GENERALIZED BAER RINGS

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We investigate the question whether the p.q.-Baer center of a ring R can be extended to R . We give several counterexamples to this question and consider some conditions under which the answer may be affirmative. The concept of a generalized p.q.-Baer property which is a generalization of Baer property of a ring is also introduced.

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1. Introduction

In [15], Kaplansky introduced *Baer rings* as rings in which every right (left) annihilator ideal is generated by an idempotent. According to Clark [9], a ring R is called *quasi-Baer* if the right annihilator of every right ideal is generated (as a right ideal) by an idempotent. Further works on quasi-Baer rings appear in [4, 6, 17]. Recently, Birkenmeier et al. [8] called a ring R to be a *right* (resp., *left*) *principally quasi-Baer* (or simply *right* (resp., *left*) *p.q.-Baer*) ring if the right (resp., left) annihilator of a principal right (resp., left) ideal is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. The class of right or left p.q.-Baer rings is a nontrivial generalization of the class of quasi-Baer rings. For example, if R is a commutative von Neumann regular ring which is not complete, then R is p.q.-Baer but not quasi-Baer. Observe that every biregular ring is also a p.q.-Baer ring.

A ring satisfying a generalization of Rickart's condition (i.e., every right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right *PP*-ring which is also another generalization of a Baer ring. A ring R is called a *right* (resp., *left*) *PP*-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). R is called a *PP*-ring (also called a *Rickart* ring [3, page 18]) if it is both right and left *PP*. Baer rings are clearly right (left) *PP*-rings, and von Neumann regular rings are also right (left) *PP*-rings by Goodearl [10, Theorem 1.1]. Note that the conditions right *PP* and right p.q.-Baer are distinct

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[8, Examples 1.3 and 1.5], but R is an abelian *PP*-ring if and only if R is a reduced p.q.-Baer ring [8, Corollary 1.15].

Throughout this paper, R denotes an associative ring with identity. For a nonempty subset X of R , we write $r_R(X) = \{a \in R \mid Xa = 0\}$ and $\ell_R(X) = \{a \in R \mid aX = 0\}$, which are called the right and left annihilators of X in R , respectively.

2. Principally quasi-Baer centers

As a motivation for this section, we recall the following results.

- (1) The center of a Baer ring is Baer [15, Theorem 7].
- (2) The center of a quasi-Baer ring is quasi-Baer [7, Proposition 1.8].
- (3) The center of a right p.q.-Baer ring is *PP* (hence p.q.-Baer) [8, Proposition 1.12].
- (4) Every reduced PI-ring with the Baer center is a Baer ring [1, Theorem D].

It is natural to ask if the p.q.-Baer center of a ring R can be extended to R . In this section, we show that this question has a negative answer, and so we investigate the class of rings with some conditions under which the answer to this question is affirmative.

Let $C(R)$ denote the center of a ring R .

Example 2.1. (1) Let K be a field. We consider the ring $R = K[X, Y, Z]$ with $XY = XZ = ZX = YX = 0$ and $YZ \neq ZY$. Then R is reduced and $C(R) = K[X]$ is Baer and so p.q.-Baer. But $r_R(Y)$ has no idempotents. Thus R is not right p.q.-Baer. Note that

$$I = \{f(Y, Z) \in K[Y, Z] \mid f(0, 0) = 0\} \quad (2.1)$$

is a two-sided ideal of R and $I \cap C(R) = 0$.

(2) Let

$$R = \left\{ \begin{pmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & v \end{pmatrix} \mid x, y, z, u, v \in \mathbb{R} \right\} \subseteq \text{Mat}_3(\mathbb{R}), \quad (2.2)$$

where $\mathbb{R}$ denotes the set of real numbers. Then R is a PI-ring which is not semiprime. Then we see that

$$r_R \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \right) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}. \quad (2.3)$$

But this cannot be generated by an idempotent. Hence R is not right p.q.-Baer. On the other hand,

$$C(R) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x \in \mathbb{R} \right\} \cong \mathbb{R}. \quad (2.4)$$

Therefore $C(R)$ is Baer.

