C_0 -SEMIGROUPS OF LINEAR OPERATORS ON SOME ULTRAMETRIC BANACH SPACES

TOKA DIAGANA

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C_0 -semigroups of linear operators play a crucial role in the solvability of evolution equations in the classical context. This paper is concerned with a brief conceptualization of C_0 -semigroups on (ultrametric) free Banach spaces $\mathbb{E}$. In contrast with the classical setting, the parameter of a given C_0 -semigroup belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we will discuss the solvability of some homogeneous p -adic differential equations.

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1. Introduction

Let $(\mathbb{K}, +, \cdot, |\cdot|)$ be a (complete) ultrametric-valued field and let Ω_r be the closed ball of $\mathbb{K}$ centered at 0 with radius $r > 0$, that is, $\Omega_r = \{\kappa \in \mathbb{K} : |\kappa| \leq r\}$. It is well known that Ω_r is also open in $\mathbb{K}$; for this reason, Ω_r is called a *clopen*. Recall that each ball Ω_r is an additive subgroup of $\mathbb{K}$. From now on, the radius r of the ball Ω_r will be suitably chosen so that the series, which defines the p -adic exponential, converges. Indeed, let $\mathbb{K} = \mathbb{Q}_p$ be the field of p -adic numbers ($p \geq 2$ being a prime) equipped with the p -adic valuation $|\cdot|$ and let $\Omega_r = \{q \in \mathbb{Q}_p : |q| \leq r\}$. In contrast with the classical context, the p -adic exponential given by

$$e^q := \sum_{n \geq 0} \frac{q^n}{n!} \quad (1.1)$$

is not always well defined and analytic for each $q \in \mathbb{Q}_p$. However, it does converge for all $q \in \mathbb{Z}_p$ such that $|q| < r = p^{-1/(p-1)}$, where $\mathbb{Z}_p$ denotes the ring of p -adic integers. (The ring of p -adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$ centered at zero, that is, the set of all $x \in \mathbb{Q}_p$ such that $|x| \leq 1$, where $|\cdot|$ is the p -adic valuation of $\mathbb{Q}_p$). For more on these and related issues, we refer the reader to [1, 7, 8, 18].

In this paper, we provide the reader with a brief conceptualization of ultrametric counterparts of C_0 -semigroups in connection with the formalism of linear operators on free Banach and non-Archimedean Hilbert spaces, recently developed in [2–6].

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The present paper is mainly motivated by the solvability of p -adic differential and partial differential equations [9, 11–13, 18] as strong (mild) solutions to the Cauchy problem related to several classes of differential and partial differential equations in the classical setting which can be expressed through C_0 -semigroups, see, for example, [15, 16].

As for the p -adic exponential defined above, here, the parameter of a given C_0 -semigroup belongs to one of those clopen balls Ω_r whose radius r will be suitably chosen. Let us mention however that the idea of considering one-parameter families of bounded linear operators on balls such as Ω_r was first initiated in [1] for bounded symmetric operators defined on $\mathbb{Q}_p$. Here, we consider those issues within the framework of free Banach and non-Archimedean Hilbert spaces, while a development of a theory of linear operators on those ultrametric spaces is underway. One of the consequences of the ongoing discussion is that if $\mathbb{K} = \mathbb{Q}_p$ and if A is a bounded linear operator on a free Banach space $\mathbb{E}$ satisfying $\|A\| \leq r$ with $r = p^{-1/(p-1)}$, then the function defined by

$$v(q) = \left(\sum_{n \geq 0} \frac{(qA)^n}{n!} \right) u_0, \quad q \in \Omega_r, \quad (1.2)$$

for a fixed $u_0 \in \mathbb{E}$ is the solution to the homogeneous p -adic differential equation given by

$$\frac{du}{dq} = Au, \quad u(0) = u_0. \quad (1.3)$$

This paper is organized as follows: Section 2 is devoted to the required background needed in the sequel. In Section 3, we study C_0 -semigroups and consider the solvability of homogeneous p -differential equations involving both bounded and unbounded linear operators on a free Banach space $\mathbb{E}$.

