathbb{I}^m \phi \rangle \quad \text{for every } \phi \in A \tag{7.3}$$

and $\mathbb{I}^m f \in A'$.

Therefore

$$\langle \mathbb{I}^m f, \psi_n \rangle = \langle f, \mathbb{I}^m \psi_n \rangle = (-1)^m \lambda_n^{2m} \langle f, \psi_n \rangle. \tag{7.4}$$

That is,

$$S'(\mathbb{I}^m f)(n) = (-1)^m \lambda_n^{2m} S'(f)(n) \tag{7.5}$$

which gives an operation transform formula.

Now consider the partial differential equation of the form

$$G(\mathbb{I})f = h, \tag{7.6}$$

where given h and unknown f are required to be in A', and G is a polynomial such that $G(-\lambda_n^2) \neq 0$, $n = 1,2,3,\ldots$.

By applying the operation transform formula (7.5) to (7.6), we obtain

$$\begin{gathered} G(-\lambda_n^2)F(n) = H(n), \quad F(n) = (S'f)(n),\ H(n) = (S'h)(n), \\ F(n) = \frac{H(n)}{G(-\lambda_n^2)}. \end{gathered} \tag{7.7}$$

By applying the inversion theorem (Theorem 5.3), we get

$$f = \sum_{n=1}^{\infty} c_n \frac{H(n)}{G(-\lambda_n^2)} \psi_n, \quad G(-\lambda_n^2) \neq 0, \text{ for } n = 1,2,\ldots. \tag{7.8}$$

8. Application

In this section we apply the present theory to find the temperature inside an equilateral prism of semi-infinite length. The formulation of the problem is given below.

Find the conventional function $v(x,z)$ on the domain

$$R \equiv \{(x,z) = (x_1,x_2,x_3,z) \,/\, 0 < x_i < p,\ i = 1,2,3,\ x_1 + x_2 + x_3 = p,\ 0 < z < \infty\} \tag{8.1}$$

that satisfies Laplace equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} - \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 v}{\partial x_1 \partial x_3} - \frac{\partial^2 v}{\partial x_2 \partial x_3} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{8.2}$$

in R and the following boundary conditions:

(i) as $z \to 0+$, $v(x,z) \to f(x) \in A'$ in the sense of convergence in A';

(ii) as $x_i \to 0+$, $i = 1,2,3$, $v(x,z)$, converges to zero uniformly on $Z \leq z < \infty$ for each $Z > 0$;

(iii) as $z \to \infty$, $v(x,z)$ converges uniformly to zero uniformly on $0 < x_i < p$, $i = 1,2,3$.

Every section of the prism by a plane perpendicular to z-axis is an equilateral triangle with its centroid on the z-axis,

$$\text{set } V(n,z) = S'(v(x,z)) = \langle v, \psi_n \rangle. \tag{8.3}$$

By applying the finite symmetric trilinear integral transform S' to (8.2), we arrive at

$$-\lambda_n^2 V(n,z) + \frac{\partial^2}{\partial z^2} V(n,z) = 0, \tag{8.4}$$

whose general solution is

$$V(n,z) = A(n)e^{\lambda_n z} + B(n)e^{-\lambda_n z}, \quad A(n) \text{ and } B(n) \text{ do not depend on } z. \tag{8.5}$$

In view of boundary condition (iii), it is reasonable to choose $A(n) = 0$ and $B(n) = F(n)$ because of boundary condition (i). Therefore,

$$V(n,z) = F(n)e^{-\lambda_n z}. \tag{8.6}$$

Applying the inversion theorem (Theorem 5.3) to the above equation, we get

$$v(x,z) = \sum_{n=1}^{\infty} c_n F(n) e^{-\lambda_n z} \psi_n(x). \tag{8.7}$$

We now verify that (8.7) is truly a solution of (8.2).