Observe that Example 2.1(2) also shows that there exists a PI-ring R with the Baer center, but R is not right p.q.-Baer.

However, we have the following results.

LEMMA 2.2 [8, Proposition 1.7]. *R is a right p.q.-Baer ring if and only if the right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*

PROPOSITION 2.3. *Let R be a ring with the p.q.-Baer center $C(R)$. If R satisfies any of the following conditions for any nonzero two-sided ideal I of R , then R is quasi-Baer (and hence right p.q.-Baer):*

- (1) $I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$;
- (2) $I \cap C(R) \neq 0$ and every central idempotent of R is orthogonal;
- (3) $I \cap C(R) \neq 0$ and every right ideal of R generated by a central element contains $C(R)$.

Proof. Let I be a nonzero two-sided ideal of R . If $r_R(I) = 0$, then we are done. Thus we assume that $r_R(I) \neq 0$.

(1) By hypothesis and Lemma 2.2, $I \cap C(R) \neq 0$ and $r_{C(R)}(I \cap C(R)) = eC(R)$ for some $e^2 = e \in C(R)$. We claim that $r_R(I) = eR$. If $Ie \neq 0$, then Ie is a nonzero two-sided ideal of R . Thus, by hypothesis, $0 \neq Ie \cap C(R) \subseteq I \cap C(R)$. Let $0 \neq x \in Ie \cap C(R)$. Then $x = ye \in I \cap C(R)$ for some $y \in I$, and so $x = xe = 0$; which is a contradiction. Hence $eR \subseteq r_R(I)$, and then $r_R(I) = R \cap r_R(I) = (eR \oplus (1-e)R) \cap r_R(I) = eR \oplus ((1-e)R \cap r_R(I))$. We show that $(1-e)R \cap r_R(I) = 0$. Suppose that $0 \neq (1-e)R \cap r_R(I)$. Then $(1-e)R \cap C(R)$ is a nonzero two-sided ideal of R . Thus, by hypothesis, $0 \neq (1-e)R \cap r_R(I) \cap C(R) = (1-e)R \cap r_{C(R)}(I) \subseteq (1-e)R \cap r_{C(R)}(I \cap C(R)) \subseteq (1-e)R \cap eC(R) \subseteq (1-e)R \cap eR = 0$; which is also a contradiction. Therefore $r_R(I) = eR$, and thus R is quasi-Baer.

(2) There exists $0 \neq a \in C(R)$ such that $a \in I$, and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$ by hypothesis. Then $r_R(aR) = eR$. Since $r_R(aR) \cap C(R) = r_{C(R)}(aC(R)) = eC(R)$, $e \in r_R(aR)$, and so $eR \subseteq r_R(aR)$, and thus $r_R(aR) = eR$ by the similar method to (1). Hence $r_R(I) \subseteq eR$. Now, we claim that $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in R$ such that $x \in I \cap C(R)$ by the same arguments as in (1). Then $r_{C(R)}(xC(R)) = fC(R)$ for some $f^2 = f \in C(R)$, and so $r_R(xR) = fR$. Hence $r_R(I) \subseteq fR \cap eR = 0$; which is a contradiction. Thus $r_R(I) = eR$ for some $e^2 = e \in R$, and therefore R is a quasi-Baer ring.

(3) By hypothesis, there exists $0 \neq a \in I \cap C(R)$, and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$. Then $r_R(aR) = eR$, and this implies that $r_R(I) \subseteq eR$ by the same method as in (2). Now, we claim that $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in Ie \cap C(R) \subseteq I \cap aR \subseteq aR$, by hypothesis. We put $x = ye \in C(R)$ for some $y \in I$. Since $r_R(x) \supseteq r_R(aR) = eR$, we obtain $x = xe = 0$; which is a contradiction. Thus $eR \subseteq r_R(I)$, and consequently $r_R(I) = eR$. Therefore R is a quasi-Baer ring. $\square$

COROLLARY 2.4. *Let R be a semiprime PI-ring with the p.q.-Baer center $C(R)$. If either every central idempotent of R is orthogonal or every right ideal of R generated by a central element contains $C(R)$, then R is quasi-Baer.*

The proof follows from [18, Theorem 6.1.28] and Proposition 2.3.