2. Preliminaries

2.1. Free Banach spaces

Defintion 2.1. Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean field and let $\mathbb{E}$ be a vector space over $\mathbb{K}$. A nonnegative real-valued function $\|\cdot\|$ over $\mathbb{E}$ is called an ultrametric norm if

- (a) $\|x\| = 0$ if and only if $x = 0$;
- (b) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in \mathbb{E}$;
- (c) $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in \mathbb{E}$ with equality holding if $\|x\| \neq \|y\|$.

Defintion 2.2. An ultrametric Banach space is a vector space endowed with an ultrametric norm, which is complete.

For details on ultrametric Banach spaces and related issues, see, for example, [14, 17].

Example 2.3. Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric field and let $\rho = (\rho_i)_{i \in I} \subset \mathbb{R}^+ - \{0\}$ be real numbers, where I is a given index set.

Define

$$l^\infty(I, \mathbb{K}, \rho) := \left\{ x = (x_i)_{i \in I} \in \mathbb{K}^I : \sup_{i \in I} |x_i| \rho_i < \infty \right\}. \quad (2.1)$$

One equips $l^\infty(I, \mathbb{K}, \rho)$ with the ultrametric norm defined by $\|x\| := \sup_{i \in I} |x_i| \rho_i$. It is well known that $(l^\infty(I, \mathbb{K}, \rho), \|\cdot\|)$ is an ultrametric Banach space, see [5, 6].

Example 2.4. Let $c_0(I, \mathbb{K}, \rho) \subset l^\infty(I, \mathbb{K}, \rho)$ be a subspace defined by

$$c_0(I, \mathbb{K}, \rho) := \left\{ x = (x_i)_{i \in I} \in \mathbb{K}^I : \lim_{i \in I} |x_i| \rho_i = 0 \right\}. \quad (2.2)$$

Clearly, $(c_0(I, \mathbb{K}, \rho), \|\cdot\|)$, where $\|\cdot\|$ is the ultrametric norm given in Example 2.3, is a closed subspace of $(l^\infty(I, \mathbb{K}, \rho), \|\cdot\|)$, and hence it is an ultrametric Banach space.

Defintion 2.5. An ultrametric Banach space $(\mathbb{E}, \|\cdot\|)$ over a (complete) field $(\mathbb{K}, |\cdot|)$ is said to be a free Banach space if there exists a family $(e_i)_{i \in I}$ of elements of $\mathbb{E}$ such that each element $x \in \mathbb{E}$ can be written in a unique fashion as a pointwise convergent series defined by $x = \sum_{i \in I} x_i e_i$ with $\lim_{i \in I} x_i e_i = 0$, and $\|x\| = \sup_{i \in I} |x_i| \|e_i\|$.

The family $(e_i)_{i \in I}$ is then called an *orthogonal basis* for $\mathbb{E}$. If $\|e_i\| = 1$, for all $i \in I$, then $(e_i)_{i \in I}$ is called an *orthonormal basis* for $\mathbb{E}$.

Example 2.6. Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric field and let M be a compact (topological) space. Let $C(M, \mathbb{K})$ denote the space of continuous functions which go from M into $\mathbb{K}$. The space $C(M, \mathbb{K})$ is equipped with the sup norm defined by

$$\|u\|_\infty := \sup_{m \in M} |u(m)|. \quad (2.3)$$

It can be shown that $(C(M, \mathbb{K}), \|\cdot\|_\infty)$ is an ultrametric Banach space. In particular, when $M = \mathbb{Z}_p$ and $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$, where $p \geq 2$ is a prime number, then the resulting space $(C(\mathbb{Z}_p, \mathbb{Q}_p), \|\cdot\|_\infty)$ is a free Banach space. Indeed, consider the sequence of functions defined by

$$f_n(x) = \frac{x(x-1)(x-2)(x-3) \cdots (x-n+1)}{n!}, \quad n \geq 1, \quad f_0(x) = 1. \quad (2.4)$$

It is well known [10] that the family $(f_n)_{n \in \mathbb{N}}$ is an orthonormal base, that is, $\|f_n\|_\infty = 1$, and that every function $u \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ has a unique uniformly convergent decomposition defined by $u(x) = \sum_{n=0}^\infty c_n f_n(x)$, $c_p \in \mathbb{Q}_p$, with $|c_n| \mapsto 0$ as $n \mapsto \infty$ and $\|u\|_\infty = \sup_{n \in \mathbb{N}} |c_n|$.

