CONVERGENCE OF p -SERIES REVISITED WITH APPLICATIONS

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We construct two adjacent sequences that converge to the sum of a given convergent p -series. In case of a divergent p -series, lower and upper bounds of the (kn) th partial sum are constructed. In either case, we extend the results obtained by Hansheng and Lu (2005) to any integer $k \geq 2$. Some numerical examples are given.

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Theorem 2 is the main result in this note. Lemma 1 is key to our main result. It is different from the result in [2] because it does not restrict the number of terms in the partial sum to an even one. Also, our inequalities (2) should be compared with the corresponding result in [2] for $k = 2$.

LEMMA 1. *Let $s_n(p)$ be the n th partial sum of the p -series $\sum_{i=1}^{\infty} (1/i^p)$, and let k be any integer greater than 1.*

(a) If $p > 0$, then

$$s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p} s_n(p) < s_{kn}(p) < s_{k-1}(p) + \frac{k}{k^p} s_n(p). \quad (1)$$

(b) If $p < 0$, then

$$k + \frac{k}{k^p} s_{n-1}(p) < s_{kn}(p) < 1 - \frac{1}{k^p} + \frac{k}{k^p} s_n(p). \quad (2)$$

Proof. Let us observe that by the definition of $s_n(p)$, we have

$$s_{kn}(p) = 1 + \frac{1}{2^p} + \cdots + \frac{1}{(kn)^p} = \sum_{j=0}^{k-1} s_{kn}^j, \quad (3)$$

where

$$s_{kn}^j = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p}. \quad (4)$$

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In particular,

$$s_{kn}^0 = \frac{1}{k^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p} = \frac{1}{k^p} s_n(p). \quad (5)$$

(a) Assume that $p > 0$, and k is any integer greater than 1. For $j = 1, \dots, k-1$,

$$s_{kn}^j = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} > \frac{1}{(k-j)^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p}. \quad (6)$$

Therefore,

$$s_{kn}^j > \frac{1}{(k-j)^p} - \frac{1}{k^p} + \frac{1}{k^p} s_n(p). \quad (7)$$

Finally,

$$\sum_{j=0}^{k-1} s_{kn}^j > s_{kn}^0 + \sum_{j=1}^{k-1} \left(\frac{1}{(k-j)^p} - \frac{1}{k^p} + \frac{1}{k^p} s_n(p) \right) = s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p} s_n(p), \quad (8)$$

which concludes the proof of the left inequality of (1).

Now, for $j = 1, \dots, k-1$, we have

$$s_{kn}^j = \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} < \frac{1}{(k-j)^p} + \frac{1}{k^p} + \cdots + \frac{1}{(nk)^p}, \quad (9)$$

so that

$$s_{kn}^j < \frac{1}{(k-j)^p} + \frac{1}{k^p} s_n(p). \quad (10)$$

It follows that

$$s_{kn}(p) < s_{kn}^0 + \sum_{j=1}^{k-1} \left(\frac{1}{(k-j)^p} + \frac{1}{k^p} s_n(p) \right) = s_{k-1}(p) + \frac{k}{k^p} s_n(p), \quad (11)$$

which concludes the proof of the right inequality of (1). The proof of (1) is complete.

(b) Assume now that $p < 0$, and k is any integer greater than 1. We have

$$\begin{aligned} s_{kn}^j &= \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} \\ &> 1 + \frac{1}{k^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(k(n-1))^p} > 1 + \frac{1}{k^p} s_{n-1}(p) \end{aligned} \quad (12)$$

for $j = 0, \dots, k-1$.

It now follows that

$$s_{kn}(p) = \sum_{j=0}^{k-1} s_{kn}^j > \sum_{j=0}^{k-1} \left(1 + \frac{1}{k^p} s_{n-1}(p)\right) > k + \frac{k}{k^p} s_{n-1}(p). \quad (13)$$

This completes the proof of the left inequality of (2).