Part (1) of the following example shows that the condition “ $I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$ ” and the condition “every central idempotent of R is orthogonal” in Proposition 2.3(1) and (2) are not superfluous, respectively, and parts (2) and (3) show that in Proposition 2.3, the condition (1) is not equivalent to the condition (2).

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Example 2.5. (1) Let $R = \{\langle a_i \rangle \in \prod_{i=1}^{\infty} T_i \mid a_i \text{ is eventually constant}\}$, where $T_i = \text{Mat}_2(F)$ for all i and F is a field. For a two-sided ideal $I = \{\langle a_i \rangle \in R \mid a_i = 0 \text{ if } i \text{ is even}\}$, $r_R(I) = \{\langle b_j \rangle \in R \mid b_j = 0 \text{ if } j \text{ is odd}\}$. Since

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle \notin R, \quad (2.5)$$

$r_R(I)$ cannot be generated by an idempotent of R . Thus R is not quasi-Baer. Note that

$$C(R) = \left\{ \langle a_i \rangle \in R \mid a_i = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \text{ for some } k \in F \right\} \quad (2.6)$$

is p.q.-Baer. Now,

$$I \cap C(R) = \left\{ \langle a_i \rangle \in R \mid a_i = 0 \text{ if } i \text{ is even, } a_i = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \text{Mat}_2(F) \text{ if } i \text{ is odd} \right\} \quad (2.7)$$

is not finitely generated. Moreover,

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle, \quad \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle \quad (2.8)$$

are idempotents, but they are not orthogonal.

(2) Let $R = F[x_1, x_2, \dots]$, where F is a field. Then R is a commutative quasi-Baer ring whose only idempotents 0 and 1 are orthogonal, but the two-sided ideal $\langle x_1^2, x_2^2, \dots \rangle$ of R is not finitely generated.

(3) Let $R = \mathbb{Z} \oplus \mathbb{Z}$. Then R is a commutative quasi-Baer ring. Since R is Noetherian, every two-sided ideal of R is finitely generated. But the central idempotents $(1, 0)$ and $(1, 1)$ are not orthogonal.

Related to the result of [1, Theorem D], we have the next example.

Example 2.6. (1) Let $R = \mathcal{C}[0, 1]$ be the ring of all real-valued continuous functions on $[0, 1]$. Then R is commutative (and so PI) and reduced. But R is not p.q.-Baer. Let

$$f : [0, 1] \longrightarrow \mathbb{R} \quad (2.9)$$

be defined by

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} < x \leq 1. \end{cases} \quad (2.10)$$

Then $f \in R$, and so

$$r_R(f) = \left\{ g \in R \mid g \left(\left(\frac{1}{2}, 1 \right] \right) = 0 \right\} \neq 0. \quad (2.11)$$

Suppose that $r_R(f) = eR$ for some nonzero idempotent $e \in R$. Then $e(x)^2 = e(x)$, for each $x \in [0, 1]$. Thus $e(x) = 0$ or $e(x) = 1$. Since $e \in r_R(f)$, $e((1/2, 1]) = \{0\}$. But e is continuous, and so $e(x) = 0$ for each $x \in [0, 1]$. Hence $r_R(f) = 0$; which is a contradiction. Thus R is a reduced PI-ring which is not right p.q.-Baer.

(2) We take the ring in [12, Example 2(1)]. Let $\mathbb{Z}$ be the ring of integers and $\text{Mat}_2(\mathbb{Z})$ the 2×2 full matrix ring over $\mathbb{Z}$. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \mid a - d \equiv b \equiv c \equiv 0 \pmod{2} \right\}. \quad (2.12)$$

Then R is right p.q.-Baer, but R is neither right PP nor left PP by [12, Example 2(1)]. Moreover, it can be easily checked that R is an abelian PI-ring with the PP center.