Example 2.7. Let $\mathbb{K}$ be a field which is complete with respect to a non-Archimedean valuation which will be denoted $|\cdot|$. Fix once and for all a sequence $\omega = (\omega_s)_{s \in \mathbb{N}}$ of nonzero elements of $\mathbb{K}$. Set $\mathbb{E}_\omega = c_0(\mathbb{N}, \mathbb{K}, (\|e_i\|_{i \in \mathbb{N}}))$, where $\|e_i\| = |\omega_i|^{1/2}$ for each $i \in \mathbb{N}$. As mentioned above, an (ultrametric) norm is defined on $\mathbb{E}_\omega$ by

$$x = (x_s)_{s \in \mathbb{N}}, \quad \|x\| := \sup_{s \in \mathbb{N}} |x_s| |\omega_s|^{1/2}. \quad (2.5)$$

Note that $\mathbb{E}_\omega$ is a free Banach space—it has a canonical orthogonal base—namely, $(e_i)_{i \in \mathbb{N}}$, where e_i is the sequence all of whose terms are 0 except the i th term which is 1, and

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$\langle e_i, e_j \rangle = \omega_i \delta_{ij}$, where δ_{ij} is the Kronecker symbol. An inner product (symmetric, bilinear, nondegenerate form) is defined by: for all $x = (x_s)_{s \in \mathbb{N}}$, $y = (y_s)_{s \in \mathbb{N}} \in \mathbb{E}_\omega$,

$$\langle x, y \rangle := \sum_{s=0}^{\infty} x_s y_s \omega_s. \quad (2.6)$$

The space $(\mathbb{E}_\omega, \|\cdot\|, \langle \cdot, \cdot \rangle)$ is then called a non-Archimedean (or p -adic) Hilbert space, see, for example, [2, 5, 6].

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$ denote its (topological) dual and $B(\mathbb{E})$ the Banach space of all bounded linear operators on $\mathbb{E}$, see [2, 3, 5, 6]. Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are equipped with their respective natural norms. For $(u, v) \in \mathbb{E} \times \mathbb{E}^*$, we define the linear operator $(v \otimes u)$ by setting

$$\forall x \in \mathbb{E}, \quad (v \otimes u)(x) := v(x)u = \langle v, x \rangle u. \quad (2.7)$$

It follows that $\|v \otimes u\| = \|v\| \cdot \|u\|$.

Let $(e_i)_{i \in \mathbb{N}}$ be an orthogonal basis for $\mathbb{E}$. Define $e'_i \in \mathbb{E}^*$ by $x = \sum_{i \in \mathbb{N}} x_i e_i$ with $e'_i(x) = x_i$. It turns out that $\|e'_i\| = 1 \|e_i\|$. Furthermore, every $x' \in \mathbb{E}^*$ can be expressed as a pointwise convergent series: $x' e = \sum_{i \in \mathbb{N}} \langle x', e_i \rangle e_i$. Moreover,

$$\|x'\| := \sup_{i \in \mathbb{N}} \left[\frac{|\langle x', e_i \rangle|}{\|e_i\|} \right]. \quad (2.8)$$

Now let us recall that every bounded linear operator A on $\mathbb{E}$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{ij} a_{ij} (e'_j \otimes e_i), \quad (2.9)$$

and for any $j \in \mathbb{N}$,

$$\lim_{i \rightarrow \infty} |a_{ij}| \|e_i\| = 0. \quad (2.10)$$

Moreover, for each $j \in \mathbb{N}$, $A e_j = \sum_{i \in \mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\|A\| := \sup_{i,j} \left[\frac{|a_{ij}| \|e_i\|}{\|e_j\|} \right]. \quad (2.11)$$

2.2. Unbounded linear operators on free Banach spaces. Let $\mathbb{E}, \mathbb{F}$ be free Banach spaces. Suppose that $(e_i)_{i \in \mathbb{N}}$ and $(h_j)_{j \in \mathbb{N}}$ are, respectively, the canonical orthogonal bases associated with the free Banach spaces $\mathbb{E}$ and $\mathbb{F}$.

For details on the next definition, see [3, 4], in which a similar definition appears on non-Archimedean Hilbert spaces.