The proof of the right inequality of (2) follows from the inequalities below:

$$\begin{aligned} s_{kn}^{k-1} &= 1 + \frac{1}{(k+1)^p} + \cdots + \frac{1}{(k(n-1)+1)^p} < 1 + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p} = 1 - \frac{1}{k^p} + \frac{1}{k^p} s_n(p), \\ s_{kn}^j &= \frac{1}{(k-j)^p} + \frac{1}{(2k-j)^p} + \cdots + \frac{1}{(nk-j)^p} < \frac{1}{k^p} + \frac{1}{(2k)^p} + \cdots + \frac{1}{(nk)^p} = \frac{1}{k^p} s_n(p) \end{aligned} \quad (14)$$

for $j = 1, \dots, k-2$.

It is now clear that

$$s_{kn}(p) = s_{kn}^0 + s_{kn}^{k-1} + \sum_{j=1}^{k-2} s_{kn}^j < \frac{1}{k^p} s_n(p) + 1 - \frac{1}{k^p} + \frac{1}{k^p} s_n(p) + \sum_{j=1}^{k-2} \frac{1}{k^p} s_n(p) < 1 - \frac{1}{k^p} + \frac{k}{k^p} s_n(p). \quad (15)$$

The proof of the right inequality of (2) is complete; so are the proofs of (2) and Lemma 1. $\square$

Let us now state and prove our main theorem. Like Lemma 1, Theorem 2 generalizes the results contained in [2] to any integer $k \geq 2$. In addition to that, it extends the results in [2] from the computational point of view.

THEOREM 2. *Let k be any integer greater than 1.*

(a) *For $p \leq 1$, the p -series is divergent and*

$$\lim_{n \rightarrow \infty} \frac{s_{kn}(p)}{s_n(p)} = k^{1-p}. \quad (16)$$

(b) *For $p > 1$, the p -series converges and*

$$l_k(p) \leq \lim_{n \rightarrow \infty} s_n(p) \leq u_k(p), \quad (17)$$

where

$$l_k(p) = \frac{k^p}{k^p - k} \left(s_{k-1}(p) - \frac{k-1}{k^p} \right), \quad u_k(p) = \frac{k^p}{k^p - k} s_{k-1}(p). \quad (18)$$

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Proof. (a) Assume $p \leq 1$.

For $p \leq 0$, the general term of the p -series does not go to 0 as n goes to ∞ and thus, the p -series diverges.

For $0 < p \leq 1$, let us assume that the p -series converges to $S(p)$. By taking the limit as n goes to ∞ of (1) and solving for $S(p)$ the left inequality, one obtains

$$0 < s_{k-1}(p) - \frac{k-1}{k^p} \leq \frac{k^p - k}{k^p} S(p) \leq 0. \quad (19)$$

The contradiction displayed by (19) shows that the p -series diverges for $0 < p \leq 1$. This completes the proof that the p -series is divergent for $p \leq 1$.

Now, assume that the p -series diverges. By dividing (1) and (2) by $s_n(p)$, and taking the limit as n goes to ∞ of the newly obtained inequalities, the squeeze theorem shows that

$$\lim_{n \rightarrow \infty} \frac{s_{kn}(p)}{s_n(p)} = \frac{k}{k^p} = k^{1-p}. \quad (20)$$

This proves (16).

(b) Assume now that $p > 1$.

From (1) and the fact that $s_{k-1}(p) \leq k-1$ for each $k \geq 2$, one can write

$$s_n(p) \leq k-1 + \frac{k}{k^p} s_n(p). \quad (21)$$

Solving (21) for $s_n(p)$, one obtains

$$s_n(p) \leq \frac{(k-1)k^{p-1}}{k^{p-1} - 1}, \quad (22)$$

which shows that the sequence of the partial sums of the p -series is bounded above. Since it is also increasing as the sum of positive numbers, it is convergent. This concludes the proof of the convergence of the p -series for $p > 1$.