3. Generalized p.q.-Baer rings

Regarding a generalization of Baer rings as well as a PP -ring, recall that a ring R is called a *generalized right PP -ring* if for any $x \in R$, the right ideal $x^n R$ is projective for some positive integer n , depending on x , equivalently, if for any $x \in R$, the right annihilator of x^n is generated by an idempotent for some positive integer n , depending on x . Left cases may be defined analogously. A ring R is called a *generalized PP -ring* if it is both generalized right and left PP -ring. Right PP -rings are generalized right PP obviously. A number of papers have been written on generalized PP -rings. For basic and other results on generalized PP -rings, see, for example, [11, 14, 16].

As a parallel definition to the generalized PP -property related to the p.q.-Baer property, we define the following.

Definition 3.1. A ring R is called a *generalized right p.q.-Baer ring* if for any $x \in R$, the right annihilator of $x^n R$ is generated by an idempotent for some positive integer n , depending on x . Left cases is defined analogously. A ring R is called a *generalized p.q.-Baer ring* if it is both generalized right and left p.q.-Baer ring.

We have the following connections.

LEMMA 3.2 [12, Lemma 1]. *Let R be a reduced ring. The following are equivalent:*

- (1) R is right PP ;
- (2) R is PP ;
- (3) R is generalized right PP ;
- (4) R is generalized PP ;
- (5) R is right p.q.-Baer;

- (6) R is p.q.-Baer;
- (7) R is generalized right p.q.-Baer;
- (8) R is generalized p.q.-Baer;

Shin [19] defined that a ring R satisfies (SI) if for each $a \in R$, $r_R(a)$ is a two-sided ideal of R , and proved that R satisfies (SI) if and only if $ab = 0$ implies that $aRb = 0$ for $a, b \in R$ [19, Lemma 1.2]. The (SI) property was studied in the context of near-rings by Bell, in [2], where it is called the insertion of factors principle (IFP). It is well known that every reduced ring has the IFP, and if R has the IFP then it is abelian, but the converses do not hold, respectively.

Recall from [8, Corollary 1.15] that R is an abelian PP-ring if and only if R is a reduced p.q.-Baer ring. Similarly, we have the following.

PROPOSITION 3.3. *Let a ring R have the IFP. Then R is a generalized right PP-ring if and only if R is a generalized right p.q.-Baer ring.*

Proof. For any $x \in R$ and positive integer n , $r_R(x^n) = r_R(x^n R)$ since R has the IFP. □

Every right p.q.-Baer rings is a generalized right p.q.-Baer, but the converse does not hold, by the next example.

Given a ring R and an (R, R) -bimodule M , the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2). \quad (3.1)$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$ and the usual matrix operations are used.

Example 3.4 [14, Example 2]. Let D be a domain and let $R = T(D, D)$ be the trivial extension of D . Then R has the IFP and R is a generalized right PP-ring, but it is not a right PP-ring. Thus R is a generalized right p.q.-Baer ring by Proposition 3.3, but it is not right p.q.-Baer by [8, Proposition 1.14].

Recall from [5] that an idempotent $e \in R$ is called *left* (resp., *right*) *semicentral* if $xe = exe$ (resp., $ex = exe$) for all $x \in R$. The set of left (resp., right) semicentral idempotents of R is denoted by $S_\ell(R)$ (resp., $S_r(R)$). Note that $S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$, where $\mathbf{B}(R)$ is the set of all central idempotents of R , and if R is semiprime then $S_\ell(R) = S_r(R) = \mathbf{B}(R)$. Some of the basic properties of these idempotents are indicated in the following.

LEMMA 3.5 [7, Lemma 1.1]. *For an idempotent $e \in R$, the following are equivalent:*

- (1) $e \in S_\ell(R)$;
- (2) $1 - e \in S_r(R)$;
- (3) $(1 - e)Re = 0$;
- (4) eR is a two-sided ideal of R ;
- (5) $R(1 - e)$ is a two-sided ideal of R .

The following example shows that the condition “ R has the IFP” in Proposition 3.3 cannot be dropped.