Definition 2.8. An unbounded linear operator A from $\mathbb{E}$ into $\mathbb{F}$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}$ (called the domain of A), and a (possibly not continuous)

linear transformation $A : D(A) \subset \mathbb{E} \rightarrow \mathbb{F}$, such that the domain $D(A)$ contains the basis $(e_i)_{i \in \mathbb{N}}$ and consists of all $u = (u_i)_{i \in \mathbb{N}} \in \mathbb{E}$, such that $Au = \sum_{i \in \mathbb{N}} u_i A e_i$ converges in $\mathbb{F}$, that is,

$$\begin{aligned} D(A) &:= \left\{ u = (u_i)_{i \in \mathbb{N}} \in \mathbb{E} : \lim_{i \rightarrow \infty} \|u_i\| \|A e_i\| = 0 \right\}, \\ A &= \sum_{i,j \in \mathbb{N}} a_{i,j} e'_j \otimes h_i, \quad \forall j \in \mathbb{N}, \lim_{i \rightarrow \infty} \|a_{i,j}\| \|h_i\| = 0. \end{aligned} \quad (2.12)$$

The collection of those unbounded linear operators is denoted by $U(\mathbb{E}, \mathbb{F})$.

For more on these and related issues, we refer the reader to [3, 4].

3. p -adic C_0 -semigroup of bounded linear operators

Let $(\mathbb{E}, \|\cdot\|)$ be a free Banach space. Throughout the rest of this paper, we consider families $(T(\kappa))_{\kappa \in \Omega_r} : \mathbb{E} \rightarrow \mathbb{E}$ of bounded linear operators. We always suppose that r is suitably chosen so that $\kappa \mapsto T(\kappa)$ is well defined.

Definition 3.1. Let $r > 0$ be a real number. A family $(T(\kappa))_{\kappa \in \Omega_r} : \mathbb{E} \rightarrow \mathbb{E}$ of bounded linear operators will be called a semigroup of bounded linear operators on $\mathbb{E}$ if

- (i) $T(0) = I_{\mathbb{E}}$, where $I_{\mathbb{E}}$ is the unit operator of $\mathbb{E}$;
- (ii) $T(\kappa + \kappa') = T(\kappa)T(\kappa')$ for all $\kappa, \kappa' \in \Omega_r$.

The semigroup $(T(\kappa))_{\kappa \in \Omega_r}$ will be called of class C_0 or strongly continuous if the following additional condition holds:

- (iii) $\lim_{\kappa \rightarrow 0} \|T(\kappa)x - x\| = 0$ for each $x \in \mathbb{E}$.

Remark 3.2. A semigroup $(T(\kappa))_{\kappa \in \Omega_r}$ will be called uniformly continuous if the following additional condition holds:

- (iv) $\lim_{\kappa \rightarrow 0} \|T(\kappa) - I_{\mathbb{E}}\| = 0$.

Remark 3.3. One should point out that a semigroup $(T(\kappa))_{\kappa \in \Omega_r}$ of bounded linear operators is not only a semigroup, but also a group. Every $T(\kappa)$ is invertible, the inverse being $T(-\kappa)$, according to Definition 3.1(i) and (ii). Moreover, it is an infinite abelian group.

Definition 3.4. If $(T(\kappa))_{\kappa \in \Omega_r}$ is a semigroup, then the linear operator A defined by

$$\begin{aligned} D(A) &= \left\{ x \in \mathbb{E} : \lim_{\kappa \rightarrow 0} \left(\frac{T(\kappa)x - x}{\kappa} \right) \text{ exists} \right\}, \\ Ax &= \lim_{\kappa \rightarrow 0} \left(\frac{T(\kappa)x - x}{\kappa} \right), \quad \text{for each } x \in D(A), \end{aligned} \quad (3.1)$$

is called the infinitesimal generator associated with the semigroup $(T(\kappa))_{\kappa \in \Omega_r}$.