By Theorem 6.2, there exist $C > 0$ and $k \in \mathbb{N}$ such that $|F(n)| \le C\lambda_n^{2k}$. For $Z \le z < \infty$ where $Z > 0$, the nth term of the series (8.7) satisfies the condition

$$\left| c_n F(n) e^{-\lambda_n z} \psi_n(x) \right| \le 3c_n C \lambda_n^{2k} e^{-\lambda_n Z}. \tag{8.8}$$

Using $c_n = 2/3p^3$, $\lambda_n = 2n\pi/p$, and $e^{-\lambda_n z} < (2k+2)!/\lambda_n^{2k+2} Z^{2k+2}$, we get

$$\left| c_n F(n) e^{-\lambda_n z} \psi_n(x) \right| < C^* \frac{1}{n^2}, \quad \text{where } C^* < \frac{(2k+2)!C}{2\pi^2 P \cdot Z^{2k+2}}. \tag{8.9}$$

By Weierstrass M test, the series on the right-hand side of (8.7) converges absolutely and uniformly over R. The factor $e^{-\lambda_n z}$ ensures the uniform convergence of any series obtained by term-by-term differentiation of (8.7) with respect to x_i, $i = 1,2,3$, or z. We may apply the operator $\mathbb{I} + D_z^2$ under the summation sign in (8.7). Since $e^{-\lambda_n z}\psi_n(x_1,x_2,x_3)$ satisfies Laplace equation, so does v. Thus the differential equation (8.2) is satisfied in the conventional sense.

To verify the boundary condition (i), we have to show that for each $\phi \in A$,

$$\langle v(x,z), \phi(x) \rangle \longrightarrow \langle f, \phi \rangle \quad \text{as } z \longrightarrow 0+. \tag{8.10}$$

Now, for any fixed $z > 0$, the series (8.7) defines a function in A_0, and by Proposition 4.2 we have for every $\phi \in A$,

$$\begin{aligned}\langle v, \phi \rangle &= \int_0^p \int_0^p \int_0^p \left[\sum_{n=1}^{\infty} c_n F(n) e^{-\lambda_n z} \psi_n(x) \right] \phi(x) dx_1 dx_2 dx_3 \\ &= \sum_{n=1}^{\infty} c_n F(n) e^{-\lambda_n z} S(\phi)(n).\end{aligned} \tag{8.11}$$

The series in (8.11) converges uniformly for all $z > 0$. By taking the limit as $z \to 0+$, one has

$$\begin{aligned}\lim_{z \to 0+} \langle v, \phi \rangle &= \sum_{n=1}^{\infty} c_n F(n) S(\phi)(n) \\ &= \lim_{\mathbb{N} \to \infty} \left\langle \sum_{n=1}^{\mathbb{N}} c_n F(n) \psi_n(x), \phi(x) \right\rangle \\ &= \langle f, \phi \rangle \quad \text{by virtue of Theorem 5.3.}\end{aligned} \tag{8.12}$$

Finally, for $Z \le z \le \infty$ $(Z > 0)$, we have from (8.7) that

$$|\, v(x,z) \,| \le \sum_{n=1}^{\infty} c_n \,|\, F(n) \,|\, e^{-\lambda_n Z} \,|\, \psi_n(x) \,|. \tag{8.13}$$

The series in (8.13) converges absolutely and uniformly on T. So we may take limit as $x_i \to 0+$ under the summation sign in (8.13), which verifies boundary condition (ii).

In the same way we have

$$|\, v(x,z) \,| \le 3 \sum_{n=1}^{\infty} c_n \,|\, F(n) \,|\, e^{-\lambda_n z}. \tag{8.14}$$

The series in (8.14) converges uniformly on $0 < z < \infty$. By taking the limit as $z \to \infty$ under the summation sign in (8.14), one verifies boundary condition (iii).

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References

[1] J. J. Betancor, M. Linares, and J. M. R. Méndez, *A distributional convolution for a generalized finite Fourier transformation*, Proceedings of the American Mathematical Society **128** (2000), no. 2, 547–556.

[2] L. S. Dube, *On finite Hankel transformation of generalized functions*, Pacific Journal of Mathematics **62** (1976), no. 2, 365–378.

[3] R. J. Iorio Jr. and V. M. Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge Studies in Advanced Mathematics, vol. 70, Cambridge University Press, Cambridge, 2001.

[4] P. B. Patil, *Methods of mathematical physics and their applications*, Ph.D. thesis, Marathwada University (M.S.), Aurangabad, 1975.
[5] W. Rudin, *Functional Analysis*, Tata McGraw-Hill, New Delhi, 1976.
[6] B. Sen, *Trilinear co-ordinates and boundary value problems*, Bulletin of the Calcutta Mathematical Society **60** (1968), no. 1-2, 25–30.
[7] G. L. Waghmare and S. V. More, *On finite symmetric trilinear integral transform of distributions*, Journal of Indian Academy of Mathematics **26** (2004), no. 1, 137–166.
[8] A. H. Zemanian, *Generalized Integral Transformations*, John Wiley & Sons, New York, 1968.

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