Now, let $S(p)$ be the sum of the p -series. By taking the limit as n goes to ∞ of the inequalities (1), one obtains

$$s_{k-1}(p) - \frac{k-1}{k^p} + \frac{k}{k^p} S(p) < S(p) < s_{k-1}(p) + \frac{k}{k^p} S(p) \quad (23)$$

or equivalently

$$\frac{k^p}{k^p - k} \left(s_{k-1}(p) - \frac{k-1}{k^p} \right) \leq \lim_{n \rightarrow \infty} s_n(p) \leq \frac{k^p}{k^p - k} s_{k-1}(p). \quad (24)$$

The proof of Theorem 2 is complete. □

COROLLARY 3. For a divergent p -series,

$$\lim_{n \rightarrow \infty} \frac{s_{kn}(p)}{s_{mn}(p)} = \left(\frac{k}{m}\right)^{1-p}. \quad (25)$$

Proof.

$$\lim_{n \rightarrow \infty} \frac{s_{kn}(p)}{s_{mn}(p)} = \lim_{n \rightarrow \infty} \frac{s_{kn}(p)}{s_n(p)} \times \lim_{n \rightarrow \infty} \frac{s_n(p)}{s_{mn}(p)} = k^{1-p} \times m^{-(1-p)} = \left(\frac{k}{m}\right)^{1-p}. \quad (26)$$

□

Example 4.

$$\lim_{n \rightarrow \infty} \frac{1 + 1/\sqrt[5]{2} + 1/\sqrt[5]{3} + \cdots + 1/\sqrt[5]{7n}}{1 + 1/\sqrt[5]{2} + 1/\sqrt[5]{3} + \cdots + 1/\sqrt[5]{3n}} = \lim_{n \rightarrow \infty} \frac{s_{7n}(1/5)}{s_{3n}(1/5)} = \left(\frac{7}{3}\right)^{4/5}. \quad (27)$$

LEMMA 5. For $x \geq 2$ and $p > 1$, the function

$$f(x) = \frac{x^p}{x^p - x} \quad (28)$$

is decreasing.

Proof. Assume $x \geq 2$ and $p > 1$. Then

$$f'(x) = \frac{(1-p)x^p}{(x^p - x)^2} < 0. \quad (29)$$

□

LEMMA 6. For $x \geq 2$ and $p > 1$,

$$\left(\frac{(x+1)^p}{(x+1)^p - (x+1)} - \frac{x^p}{x^p - x}\right) + \left(\frac{(x+1)^p}{(x+1)^p - (x+1)}\right) \frac{1}{x^p} \leq 0. \quad (30)$$

Proof. Assume $x \geq 2$ and $p > 1$; let

$$g(x) = \left(\frac{(x+1)^p}{(x+1)^p - (x+1)} - \frac{x^p}{x^p - x}\right) + \left(\frac{(x+1)^p}{(x+1)^p - (x+1)}\right) \frac{1}{x^p}. \quad (31)$$

Then

$$\lim_{x \rightarrow \infty} g(x) = 0. \quad (32)$$

As in Lemma 5, one can show that $g(x)$ is increasing. This shows that

$$g(x) \leq 0. \quad (33)$$

□

We would like to point out that $l_k(p)$ and $u_k(p)$ as defined in (18) are, respectively, a lower estimate and an upper estimate of the sum of a convergent p -series. They are the general terms of sequences that enjoy some interesting properties from the computational

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point of view. We would like to study some of these properties in the next theorem and its corollary.

THEOREM 7. *The sequences $(l_k(p))_{k=2}^{\infty}$ and $(u_k(p))_{k=2}^{\infty}$ are adjacent.*

Proof. (i) By construction,

$$l_k(p) \leq u_k(p), \quad (34)$$

for each $k \geq 2$.