Remark 3.5. (i) Note that if $(T(\kappa))_{\kappa \in \Omega_r}$ is a semigroup on $\mathbb{E}$ and if $(e_i)_{i \in \mathbb{N}}$ denotes the orthogonal basis for $\mathbb{E}$, then $T(\kappa)$ for each $\kappa \in \Omega_r$ can be expressed [3, 5, 6], for any $x = \sum_{i \in \mathbb{N}} x_i e_i \in \mathbb{E}$, by $T(\kappa)x = \sum_{i \in \mathbb{N}} x_i T(\kappa)e_i$, where

$$\forall s \in \mathbb{N}, \quad T(\kappa)e_s = \sum_{i \in \mathbb{N}} a_{i,s}(\kappa)e_i \quad \text{with} \quad \lim_{i \rightarrow \infty} |a_{i,s}(\kappa)| \|e_i\| = 0. \quad (3.2)$$

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(ii) Using (i), one can easily see that for each $0 \neq \kappa \in \Omega_r$,

$$\forall s \in \mathbb{N}, \quad \left(\frac{T(\kappa) - I_{\mathbb{E}}}{\kappa} \right) e_s = \left(\frac{a_{s,s}(\kappa) - 1}{\kappa} \right) e_s + \sum_{i \neq s} \frac{a_{i,s}(\kappa)}{\kappa} e_i \quad (3.3)$$

with $\lim_{i \neq s, i \rightarrow \infty} |a_{i,s}(\kappa)| \|e_i\| = 0$.

(iii) If $(T(\kappa))_{\kappa \in \Omega_r}$ is a semigroup on $\mathbb{E}$, then its infinitesimal generator A may or may not be a bounded linear operator on $\mathbb{E}$.

In this paper, we mainly focus on general semigroups and strongly continuous semigroups of bounded linear operators on general free Banach spaces.

We begin with the following example.

Example 3.6. Take $\mathbb{K} = \mathbb{Q}_p$ the field of p -adic numbers. Consider the ball Ω_r of $\mathbb{Q}_p$ with $r = p^{-1/(p-1)}$. Let $\mathbb{E}$ be a free Banach space over $\mathbb{Q}_p$ and let $(e_i)_{i \in \mathbb{N}}$ be the canonical orthogonal base. Define for each $q \in \Omega_r$ and for $x = \sum_{i \geq 0} x_i e_i \in \mathbb{E}$ the family of linear operators $T(q)x = \sum_{i \geq 0} x_i e^{\mu_i q} e_i$, where $(\mu_i)_{i \in \mathbb{N}} \subset \Omega_r$ is a sequence of nonzero elements.

It is routine to check that the family $(T(q))_{q \in \Omega_r}$ is well defined.

PROPOSITION 3.7. *The family $(T(q))_{q \in \Omega_r}$ of linear operators given above is a C_0 -semigroup of bounded linear operators, whose infinitesimal generator is the (bounded) diagonal operator A defined by $Ax = \sum_{i \geq 0} \mu_i x_i e_i$ for each $x = \sum_{i \geq 0} x_i e_i \in \mathbb{E}$.*

Proof. First, note that $T(q)$ is analytic on the ball Ω_r . It is routine to check that $(T(q))_{q \in \Omega_r}$ is a family of bounded linear operators on $\mathbb{E}$. Indeed, for each $q \in \Omega_r$,

$$T(q)e_i = e^{\mu_i q} e_i = \left(\sum_{n \geq 0} \frac{\mu_i^n q^n}{n!} \right) e_i, \quad \forall i \in \mathbb{N}, \quad (3.4)$$

and hence, $\|T(q)\| = |(\sum_{n \geq 0} (\mu_i^n q^n)/n!)| < \infty$, by the fact that $q\mu_i \in \Omega_r$ for each $i \in \mathbb{N}$. Furthermore, one can easily check that $T(0) = I_{\mathbb{E}}$, $T(q + q') = T(q)T(q')$ for all $q, q' \in \Omega_r$, and that $\lim_{q \rightarrow 0} \|T(q)x - x\| = 0$ for each $x \in \mathbb{E}$, and hence, $(T(q))_{q \in \Omega_r}$ is a C_0 -semigroup of bounded linear operators.