Example 3.6 [8, Example 1.6]. For a field F , take $F_n = F$ for $n = 1, 2, \dots$, and let

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \left\langle \bigoplus_{n=1}^{\infty} F_n, 1 \right\rangle \end{pmatrix}, \quad (3.2)$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $\langle \bigoplus_{n=1}^{\infty} F_n, 1 \rangle$ is the F -algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and 1. Then R is a regular ring by [10, Lemma 1.6], and so R is a generalized PP -ring.

Let $a \in (a_n) \in \prod_{n=1}^{\infty} F_n$ such that $a_n = 1$ if n is odd and $a_n = 0$ if n is even, and let $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$. Now we assume that there exists an idempotent $e \in R$ such that $r_R(\alpha^k R) = eR$ for a positive integer k . Then e is left semicentral, and so e is central since R is semiprime, but this is impossible. Thus R is not generalized right p.q.-Baer. Similarly R is not generalized left p.q.-Baer.

PROPOSITION 3.7. *Let R be a ring. The following are equivalent:*

- (1) R is generalized right p.q.-Baer;
- (2) for any principal ideal I of the form Ra^nR of R , where n is a positive integer, there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1 - e)Re$.

Proof. The proof is an adaptation from [8, Proposition 1.9]. (1) $\Rightarrow$ (2). Assume (1) holds. Then $r_R(I) = r_R(Ra^nR) = r_R(a^nR) = fR$ with $f \in S_\ell(R)$. So $I \subseteq \ell_R(r_R(I)) = R(1 - f)$. Let $e = 1 - f$, then $e \in S_r(R)$. Hence $r_R(I) \cap Re = (1 - e)R \cap Re = (1 - e)Re$.

(2) $\Rightarrow$ (1). Assume (2) holds. Clearly $(1 - e)R \subseteq r_R(I)$ for any ideal I of the form Ra^nR . Let $\alpha \in r_R(I)$, then $\alpha e = e\alpha e + (1 - e)\alpha e \in r_R(I) \cap Re = (1 - e)Re$. So $e\alpha = e\alpha e = 0$. Hence $\alpha = (1 - e)\alpha \in (1 - e)R$. Thus $r_R(I) = (1 - e)R$, and therefore R is generalized right p.q.-Baer. $\square$

COROLLARY 3.8. *Let R be a generalized right p.q.-Baer ring. If I is a principal ideal of the form Ra^nR of R , then there exists $e \in S_r(R)$ such that $I \subseteq Re$, $(1 - e)Re$ is an ideal of R , and $I + (1 - e)Re$ is left essential in Re .*

As a parallel result to [8, Proposition 1.12], we have the following whose proof is also an adaptation from [8].

PROPOSITION 3.9. *If R is a generalized right p.q.-Baer ring, then the center $C(R)$ of R is a generalized PP -ring.*

Proof. Let $a \in C(R)$. For any positive integer n , there exists $e \in S_\ell(R)$ such that $\ell_R(a^n) = \ell_R(Ra^n) = r_R(a^n) = r_R(a^nR) = eR$. Observe that $\ell_R(Ra^n) = \ell_R r_R \ell_R(Ra^n) = \ell_R r_R(eR)$. Let $r_R(eR) = r_R(e^nR) = fR$ with $f \in S_\ell(R)$, then $1 - f \in S_r(R)$. Hence $eR = \ell_R(Ra^n) = \ell_R r_R(eR) = \ell_R(fR) = R(1 - f)$. So there exists $x \in R$ such that $e = x(1 - f)$, and hence $ef = x(1 - f)f = 0$. Now $fe = efe = 0$ because $e \in S_\ell(R)$, and so $ef = fe = 0$. Since $eR = R(1 - f)$, there is $y \in R$ such that $1 - f = ey$, and so $e = e(1 - f) = ey = 1 - f$. Thus $e \in S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$. Consequently, $r_C(R)(a^n) = r_R(a^n) \cap C(R) = eR \cap C(R) = eC(R)$. Therefore the center $C(R)$ of R is a generalized PP -ring. $\square$

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The following example shows that there exists a semiprime ring $\mathcal{R}$ whose center is a generalized PP , but $\mathcal{R}$ is not a generalized right p.q.-Baer.

