FINITE RANK INTERMEDIATE HANKEL OPERATORS AND THE BIG HANKEL OPERATOR

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Let L_a^2 be a Bergman space. We are interested in an intermediate Hankel operator H_ϕ^M from L_a^2 to a closed subspace M of L^2 which is invariant under the multiplication by the coordinate function z . It is well known that there do not exist any nonzero finite rank big Hankel operators, but we are studying same types in case H_ϕ^M is close to big Hankel operator. As a result, we give a necessary and sufficient condition about M that there does not exist a finite rank H_ϕ^M except $H_\phi^M = 0$.

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Let D be the open unit disc in $\mathbb{C}$ and let dA be the normalized area measure on D . When $dA = r dr d\theta/\pi$, let $L^2 = L^2(D, dA)$ be the Lebesgue space on the open unit disc D and let $L_a^2 = L^2 \cap \text{Hol}(D)$ be a Bergman space on D . When M is the closed subspace of L^2 and $zM \subseteq M$, M is called an invariant subspace. Suppose that $zL_a^2 \subseteq M$. P^M denotes the orthogonal projection from L^2 onto M . For ϕ in L^∞ , the intermediate Hankel operator H_ϕ^M is defined by

$$H_\phi^M f = (I - P^M)(\phi f) \quad (f \in L_a^2). \quad (1)$$

When $M = L_a^2$, H_ϕ^M is called a big Hankel operator and when $M = (\bar{z}L_a^2)^\perp$, H_ϕ^M is called small Hankel operator. L^2 has the following orthogonal decomposition:

$$L^2 = \sum_{j=-\infty}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \quad (2)$$

where $\mathcal{L}^p = L^p([0, 1), 2r dr)$ for $1 \leq p \leq \infty$. Set

$$\mathbf{H}^2 = \sum_{j=0}^{\infty} \oplus \mathcal{L}^2 e^{ij\theta}, \quad (3)$$

then $L_a^2 \subset \mathbf{H}^2 \subset (\bar{z}L_a^2)^\perp$.

2 Finite rank intermediate Hankel operators

For an invariant subspace M , set

$$M_j = \left\{ f_j \in \mathcal{L}^2; f \in M, f(z) = \sum_{j=-\infty}^{\infty} f_j(r) e^{ij\theta} \right\}. \quad (4)$$

We call $\{M_j\}_{j=-\infty}^{\infty}$ the Fourier coefficients of M and then $rM_j \subseteq M_{j+1}$. If $M_j e^{ij\theta}$ belongs to M for any j , then M has the following decomposition:

$$M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}. \quad (5)$$

When $M \subseteq \mathbf{H}^2$, H_ϕ^M is close to big Hankel operator. In this case, we give a necessary and sufficient condition about M that there does not exist a finite rank H_ϕ^M except $H_\phi^M = 0$.

The following lemma is proved in previous paper [1].

LEMMA 1. Suppose M is an invariant subspace which contains zL_a^2 , and ϕ is a function in L^∞ . H_ϕ^M is of finite rank $\leq \ell$ if and only if ϕ belongs to $M^{\infty, \ell}$, where

$$M^{\infty, \ell} = \left\{ \phi \in L^\infty; b\phi(z) \in M, b(z) = \sum_{j=0}^{\ell} b_j z^j \text{ and } b_j \in \mathbb{C} \right\}. \quad (6)$$

Note that we have proved in the previous paper [1, Theorem 5.4(1)] only when $k = 0$. We improve [1, Theorem 5.4]. That is the following theorem.

THEOREM 2. Suppose M is an invariant subspace between zL_a^2 and $e^{-ik\theta}\mathbf{H}^2$ where $k \geq 0$, and $\phi = \sum_{j=1+k}^{\infty} \phi_{-j}(r) e^{-ij\theta}$ is a function in L^∞ . Then there does not exist any finite rank H_ϕ^M except for $H_\phi^M = 0$ if and only if $M_{-(k-j)} \cap r^{j+1}\mathcal{L}^\infty = \{0\}$ for any $j \geq 0$.