Now, let B be the infinitesimal generator of $(T(q))_{q \in \Omega_r}$. It remains to show that $A = B$. First of all, let us show that $D(B) = \mathbb{E} (= D(A))$. Clearly, for each $0 \neq q \in \Omega_r$, $(T(q)e_i - e_i)/q = ((e^{\mu_i q} - 1)/q)e_i$ for each $i \in \mathbb{N}$, and hence

$$D(B) = \left\{ x = (x_i)_{i \in \mathbb{N}} : \lim_{i \rightarrow \infty} |x_i| \cdot \left\| \frac{T(q)e_i - e_i}{q} \right\| = 0 \right\} = \mathbb{E} \quad (3.5)$$

by $|x_i| \cdot \|(T(q)e_i - e_i)/q\| \leq (|x_i| \|e_i\|)/|q| \rightarrow 0$ as $i \rightarrow \infty$, for each $x = \sum_{i \in \mathbb{N}} x_i e_i \in \mathbb{E}$.

To complete the proof, it suffices to prove that

$$\left\| Ae_i - \left(\frac{T(q)e_i - e_i}{q} \right) \right\| \rightarrow 0 \quad \text{as } q \rightarrow 0. \quad (3.6)$$

The latter is actually obvious since $((e^{\mu_i q} - 1)/q) \rightarrow \mu_i$ as $q \rightarrow 0$, and hence $B = A$ is the infinitesimal generator of the C_0 -semigroup $(T(q))_{q \in \Omega_r}$. $\square$

In the next theorem, we suppose $\mathbb{K} = \mathbb{Q}_p$, where $p \geq 2$ is prime. Note also that it is a natural generalization of Example 3.6.

THEOREM 3.8. *Let A be a bounded linear operator on $\mathbb{E}$ such that $\|A\| < r = p^{-1/(p-1)}$. Then, A is the infinitesimal generator of a uniformly continuous semigroup of bounded operators $(T(q))_{q \in \Omega_r}$.*

Proof. Suppose that A is a bounded linear operator on $\mathbb{E}$ with $\|A\| < r = p^{-1/(p-1)}$, and set, for each $q \in \Omega_r$,

$$T(q) = \sum_{n \geq 0} \frac{(qA)^n}{n!}. \quad (3.7)$$

Clearly, the series given by (3.7) converges in norm and defines a family of bounded linear operators on $\mathbb{E}$, by $|q| \cdot \|A\| < r$. It is also routine to check that $T(0) = I_{\mathbb{E}}$, $T(q + q') = T(q)T(q')$ for all $q, q' \in \Omega_r$.

It remains to show that $(T(q))_{q \in \Omega_r}$ given above is uniformly continuous. Indeed, $0 \neq q \in \Omega_r$; one has $T(q) - I_{\mathbb{E}} = qA \{ \sum_{n \geq 0} ((qA)^n / (n+1)!) \}$, and hence

$$\left\| \frac{T(q) - I_{\mathbb{E}}}{q} - A \right\| \leq \|A\| \cdot \|T(q) - I_{\mathbb{E}}\| < \|T(q) - I_{\mathbb{E}}\|. \quad (3.8)$$

Now, $\|T(q) - I_{\mathbb{E}}\| \leq |q| \cdot \|A\| \cdot \|\zeta(q)\|$, where $\zeta(q) = \sum_{n \geq 0} ((qA)^n / (n+1)!) converges, and hence$

$$\lim_{q \rightarrow 0} \|T(q) - I_{\mathbb{E}}\| = 0. \quad (3.9)$$

Consequently,

$$\lim_{q \rightarrow 0} \left\| \frac{T(q) - I_{\mathbb{E}}}{q} - A \right\| = 0 \quad (3.10)$$

by both (3.8) and (3.9). □

Remark 3.9. (i) Note that the mapping $\Omega_r \rightarrow B(\mathbb{E})$, $q \mapsto T(q)$ is analytic. Furthermore, $dT(q)/dt = AT(q) = T(q)A$.

(ii) An abstract version of Theorem 3.8, that is, in a general ultrametric-valued field $\mathbb{K}$, remains an unsolved problem.

Now, let $\mathbb{K}$ be a (complete) ultrametric-valued field and let $\Omega_r \subset \mathbb{K}$ be a clopen, where r is chosen so that $\Omega_r \rightarrow B(\mathbb{E})$, $\kappa \mapsto T(\kappa)$ is well defined.

We have the following theorem.