(ii) Let us show that $(u_k(p))_{k=2}^{\infty}$ is nonincreasing. For each $k \geq 2$, we have

$$\begin{aligned} u_{k+1}(p) - u_k(p) &= \frac{(k+1)^p}{(k+1)^p - (k+1)} s_k(p) - \frac{k^p}{k^p - k} s_{k-1}(p) \\ &= \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} - \frac{k^p}{k^p - k} \right) s_{k-1}(p) + \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} \right) \frac{1}{k^p}. \end{aligned} \quad (35)$$

Lemma 5 implies that

$$\left(\frac{(k+1)^p}{(k+1)^p - (k+1)} - \frac{k^p}{k^p - k} \right) s_{k-1}(p) + \left(\frac{(k+1)^p}{(k+1)^p - (k+1)} \right) \frac{1}{k^p} \leq 0, \quad (36)$$

which shows that $(u_k(p))_{k=2}^{\infty}$ is a nonincreasing sequence.

(iii) The proof that $(l_k(p))_{k=2}^{\infty}$ is nondecreasing is similar to the one given in part (ii).

(iv) From (18), we have

$$\lim_{k \rightarrow \infty} (u_k(p) - l_k(p)) = \lim_{k \rightarrow \infty} \frac{k^p}{k^p - k} \left(\frac{k-1}{k^p} \right) = \lim_{k \rightarrow \infty} \frac{k-1}{k^p - k} = 0. \quad (37)$$

The proof of Theorem 7 is complete. $\square$

The following corollary shows the computational importance of Theorem 7. It will be used to illustrate the results obtained in this note.

COROLLARY 8. *If the p -series converges to $S(p)$, then*

$$\lim_{k \rightarrow \infty} u_k(p) = \lim_{k \rightarrow \infty} l_k(p) = S(p). \quad (38)$$

Proof. The convergence of $(u_k(p))_{k=2}^{\infty}$ follows from the fact that it is nonincreasing and bounded below by any term of the sequence $(l_k(p))_{k=2}^{\infty}$. Likewise, the convergence of $(l_k(p))_{k=2}^{\infty}$ follows from the fact that it is nondecreasing and bounded above by any term of $(u_k(p))_{k=2}^{\infty}$. Now, from (37), one concludes that $(u_k(p))_{k=2}^{\infty}$ and $(l_k(p))_{k=2}^{\infty}$ have the same limit $S(p)$. $\square$

Example 9. The p -series $\sum_{i=1}^{\infty} 1/i^2$ with $p = 2$ is known to converge to $\pi^2/6$. Let us sum it correct to one decimal place using the sequences $(l_k(p))_{k=2}^{\infty}$ and $(u_k(p))_{k=2}^{\infty}$.

Table 1

k	$l_k(2)$	$u_k(2)$	$u_k(2) - l_k(2)$
2	1.50	2.00	0.50
3	1.54	1.88	0.33
4	1.56	1.81	0.25
5	1.58	1.78	0.20
6	1.59	1.76	0.17
7	1.60	1.74	0.14
8	1.60	1.73	0.13
9	1.61	1.72	0.11
10	1.61	1.71	0.10
11	1.61	1.70	0.09

Solving the inequality obtained from (37) for $p = 2$,

$$\frac{k-1}{k^2-k} < 10^{-1}, \quad (39)$$

we have

$$k > 10. \quad (40)$$

Table 1 shows the details of the computation as obtained using *Microsoft Excel*.

By averaging $l_{11}(2)$ and $u_{11}(2)$, we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \approx 1.655, \quad (41)$$

while $\pi^2/6 \approx 1.645$. It is important to realize that the values of $l_2(2)$ and $u_2(2)$ are the same as those obtained in [2]; however, unlike in [2], our algorithm allows any given accuracy, thanks to the adjacent sequences $(l_k(p))_{k=2}^{\infty}$ and $(u_k(p))_{k=2}^{\infty}$ that it generates.

Conclusion. While the convergence of p -series has been extensively studied in the literature with different levels of sophistication (see, e.g., [1, 3, 4]), the generalization of the elegant approach developed in [2] has led us to two types of results that translate into the following applications: the estimation of the sum of a convergent p -series as the limit of adjacent sequences and the limit of the ratio of partial sums containing different multiples of n terms.

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