Example 3.10. Let $\mathcal{R} = R \oplus \text{Mat}_2(\mathbb{Z}[x])$, where

$$R = \left(\begin{array}{cc} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \left\langle \bigoplus_{n=1}^{\infty} F_n, 1 \right\rangle \end{array} \right), \quad (3.3)$$

in Example 3.6. Then the center of $\mathcal{R}$ is generalized PP . Since R is not a generalized right p.q.-Baer by Example 3.6, $\mathcal{R}$ is not a generalized right p.q.-Baer either. Furthermore, due to [14, Example 4], $\text{Mat}_2(\mathbb{Z}[x])$ is not a generalized right PP . Thus $\mathcal{R}$ is not generalized right PP .

Note that given a reduced ring R , the trivial extension of R (by R) has the IFP by simple computations. However, the trivial extension of a ring R which has the IFP does not have the IFP by [13, Example 11]. We give examples of generalized right p.q.-Baer rings, which are extensions of the trivial extension, as in the following.

LEMMA 3.11. *Let S be a ring and for $n \geq 2$,*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}. \quad (3.4)$$

If S has the IFP, then for any $A \in R_n$ and any $E^2 = E \in R_n$, $AE = \mathbf{0}$ implies that $AR_nE = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix in R_n .

Proof. Note that every idempotent E in R_n is of the form

$$\begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \quad (3.5)$$

with $e^2 = e \in S$ by [14, Lemma 2]. Suppose that $AE = \mathbf{0}$ for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n. \quad (3.6)$$

Then we have the following: $ae = 0$ and $a_{ij}e = 0$ for $i < j$, $1 \leq i$ and $2 \leq j$. Since S has the IFP, $aSe = 0$ and $a_{ij}Se = 0$ for $i < j$, $1 \leq i$ and $2 \leq j$. These imply that $AR_nE = \mathbf{0}$. $\square$

PROPOSITION 3.12. *Let a ring S have the IFP and let R_n for $n \geq 2$ be the ring in Lemma 3.11. Then the following are equivalent:*

- (1) S is generalized right p.q.-Baer;
- (2) R_n is generalized right PP;
- (2) R_n is generalized right p.q.-Baer.

Proof. (1) $\Rightarrow$ (2). Suppose that S is generalized right p.q.-Baer. By Proposition 3.3, S is a generalized right PP. Hence R_n is also a generalized right PP by [14, Proposition 3].

(2) $\Rightarrow$ (3). Suppose that R_n is a generalized right PP. Then for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n \quad (3.7)$$

and a positive integer k , there exists an idempotent

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n \quad (3.8)$$

with $e^2 = e \in S$ such that $r_{R_n}(A^k) = ER_n$. Note that $r_{R_n}(A^k R_n) \subseteq ER_n$. From $r_{R_n}(A^k) = ER_n$, $A^k E = \mathbf{0}$, and so $A^k R_n E = \mathbf{0}$ by Lemma 3.11. Thus we have $E \in r_{R_n}(A^k R_n)$, and so $ER_n \subseteq r_{R_n}(A^k R_n)$. Consequently, $r_{R_n}(A^k R_n) = ER_n$, and therefore R_n is generalized right p.q.-Baer.

(3) $\Rightarrow$ (1). Suppose that R_n is a generalized right p.q.-Baer. Let $a \in S$ and consider

$$A = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n. \quad (3.9)$$

Since R_n is a generalized right p.q.-Baer, $r_{R_n}(A^k R_n) = ER_n$ for some $E^2 = E \in R_n$ and a positive integer k . Then by [14, Lemma 2], there is $e^2 = e \in S$ such that

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n. \quad (3.10)$$

Hence $eS \subseteq r_S(a^k S)$. Let $b \in r_S(a^k S)$, then

$$\begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in R_n \quad (3.11)$$

is contained in $r_{R_n}(A^k R_n) = ER_n$, so $b \in eS$. Thus S is also a generalized right p.q.-Baer ring. $\square$

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