THEOREM 3.10. *Let $(T(\kappa))_{\kappa \in \Omega_r}$ be a C_0 -semigroup satisfying $\|T(\kappa)\| \leq M$ for each $\kappa \in \Omega_r \subset \mathbb{K}$ with $M > 0$, and let A be its infinitesimal generator. Then, for each $x \in D(A)$, $T(\kappa)x \in D(A)$ for each $\kappa \in \Omega_r$. Furthermore,*

$$\left(\frac{dT(\kappa)}{d\kappa} \right) x = AT(\kappa)x = T(\kappa)Ax. \quad (3.11)$$

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Proof. The proof in some extent, is similar to that of the classical one; however, for the sake of clarity, we will provide the reader with all details.

Let $x \in D(A)$ and let $0 \neq \kappa \in \Omega_r$. Using Definition 3.1, Definition 3.4, and the boundedness of the C_0 -semigroup $T(\kappa)$, it easily follows that

$$\left(\frac{T(\kappa) - I_{\mathbb{E}}}{\kappa} \right) T(\kappa')x = T(\kappa') \left(\frac{T(\kappa) - I_{\mathbb{E}}}{\kappa} \right) x \mapsto T(\kappa')Ax \quad (3.12)$$

as $\kappa \mapsto 0$.

Consequently, $T(\kappa')x \in D(A)$ and $AT(\kappa')x = T(\kappa')Ax$, by (3.12). Furthermore, since $T(\kappa')((T(\kappa) - I_{\mathbb{E}})/\kappa)x \mapsto T(\kappa')Ax$ as $\kappa \mapsto 0$, it follows that the right derivative of $T(\kappa')x$ is $T(\kappa')Ax$. Thus, to complete the proof, we have to show that for each $0 \neq \kappa' \in \Omega_r$, the left derivative of $T(\kappa')x$ exists and is $T(\kappa')Ax$. (Note that if $\sigma, \sigma' \in \Omega_r$, so is $\sigma - \sigma'$, by $|\sigma - \sigma'| \leq \max(|\sigma|, |\sigma'|) < r$.) Now

$$\begin{aligned} \lim_{\kappa \rightarrow 0} \left(\frac{T(\kappa')x - T(\kappa' - \kappa)x}{\kappa} - T(\kappa')x \right) &= \lim_{\kappa \rightarrow 0} T(\kappa' - \kappa) \left(\frac{T(\kappa)x - x}{\kappa} - Ax \right) \\ &\quad + \lim_{\kappa \rightarrow 0} [T(\kappa' - \kappa)Ax - T(\kappa')Ax]. \end{aligned} \quad (3.13)$$

Clearly, $\lim_{\kappa \rightarrow 0} T(\kappa' - \kappa)((T(\kappa)x - x)/\kappa - Ax) = 0$, by $\|T(\sigma)\| \leq M$ for each $\sigma \in \Omega_r$. Using the strong continuity of the semigroup $T(\kappa)$, it follows that

$$\lim_{\kappa \rightarrow 0} [T(\kappa' - \kappa)Ax - T(\kappa')Ax] = 0. \quad (3.14)$$

Consequently, $\lim_{\kappa \rightarrow 0} ((T(\kappa')x - T(\kappa' - \kappa)x)/\kappa - T(\kappa')x) = 0$, and hence the left derivative of $T(\kappa')x$ exists and equals $T(\kappa')Ax$. This completes the proof. $\square$

Remark 3.11. One of the consequences of Theorem 3.10 is that the function $v(\kappa) = T(\kappa)u_0$, $\kappa \in \Omega_r$, for some $u_0 \in D(A)$, is the solution to the homogeneous p -adic differential equation given by

$$\begin{aligned} \frac{d}{d\kappa} u(\kappa) &= Au(\kappa), \quad \kappa \in \Omega_r, \\ u(0) &= u_0, \end{aligned} \quad (3.15)$$

where $A : D(A) \subset \mathbb{E} \mapsto \mathbb{E}$ is the infinitesimal generator of the C_0 -semigroup $(T(\kappa))_{\kappa \in \Omega_r}$, and $u : \Omega_r \mapsto D(A)$ is an $\mathbb{E}$ -valued function.