Proof.

$$\int \sum_{j=1+k}^{\infty} \phi_{-j}(r) e^{-i(j-m)\theta} \frac{d\theta}{2\pi} = \phi_{-m}(r) \quad (1+k \leq m \leq \infty), \quad (7)$$

and so

$$\begin{aligned} |\phi_{-m}(r)| &\leq \int \left| \sum_{j=1+k}^{\infty} \phi_{-j}(r) e^{-i(j-m)\theta} \right| \frac{d\theta}{2\pi} \\ &= \int |e^{-im\theta}| \left| \sum_{j=1+k}^{\infty} \phi_{-j}(r) e^{-i(j-m)\theta} \right| \frac{d\theta}{2\pi} = \int |\phi| \frac{d\theta}{2\pi} < \infty. \end{aligned} \quad (8)$$

Hence $\phi_{-m}(r) \in \mathcal{L}^\infty$ for $1+k \leq m \leq \infty$. If $r(H_\phi^M) \leq \ell (< \infty)$ by Lemma 1 then there exist complex numbers $b_0, \dots, b_\ell$ such that $b_\ell = 1, b = \sum_{j=0}^{\ell} b_j z^j$:

$$b\phi = \sum_{n=-\infty}^{\ell-(1+k)} \left(\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \right) e^{in\theta} \in M. \quad (9)$$

Since $M \subseteq e^{-ik\theta} \mathbf{H}^2$,

$$b\phi = \sum_{n=-k}^{\ell-(1+k)} \left(\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \right) e^{in\theta} \in M \quad (- (\ell+k) \leq n-m \leq \ell-(1+k)). \quad (10)$$

Since $M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}$, by (10),

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) \in M_n \quad (-k \leq n \leq \ell-(1+k)). \quad (11)$$

As $n = \ell-(1+k)$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^{\ell} \phi_{-(1+k)}(r) \in M_{\ell-(1+k)}. \quad (12)$$

If $\phi_{-(1+k)}(r) \neq 0$, then $M_{\ell-(1+k)} \cap r^{\ell} \mathcal{L}^{\infty} \neq \{0\}$. So we assume $\phi_{-(1+k)}(r) = 0$. As $n = \ell-(2+k)$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^{\ell} \phi_{-(2+k)}(r) \in M_{\ell-(2+k)}. \quad (13)$$

If $\phi_{-(2+k)}(r) \neq 0$, then $M_{\ell-(2+k)} \cap r^{\ell} \mathcal{L}^{\infty} \neq \{0\}$ and so $M_{\ell-(2+k)} \cap r^{\ell-1} \mathcal{L}^{\infty} \neq \{0\}$. So we assume $\phi_{-(2+k)}(r) = 0$. Repeating the same way from $n = \ell-(3+k)$ to $n = \ell-(\ell-1+k)$, we can get $\phi_{-(3+k)}(r) = \cdots = \phi_{-(\ell-1+k)}(r) = 0$. As $n = -k$,

$$\sum_{m=0}^{\ell} b_m r^m \phi_{n-m}(r) = r^{\ell} \phi_{-(\ell+k)}(r) \in M_{-k}. \quad (14)$$

If $\phi_{-(\ell+k)}(r) \neq 0$, then $M_{-k} \cap r^{\ell} \mathcal{L}^{\infty} \neq \{0\}$ and so $M_{-k} \cap r^{\ell} \mathcal{L}^{\infty} \neq \{0\}$. If $\phi_{-(\ell+k)}(r) = 0$, then $\phi_{-(1+k)}(r) = \phi_{-(2+k)}(r) = \cdots = \phi_{-(\ell+k)}(r) = 0$ and $\phi = 0$ by (10). This result contradicts $H_{\phi}^M \neq 0$, and so $M_{j-k} \cap r^{j+1} \mathcal{L}^{\infty} \neq \{0\}$ for $j \geq 0$.

If $r^{j+1} f \in M_{j-k} \cap r^{j+1} \mathcal{L}^{\infty}$ ($f \in \mathcal{L}^{\infty}$), then put $\phi = f e^{-i(k+1)\theta} \in L^{\infty}$. If $f \neq 0$, then $\phi \notin M$ and

$$z^{j+1} \phi = r^{j+1} f e^{i(j-k)\theta} \in M_{j-k} e^{i(j-k)\theta}. \quad (15)$$

Since $M = \sum_{j=-\infty}^{\infty} \oplus M_j e^{ij\theta}$, $M_{j-k} e^{i(j-k)\theta} \subseteq M$ and so $z^{j+1} \phi \in M$. Lemma 1 gives a contradiction. $\square$

References

- [1] T. Nakazi and T. Osawa, *Finite-rank intermediate Hankel operators on the Bergman space*, International Journal of Mathematics and Mathematical Sciences **25** (2001), no. 1, 19–31.

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