Example 3.12. Take $\mathbb{K} = \mathbb{Q}_p$. Let A be the multiplication operator on $\mathbb{E} = C(\mathbb{Z}_p, \mathbb{Q}_p)$ defined by

$$Au = Q(x)u, \quad \forall u \in C(\mathbb{Z}_p, \mathbb{Q}_p), \quad (3.16)$$

where $Q = \sum_{n=0}^{\infty} q_n f_n \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

Suppose that $\|Q\|_{\infty} = \sup_{n \in \mathbb{N}} |q_n| < r$ with $r = p^{-1/(p-1)}$ (here, one can take $M = 1$).

In view of Theorem 3.10, the function defined by $v(q) = (\sum_{n \geq 0} ((qA)^n/n!))u_0$, $q \in \Omega_r$, for some $u_0 \in \mathbb{E}$, is the solution to the homogeneous p -adic differential equation

$$\begin{aligned} \frac{d}{d\kappa} u(\kappa) &= Q(\kappa)u(\kappa), \quad \kappa \in \Omega_r, \\ u(0) &= u_0 \in \mathbb{Q}_p. \end{aligned} \quad (3.17)$$

References

- [1] S. Albeverio and A. Khrennikov, *Representations of the Weyl group in spaces of square integrable functions with respect to p -adic valued Gaussian distributions*, Journal of Physics. A. Mathematical and General **29** (1996), no. 17, 5515–5527.
- [2] S. Basu, T. Diagana, and F. Ramaroson, *A p -adic version of Hilbert-Schmidt operators and applications*, Journal of Analysis and Applications **2** (2004), no. 3, 173–188.
- [3] T. Diagana, *Towards a theory of some unbounded linear operators on p -adic Hilbert spaces and applications*, Annales Mathématiques Blaise Pascal **12** (2005), no. 1, 205–222.
- [4] ———, *Erratum to: “Towards a theory of some unbounded linear operators on p -adic Hilbert spaces and applications”*, Annales Mathématiques Blaise Pascal **13** (2006), 105–106.
- [5] B. Diarra, *An operator on some ultrametric Hilbert spaces*, Journal of Analysis **6** (1998), 55–74.
- [6] ———, *Geometry of the p -adic Hilbert Spaces*, preprint, 1999.
- [7] F. Q. Gouvêa, *p -Adic Numbers*, 2nd ed., Universitext, Springer, Berlin, 1997.
- [8] A. Khrennikov, *p -Adic Valued Distributions in Mathematical Physics*, Mathematics and Its Applications, vol. 309, Kluwer Academic, Dordrecht, 1994.
- [9] A. N. Kochubei, *A Schrödinger-type equation over the field of p -adic numbers*, Journal of Mathematical Physics **34** (1993), no. 8, 3420–3428.
- [10] ———, *p -adic commutation relations*, Journal of Physics. A. Mathematical and General **29** (1996), no. 19, 6375–6378.
- [11] ———, *Heat equation in a p -adic ball*, Methods of Functional Analysis and Topology **2** (1996), no. 3-4, 53–58.
- [12] ———, *Differential equations for $\mathbb{F}_q$ -linear functions*, Journal of Number Theory **83** (2000), no. 1, 137–154.
- [13] ———, *Pseudo-Differential Equations and Stochastics Over Non-Archimedean Fields*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 244, Marcel Dekker, New York, 2001.
- [14] H. Ochsenius and W. H. Schikhof, *Banach spaces over fields with an infinite rank valuation, p -Adic Functional Analysis (Poznań, 1998)*, Lecture Notes in Pure and Appl. Math., vol. 207, Marcel Dekker, New York, 1999, pp. 233–293.
- [15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer, New York, 1983.
- [16] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, Texts in Appl. Math., vol. 13, Springer, New York, 1992.
- [17] A. C. M. van Rooij, *Non-Archimedean Functional Analysis*, Monographs and Textbooks in Pure and Applied Math., vol. 51, Marcel Dekker, New York, 1978.
- [18] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p -Adic Analysis and Mathematical Physics*, Series on Soviet and East European Mathematics, vol. 1, World Scientific, New Jersey, 1994.

Toka Diagana: Department of Mathematics, Howard University, 2441 6th Street,
 NW Washington, DC 20059, USA
E-mail address: tdiagana@howard.edu